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Hamiltonian quantum dynamics with separability constraints

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Abstract

Schroedinger equation on a Hilbert space $\mathcal{H}$, represents a linear Hamiltonian dynamical system on the space of quantum pure states, the projective Hilbert space $P\mathcal{H}$. Separable states of a bipartite quantum system form a special submanifold of $P\mathcal{H}$. We analyze the Hamiltonian dynamics that corresponds to the quantum system constrained on the manifold of separable states, using as an important example the system of two interacting qubits. The constraints introduce nonlinearities which render the dynamics nontrivial. We show that the qualitative properties of the constrained dynamics clearly manifest the symmetry of the qubits system. In particular, if the quantum Hamilton's operator has not enough symmetry, the constrained dynamics is nonintegrable, and displays the typical features of a Hamiltonian dynamical system with mixed phase space. Possible physical realizations of the separability constraints are discussed.

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1. Introduction

Classical and quantum descriptions of a physical system that is considered as composed of interacting subsystems have radically different features. The typical feature of quantum dynamics is the creation of specifically quantum correlations, the entanglement, among the subsystems. On the other hand, the typical property of classical description is the occur-

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rence of chaotic orbits and fractality of the phase space portrait, which can be considered as typically classical type of correlations between the subsystems. The type of correlations introduced by the dynamical entanglement does not occur in the classical description, and likewise, the type of correlations introduced by the chaotic orbits with fractal structures does not occur in the quantum description. This intriguing complementarity of the two descriptions represents a problem that is expected to be solved by a detailed formulation of the correspondence principle.

Comparison of typical features of classical and quantum mechanics is facilitated if the same mathematical framework is used in both theories. It is well known, since the work of Kibble [1–3], that the quantum evolution, determined by the linear Schroedinger equation, can be represented using the typical language of classical mechanics, that is as a Hamiltonian dynamical system on an appropriate phase space, given by the Hilbert space geometry of the quantum system. This line of research was later developed into the full geometric Hamiltonian representation of quantum mechanics [4–12]. Such geometric formulation of quantum mechanics has recently inspired natural definitions of measures of the entanglement [13], and has been used to model the spontaneous collapse of the state vector [14,15], and dynamics of decoherence [16].

It is our goal to use the geometric Hamiltonian formulation of quantum mechanics to study the relation between the dynamical entanglement and typical qualitative properties of Hamiltonian dynamics. Motivated by the fact that the Schroedinger equation can always be considered as a Hamiltonian dynamical system, and that for Hamiltonian systems the definitions and properties of the dynamical chaos are well understood, we shall seek for a formal condition that when imposed on Hamiltonian system representing the Schroedinger equation of the compound quantum system renders the Hamiltonian dynamics nonintegrable and chaotic. It is well known that the linear Schroedinger equation of quantum mechanics represents always an integrable Hamiltonian dynamical system, irrespective of the dynamical symmetries of the system. This is in sharp contrast with the Hamiltonian formulation of classical systems, where enough symmetry implies integrability and the lack of it implies the chaotic dynamics. Linearity of the quantum Hamiltonian dynamics, and the consequent integrability, is introduced in the Hamiltonian formulation by a very large dimensionality of the phase space of the quantum system. This high dimensionality can be considered as a consequence of two reasons. For a single quantum system, say a one dimensional particle in a potential, linear evolution and with it the principle of state superposition require infinite dimensional phase space of the Hamiltonian formulation. If the classical mechanical model is linear, say the harmonic oscillator, the quantum Hamiltonian dynamics can be exactly describe on the reduced finite-dimensional phase space, the real plane in the case of the harmonic oscillator. The other related reason that increases the dimensionality of the quantum phase space compared to the classical model is in the way the state space of the compound systems are formed out of the components state spaces in the two theories. In order to represents the entangled states as points of the quantum phase space the dimensionality of the quantum phase space is much larger than just the sum of the dimensions of the components phase spaces. The points in the Cartesian product of the components phase spaces represent the separable quantum states and form a subset of the full quantum phase space. Needles to say, although the separable states are the most classical-like states of the compound system, they still are quantum states with nonclassical properties like nonzero dispersion of some subsystem's variables. Our main result will be that when the quantum dynamics, represented as a

Hamiltonian system, is constrained on the manifold of separable quantum states the relation between the symmetry and the qualitative properties of the dynamics such as integrability or chaotic motion is reestablished. Thus, suppression of dynamical entanglement is enough to enable manifestations of the qualitative differences in dynamics of quantum systems and the relation between integrability and symmetry, traditionally related with classical mechanical models.

In order to study the relation between the dynamical entanglement, separability and the properties of Hamiltonian formulation of the quantum dynamics we shall use, in this paper, the simplest quantum system that displays the dynamical entanglement, that is a system of two interacting qubits

$$H = \omega\sigma^1 + \omega\sigma^2 + \mu_x\sigma_x^1\sigma_x^2 + \mu_y\sigma_y^1\sigma_y^2 + \mu_z\sigma_z^1\sigma_z^2, \tag{1}$$

where ${\sigma_{x,y,z}}^i$ are the three Pauli matrices of the ith qubit, and satisfy the usual $SU(2)$ commutation relations. In particular, we shall compare the dynamics of the system (1) in the case $\mu_z \neq 0$, $\mu_x = \mu_y = 0$ with the case when $\mu_x \neq 0$, $\mu_z = \mu_y = 0$. The former case is symmetric with respect to $SO(2)$ rotations around z-axis and the later lacks this symmetry. Besides its simplicity, the systems of the form (1) are of considerable current interest because the Hamiltonian of the universal quantum processor is of this form [17,18].

Various lines of research, during the last decade, improved the understanding of the relation between dynamical entanglement and properties of the dynamics. Strong impetus to the study of all aspects of quantum entanglement came from the theory of quantum computation [18]. Quantization of classical nonintegrable systems, and various characteristic properties of resulting quantum systems, have been studied for a long time [19]. The dependence of the dynamical entanglement, between a quantum system and its environment, on the qualitative properties of the dynamics of the system was studied indirectly, within the theory of environmental decoherence [20]. The relation between the rates of dynamical entanglement and the qualitative properties of the dynamics in the semi-classical regime was initiated in the reference [21] and various aspects of this relations have been studied since [22–31]. The relation between the symmetry of the genuinely quantum system (1) and the degree of dynamical entanglement was studied in reference [32]. As we shall see, our present analyzes is related to the quoted works, but the relation between the dynamical entanglement and symmetry is here approached from a very different angle.

The structure of the paper is as follows. We shall first recapitulate the necessary background such as: the complex symplectic and Riemannian geometry of $\mathbf{C}P^n$; Hamiltonian formulation on $\mathbf{C}P^n$ of the quantum dynamics; geometric formulation of the set of separable pure states and Hamiltonian formulation of the constrained dynamics. In parallel with the general reminder, the explicit formulas for the system of two interacting qubits will be given. These are then applied, in Section 3, to the study the qualitative properties of the separability constrained dynamics for the qubits systems. The main results are summarized and discussed in Section 4. There we also discuss a model of an open quantum system with dynamics that clearly differentiates between the symmetric and the nonsymmetric systems.

2. Geometry of the state space $\mathbf{C}P^n$

Hamiltonian formulation of quantum mechanics is based on the fact that the scalar product of vectors $|\psi\rangle$ in the Hilbert space of a quantum system can be used to represent the linear Schroedinger equation of quantum mechanics in the form of Hamilton's equations. The canonical phase space structure of this equations is determined by the imaginary part of the scalar product, and the Hamilton' s function is given by the quantum expectation $\langle\psi|H|\psi\rangle$ of the quantum Hamiltonian.

However, due to phase invariance and arbitrary normalization the proper space of pure quantum states is not the Hilbert space used to formulate the Schroedinger equation, but the projective Hilbert space which is the manifold to be used in the Hamiltonian formulation of quantum mechanics. In general, the resulting Hamiltonian dynamical system is infinite-dimensional, but we shall need the general definitions only for the case of quantum system with finite-dimensional Hilbert space, like the finite collection of qubits, in which case the quantum phase space is also finite-dimensional. We shall first review the definition of the complex projective space $\mathbf{C}P^n$, and then briefly state the basic definitions and recapitulate the formulas which are needed for the Hamiltonian formulation of the quantum dynamics on the state space and its restriction on the separable state subset. The general reference for the mathematical aspects of complex differential geometry is [33]. All concepts and formulas will be illustrated using the system of two interacting qubits.

Differential geometry of the state space $\mathbf{C}P^n$ is discussed by viewing it as a real $2n$ dimensional manifold endowed with complex, Riemannian and symplectic structure. In the case of $\mathbf{C}P^n$ this three structures are compatible.

2.1. Definition and intrinsic coordinates of $\mathbf{C}P^n$

States of a collection of $N = n + 1$ qubits are represented using normalized vectors of the complex Hilbert space $\mathbf{C}^N$. Since all quantum mechanical predictions are given in terms of the Hermitian scalar product on $\mathbf{C}^N$, and this is invariant under multiplication by a constant (vector independent) phase factor, the states of the quantum system are actually represented by equivalence classes of vectors in $\mathbf{C}^N$. Two vectors ψ_1 and ψ_2 are equivalent: $\psi_2 \sim \psi_1$ if there is a complex scalar $a \neq 0$ such that $\psi_2 = a\psi_1$. This set of equivalence classes defines the complex projective space: $\mathbf{C}P^n :\equiv (\mathbf{C}^{n+1} - 0)/\sim$. It is the state space of the system of N qubits. Global coordinates $(c^1, \ldots, c^N)$ of a vector in $\mathbf{C}^N$ that represent an equivalence class $[\psi]$, that is an element of $\mathbf{C}P^n$, are called homogeneous coordinates on $\mathbf{C}P^n$. The complex projective space is topologically equivalent to S^{2n+1}/S^1, where the $2n + 1$-dimensional sphere comes from normalization and the circle S^1 takes care of the unimportant overall phase factor.

The projective space $\mathbf{C}P^n$ is locally homeomorphic with $\mathbf{C}^n$. Intrinsic coordinates on $\mathbf{C}P^n$ are introduced as follows. A chart U_μ consists of equivalence classes of all vectors in $(\mathbf{C}^{n+1} - 0)$ such that $c_\mu \neq 0$. In the chart U_μ the local (so called inhomogeneous) coordinates ζ^v, $>v = 1, 2, \ldots, n$ are given by

$$\zeta^v = \xi^v \quad (v \leqslant \mu - 1), \qquad \zeta^v = \xi^{v+1} \quad (v > \mu), \tag{2}$$

where

$$\xi^v = c^v/c^\mu \quad v = 1, 2, \ldots, \mu - 1, \mu + 1, \ldots, n + 1. \tag{3}$$

The coordinates $\zeta^v_\mu(c)$ and $\zeta^v_{\mu'}(c)$ of a point c which belongs to the domain where two charts U_μ and $U_{\mu'}$ overlap are related by the following holomorphic transformation

$$\zeta^v_{\mu'}(c) = (c^\mu/c^{\mu'})\zeta^v_\mu(c). \tag{4}$$

As an illustration consider the system of two qubits. The Hilbert space is $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 = \mathbf{C}^2 \otimes \mathbf{C}^2 = \mathbf{C}^4$. As a basis we can choose the set of separable vectors $|\uparrow\uparrow\rangle$, $\langle|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$ or any other four orthogonal vectors. The coordinates of a vector in $\mathbf{C}^4$ with respect to a basis are denoted (c^1, c^2, c^3, c^4). The corresponding projective space is $\mathbf{C}P^3 \equiv S^7/S^1$. At least two charts are needed to define the intrinsic coordinates over all $\mathbf{C}P^3$. Consider first all vectors with a nonzero component along $|1\rangle = |\uparrow\uparrow\rangle$ that is $c^1 \neq 0$, i.e. all vectors except the vector $|\downarrow\downarrow\rangle$. Then the numbers ξ_1^ν are defined as $\xi_1^1 = c^1/c^1 = 1, \xi_1^2 = c^2/c^1, \xi_1^3 = c^3/c^1, \xi_1^4 = c^4/c^1$ and finally the three intrinsic coordinates $(\zeta_1^1, \zeta_1^2, \zeta_1^3)$ are given by relabelling of ξ_1^ν: $\zeta_1^1 = \xi_1^2, \zeta_1^2 = \xi_1^3, \zeta_1^3 = \xi_1^4$. To coordinatize the vector $|4\rangle = |\downarrow\downarrow\rangle$ we need another chart.

Quantum mean values of linear operators on $\mathbf{C}^4$ are indeed reduced to functions on $\mathbf{C}P^3$. For example, consider the following Hamiltonian operator

$$H = \omega\sigma_z \otimes \mathbf{1} + \omega\mathbf{1} \otimes \sigma_z + \mu\sigma_x \otimes \sigma_x. \tag{5}$$

In the separable bases the normalized quantum expectation $\langle\psi|H|\psi\rangle/\langle\psi|\psi\rangle$ is given by the following function of $(c^1, c^2, \ldots, \bar{c}^4)$

$$H = \frac{2\omega(c^1\bar{c}^1 - c^4\bar{c}^4) + \mu(\bar{c}^2c^3 + \bar{c}^3c^2 + \bar{c}^1c^4 + \bar{c}^4c^1)}{c^1\bar{c}^1 + c^2\bar{c}^2 + c^3\bar{c}^3 + c^4\bar{c}^4}. \tag{6}$$

In the intrinsic coordinates $\zeta^1, \zeta^2, \zeta^3$ and their conjugates this expression is given by

$$H = \frac{2\omega(1 - \zeta^3\bar{\zeta}^3) + \mu(\bar{\zeta}^1\zeta^2 + \bar{\zeta}^2\zeta^1 + \zeta^3 + \bar{\zeta}^3)}{1 + \zeta^1\bar{\zeta}^1 + \zeta^2\bar{\zeta}^2 + \zeta^3\bar{\zeta}^3}. \tag{7}$$

We shall also analyze the following Hamiltonian

$$H = \omega\sigma_z \otimes \mathbf{1} + \omega\mathbf{1} \otimes \sigma_z + \mu\sigma_z \otimes \sigma_z, \tag{8}$$

whose normalized mean value is given by

$$H = \frac{2\omega(c^1\bar{c}^1 - c^4\bar{c}^4) + \mu(c^1\bar{c}^1 + c^4\bar{c}^4 - c^2\bar{c}^2 - c^3\bar{c}^3)}{c^1\bar{c}^1 + c^2\bar{c}^2 + c^3\bar{c}^3 + c^4\bar{c}^4}. \tag{9}$$

The corresponding function on $\mathbf{C}P^3$ is, in the intrinsic coordinates, given by

$$H = \frac{\omega(1 - \zeta^3\bar{\zeta}^3) + \mu(1 + \zeta^3\bar{\zeta}^3 - \zeta^1\bar{\zeta}^1 - \zeta^2\bar{\zeta}^2)}{1 + \zeta^1\bar{\zeta}^1 + \zeta^2\bar{\zeta}^2 + \zeta^3\bar{\zeta}^3}. \tag{10}$$

2.1.1. Submanifold of separable states

Consider two quantum systems **A** and **B** with the corresponding Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. Taken together, the systems **A** and **B** form another quantum system. The statistics of measurements that could be performed on this compound system requires that the Hilbert space of the compound system is given by the direct product $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. The space of pure states of the compound system is the projective Hilbert space $P\mathcal{H}_{AB}$. In the case of finite dimensional state spaces $P\mathcal{H}_A^{n+1} = CP^n$ and $P\mathcal{H}_A^{m+1} = CP^m$ the state space of the compound system is $CP^{(m+1)(n+1)-1}$. Vectors in $\mathcal{H}_{AB}$ of the form $\psi_A \otimes \psi_B$ where $\psi_{A/B} \in \mathcal{H}_{A/B}$ are called separable. The corresponding separable states form the $(m+n)$-dimensional submanifold $CP^m \times CP^n$ embedded in $CP^{(m+1)(n+1)-1}$.

In the case of two qubits the submanifold of the separable states $CP^1 \times CP^1$ forms a quadric in the full state space CP^3, given in terms of the homogeneous coordinates (c^1, c^2, c^3, c^4) of CP^3 by the following formula

$$c^1c^4 = c^2c^3. \tag{11}$$

In terms of the intrinsic coordinates $\zeta^1, \zeta^2, \zeta^3$, in the chart with $c^1 \neq 0$, i.e. $\xi^1 = 1$, Eq. (11) is

$$\zeta^1\zeta^2 = \zeta^3. \tag{12}$$

2.2. Complex structure on $\mathbf{C}P^n$

Consider a complex manifold $\mathcal{M}$ with complex dimension $\dim_C\mathcal{M} = n$ (in particular $\mathbf{C}P^n$). We can look at $\mathcal{M}$ as a real manifold with $\dim_R\mathcal{M} = 2n$. The real coordinates $(x^1,\ldots,x^{2n})$ are related to the holomorphic $(\zeta^1,\ldots,\zeta^n)$ and anti-holomorphic $(\overline{\zeta}^1,\ldots,\overline{\zeta}^n)$ coordinates via the following formulas:

$$\begin{aligned}(x^v + \imath x^{v+n})/\sqrt{2} &= \zeta^v, \quad v = 1,2,\ldots n,\\ (x^v - \imath x^{v+n})/\sqrt{2} &= \overline{\zeta}^v, \quad v = 1,2,\ldots n,\end{aligned} \tag{13}$$

and

$$\begin{aligned}q^v \equiv x^v &= (\zeta^v + \overline{\zeta}^v)/\sqrt{2}, \quad v = 1,2,\ldots n,\\ p^v \equiv x^{v+n} &= (\overline{\zeta}^v - \overline{\zeta}^v)/\sqrt{2}, \quad v = 1,2,\ldots n.\end{aligned} \tag{14}$$

The tangent space $T_x\mathcal{M}$ is spanned by $2n$ vectors

$$\left\{\frac{\partial}{\partial q^1},\cdots\frac{\partial}{\partial q^n},\frac{\partial}{\partial p^1},\cdots\frac{\partial}{\partial p^n}\right\} \tag{15}$$

or by the basis

$$\left\{\frac{\partial}{\partial \zeta^1},\cdots\frac{\partial}{\partial \zeta_n},\frac{\partial}{\partial \overline{\zeta}^1},\cdots\frac{\partial}{\partial \overline{\zeta}_n}\right\}. \tag{16}$$

An almost complex structure on a real $2n$-dimensional manifold is given by a $(1,1)$ tensor $\mathbf{J}$ satisfying bf $\mathrm{J}^2 = \mathbf{1}$, i.e. $J^a_cJ^c_b = -\delta^a_b$. Locally, the almost complex structure J is given in the real coordinates by the following matrix

$$\begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \tag{17}$$

where $\mathbf{1}$ is n-dimensional unit matrix. If the real $2n$ manifold is actually a complex manifold, like in our case, the almost complex structure is defined globally and is called the complex structure.

2.3. Riemannian structure on $\mathbf{C}P^n$

Hermitian scalar product induces a complex Euclidean metric on $\mathbf{C}^N$. The metric induced on $\mathbf{C}P^n$ is the Fubini-Study metric, and is given, in $(\zeta,\overline{\zeta})$ coordinates, using an $n \times n$ matrix with following entries

$$g_{\mu,\overline{v}}(\zeta,\overline{\zeta}) = \frac{\delta_{\mu,v}(1+\zeta\overline{\zeta}) - \zeta^\mu\overline{\zeta}^v}{(1+\zeta\overline{\zeta})^2}, \quad \mu, v = 1,2\ldots n, \tag{18}$$

where $\zeta\overline{\zeta} \equiv \sum_\mu^n \zeta^\mu\overline{\zeta}^\mu$.

The Fubini-Study metric in $(\zeta, \overline{\zeta})$ coordinates is then given by $2n \times 2n$ matrix

$$\mathcal{G}(\zeta, \overline{\zeta}) = \frac{1}{2} \begin{pmatrix} 0 & g_{\mu,\overline{\nu}} \\ g_{\overline{\mu},\nu} & 0 \end{pmatrix}. \tag{19}$$

In the real coordinates the Fubini-Study metric is given by the standard transformation formulas

$$\mathcal{G}_{i,j}(q, \overline{p}) = \mathcal{G}_{k,l}(\zeta(q,p), \overline{\zeta}(q,p)) \frac{\partial Z_i}{\partial X_k} \frac{\partial Z_j}{\partial X_l}, \tag{20}$$

where we used $Z = (\zeta^1, \ldots, \overline{\zeta}_n)$ and $X = (q^1, \ldots, p_n)$.

In the example of two qubits the Fubini-Study metric on $\mathbf{C}P^3$ is

$$2\mathcal{G} = \begin{pmatrix} 0 & 0 & 0 & \frac{(1+\zeta\overline{\zeta})-\zeta^1\overline{\zeta}^1}{(1+\zeta\overline{\zeta})^2} & \frac{-\zeta^1\overline{\zeta}^2}{(1+\zeta\overline{\zeta})^2} & \frac{-\zeta^1\overline{\zeta}^3}{(1+\zeta\overline{\zeta})^2} \\ 0 & 0 & 0 & \frac{-\zeta^2\overline{\zeta}^1}{(1+\zeta\overline{\zeta})^2} & \frac{(1+\zeta\overline{\zeta})-\zeta^2\overline{\zeta}^2}{(1+\zeta\overline{\zeta})^2} & \frac{-\zeta^2\overline{\zeta}^3}{(1+\zeta\overline{\zeta})^2} \\ 0 & 0 & 0 & \frac{-\zeta^3\overline{\zeta}^1}{(1+\zeta\overline{\zeta})^2} & \frac{-\zeta^3\overline{\zeta}^2}{(1+\zeta\overline{\zeta})^2} & \frac{(1+\zeta\overline{\zeta})-\zeta^3\overline{\zeta}^3}{(1+\zeta\overline{\zeta})^2} \\ \frac{(1+\zeta\overline{\zeta})-\zeta^1\overline{\zeta}^1}{(1+\zeta\overline{\zeta})^2} & \frac{-\zeta^2\overline{\zeta}^1}{(1+\zeta\overline{\zeta})^2} & \frac{-\zeta^3\overline{\zeta}^1}{(1+\zeta\overline{\zeta})^2} & 0 & 0 & 0 \\ \frac{-\zeta^1\overline{\zeta}^2}{(1+\zeta\overline{\zeta})^2} & \frac{(1+\zeta\overline{\zeta})-\zeta^2\overline{\zeta}^2}{(1+\zeta\overline{\zeta})^2} & \frac{-\zeta^3\overline{\zeta}^2}{(1+\zeta\overline{\zeta})^2} & 0 & 0 & 0 \\ \frac{-\zeta^1\overline{\zeta}^3}{(1+\zeta\overline{\zeta})^2} & \frac{-\zeta^2\overline{\zeta}^3}{(1+\zeta\overline{\zeta})^2} & \frac{(1+\zeta\overline{\zeta})-\zeta^3\overline{\zeta}^3}{(1+\zeta\overline{\zeta})^2} & 0 & 0 & 0 \end{pmatrix}. \tag{21}$$

Transformation to the real coordinates, by application of the formula (20), gives

$$\begin{pmatrix} \frac{b}{a} & -\frac{p^1p^2+q^1q^2}{a} & -\frac{p^1p^3+q^1q^3}{a} & 0 & \frac{p^1q^2-p^2q^1}{a} & \frac{p^1q^3-p^3q^1}{a} \\ -\frac{p^1p^2+q^1q^2}{a} & \frac{b}{a} & \frac{p^2p^3+q^2q^3}{a} & \frac{p^2q^1-p^1q^2}{a} & 0 & \frac{p^2q^3-p^3q^2}{a} \\ -\frac{p^1p^2+q^1q^2}{a} & \frac{p^2p^3+q^2q^3}{a} & \frac{b}{a} & \frac{p^3q^1-p^1q^3}{a} & \frac{p^3q^2-p^2q^3}{a} & 0 \\ 0 & \frac{p^2q^1-p^1q^2}{a} & \frac{p^3q^1-p^1q^3}{a} & \frac{b}{a} & -\frac{p^1p^2+q^1q^2}{a} & -\frac{p^1p^3q^1q^3}{a} \\ \frac{p^1q^2-p^2q^1}{a} & 0 & \frac{p^3q^2-p^2q^3}{a} & -\frac{p^1p^2+q^1q^2}{a} & \frac{b}{a} & -\frac{p^2p^3+q^2q^3}{a} \\ \frac{p^1q^3-p^3q^1}{a} & \frac{p^2q^3-q^2p^3}{a} & 0 & -\frac{p^1p^3-q^1q^3}{a} & -\frac{p^2p^3+q^2q^3}{a} & \frac{b}{a} \end{pmatrix}, \tag{22}$$

where

$$a = (p^1)^2 + (p^2)^2 + (p^3)^2 + (q^1)^2 + (q^2)^2 + (q^3)^2 + 2,$$
$$b = (p^1)^2 + (p^3)^2 + (q^1)^2 + (q^3)^2 + 2.$$

Obviously, $\mathcal{G}$ is positive definite and symmetric.

2.4. Symplectic structure on $\mathbf{C}P^n$

The Hermitian scalar product on $\mathbf{C}^N$ is also used to define the symplectic structure on C^N and this induces the symplectic structure on $\mathbf{C}P^n$. The symplectic structure is the closed nondegenerate two form Ω on $\mathbf{C}P^n$, which is, in $(\zeta, \overline{\zeta})$ coordinates given by

$$\omega = \imath g(\zeta, \overline{\zeta})_{\mu,\overline{\nu}} \mathrm{d}\zeta^\mu \wedge \overline{\zeta}^\nu, \tag{23}$$

where $g_{\mu,\bar{\nu}}$ is the Fubini-Study metric (18). In real coordinates, the symplectic structure is given by $\Omega(q,p) = J\mathcal{G}(q,p)$ where $\mathcal{G}(q,p)$ is given by (20) and J by (17).

The symplectic form on the two qubits state space is in the real bases given by the product of matrices (17) and (22). The results is

$$\Omega = \begin{pmatrix} 0 & \frac{-p^2q^1+p^1q^2}{a^2} & \frac{-p^3q^1+p^1q^3}{a^2} & -\frac{b}{a^2} & \frac{p^1p^2+q^1q^2}{a^2} & \frac{p^1p^3+q^1q^3}{a^2} \\ \frac{p^2q^1-p^1q^2}{a^2} & 0 & \frac{p^2q^3-p^3q^2}{a^2} & \frac{p^2p^1+q^1q^2}{a^2} & -\frac{b}{a^2} & \frac{p^2p^3+q^2q^3}{a^2} \\ \frac{p^3q^1-p^1q^3}{a^2} & \frac{p^3q^2-p^2q^3}{a^2} & 0 & \frac{p^1p^3+q^1q^3}{a^2} & \frac{p^2p^3+q^2q^3}{a^2} & -\frac{b}{a^2} \\ \frac{b}{a^2} & -\frac{p^2p^1+q^1q^2}{a^2} & -\frac{p^1p^3+q^1q^3}{a^2} & 0 & \frac{p^1q^2-p^2q^1}{a^2} & \frac{p^1q^3-p^3q^1}{a^2} \\ -\frac{p^1p^2+q^1q^2}{a^2} & \frac{b}{a^2} & -\frac{p^2p^3+q^2q^3}{a^2} & \frac{p^2q^1-p^1q^2}{a^2} & 0 & \frac{p^2q^3-p^3q^2}{a^2} \\ -\frac{p^1p^3+q^1q^3}{a^2} & -\frac{p^2p^3+q^2q^3}{a^2} & \frac{b}{a^2} & \frac{p^3q^1-p^1q^3}{a^2} & \frac{p^3q^2-p^2q^3}{a^2} & 0 \end{pmatrix}. \tag{24}$$

3. Quantum Hamiltonian dynamical system on $\mathbf{C}P^n$

The Schroedinger equation on C^N is in some basis $\{|\psi_i+\rangle,\ i = 1,2\ldots,N\}$ given by

$$\imath\frac{\mathrm{d}c^i}{\mathrm{d}t} = \langle\psi_j \mid H \mid \psi_i\rangle c^j. \tag{25}$$

In the real coordinates this equation assumes the form of a Hamiltonian dynamical system on R^{2N} with a global gauge symmetry corresponding to the invariance $|\psi\rangle \to \exp(ix)|\psi\rangle$. Reduction with respect to this symmetry results in the Hamiltonian system on $\mathbf{C}P^n$, considered as a real manifold with the symplectic structure given by (23). The Hamilton equation on $\mathbf{C}P^n$, that are equivalent to the Schroedinger equation (25), are

$$\frac{\mathrm{d}x^l}{\mathrm{d}t} = 2\Omega^{l,k}\nabla_k H(x), \tag{26}$$

where $\Omega^{l,k}$ is the inverse of the symplectic form, and $H(x)$ is given by the normalized quantum expectation of Hamilton's operator $\langle\psi|H|\psi\rangle/\langle\psi|\psi\rangle$ expressed in terms of the real coordinates (14). For example, the hamiltonian (7) is given in terms of the real coordinates $q^i \equiv x^i$, $p^i \equiv x^{i+n}$, $i = 1,\ldots,n$ by

$$H = \frac{\omega}{a}\left[2 - (p^3)^2 - (q^3)^2\right] + \frac{\mu}{a}\left(p^1p^2 + q^1q^2 + \sqrt{2}q^3\right), \tag{27}$$

and the symmetric hamiltonian (9) is given by

$$\begin{aligned} H = {} & \frac{\omega}{a}\left[2 - (p^3)^2 - (q^3)^2\right] \\ & - \frac{\mu}{a}\left[(p^1)^2 + (p^2)^2 + (q^1)^2 + (q^2)^2 - (p^3)^2 - (q^3)^2 - 2\right]. \end{aligned} \tag{28}$$

Hamilton's equations (26) with the hamiltonian (27) and the symplectic form (24) assume the following form:

$$\begin{aligned}
\dot{q}^1 &= -2\omega p^1 + \mu p^2 - \mu(p^3q^1 + p^1q^3)/\sqrt{2}\\
\dot{q}^2 &= -2\omega p^2 + \mu p^2 - \mu(p^3q^2 + p^2q^3)/\sqrt{2}\\
\dot{q}^3 &= -4\omega p^3 - \sqrt{2}\mu p^3 q^3\\
\dot{p}^1 &= 2\omega q^1 - \mu q^2 + \mu(q^3q^1 - p^1p^3)/\sqrt{2}\\
\dot{p}^2 &= 2\omega q^2 - \mu q^1 + \mu(q^3q^2 - p^2p^3)/\sqrt{2}\\
\dot{p}^3 &= 4\omega q^3 + \mu((q^3)^2 - (p^3)^2 - 2)/\sqrt{2}.
\end{aligned} \tag{29}$$

The equations of motion with the symmetric hamiltonian (28) on $\mathbf{C}P^3$ are quite simple

$$\begin{aligned}
\dot{q}^1 &= -2(\omega+\mu)p^1\\
\dot{q}^2 &= -2(\omega+\mu)p^2\\
\dot{q}^3 &= -4\omega p^3\\
\dot{p}^1 &= 2(\omega+\mu)q^1\\
\dot{p}^2 &= 2(\omega+\mu)q^1\\
\dot{p}^3 &= 4\omega q^3.
\end{aligned} \tag{30}$$

3.1. *Quantum Hamiltonian system with imposed separability constraints*

Dynamics of a constrained Hamiltonian system is usually described by the method of Lagrange multipliers [34,35]. Consider a Hamiltonian system given by a symplectic manifold $\mathcal{M}$ with the symplectic form Ω and Hamilton's function H on $\mathcal{M}$. Suppose that besides the forces described by H the dynamics of the system is affected also by forces whose sole effect is to constrain the motion on a submanifold $\mathcal{N} \in \mathcal{M}$ determined by a set functional relations

$$f_1(q,p) = \ldots f_k(q,p) = 0. \tag{31}$$

The method of Lagrange multiplies assumes that the dynamics on $\mathcal{N}$ is determined by the following set of differential equations

$$\dot{X} = \Omega(\nabla X, \nabla H'), \qquad H' = H + \sum_j^k \lambda_j f_j, \tag{32}$$

which should be solved together with the equations of the constraints (31). The Lagrange multipliers λ_j are functions of (p,q) that are to be determined from the following, so called compatibility, conditions

$$\dot{f}_l = \Omega(\nabla f_l, \nabla H'), \tag{33}$$

on $\mathcal{N}$. Eq. (33) uniquely determine the functions $\lambda_1(p,q), \ldots, \lambda_k(p,q)$ if and only if the matrix of Poison brackets $\{f_i, f_j\} = \Omega(\nabla f_i, \nabla f_j)$ is nonsingular. If this is the case then all constraints (31) are called primary, and $\mathcal{N}$ is symplectic manifold with the symplectic structure determined by the so called Dirac–Poison brackets

$$\{F_1, F_2\}' = \{F_1, F_2\} + \sum_{i,j}^{k} \{f_i, F_1\}\{f_i, f_j\}^{-1}\{f_j, F_2\}. \tag{34}$$

As we shall see, this is the case in the examples of pairs of interacting qubits constrained on the manifold of separable states that we shall analyze. On the other hand, if some of the compatibility equations do not contain multipliers, than for that constrain $\dot{f}_j = \{f_j, H\} = 0$, which represents an additional constraint. These are called secondary constraints, and they must be added to the system of original constraints (31). If this enlarged set of constraints is functionally independent one can repeat the procedure. At the end one either obtains a contradiction, in which case the original problem has no solution, or one obtains appropriate multipliers λ_k such that the system (33) is compatible. In the later case the solution for λ_k might not be unique in which case the orbits of (32) and (31) are not uniquely determined by the initial conditions.

Let us apply the formalism of Lagrange multipliers on the system of two interacting qubits additionally constrained to remain on the manifold of separable pure state. The real and imaginary parts of (12) give the two constraints in terms of real coordinates $(q^1, q^2, q^3, p^1, p^2, p^3)$

$$f_1 = p^1p^2 - q^1q^2 + \sqrt{2}q^3, \qquad f_2 = \sqrt{2}q^3 - p^2q^1 - p^1q^2. \tag{35}$$

The compatibility conditions (33) assume the following form

$$\begin{aligned} \dot{f}_1 &= \Omega(\nabla f_1, \nabla H) + \lambda_2 \Omega(\nabla f_1, \nabla f_2) = 0, \\ \dot{f}_2 &= \Omega(\nabla f_2, \nabla H) + \lambda_1 \Omega(\nabla f_2, \nabla f_1) = 0, \end{aligned} \tag{36}$$

where Ω is the symplectic form (24) and $\Omega(\nabla f_1, \nabla\, H) = \Omega^{a,b}\nabla_a f_1 \nabla_b\, H$.

The matrix of Poisson brackets $\{f_i, f_j\}$ on $\mathcal{N}$ is

$$\begin{pmatrix} 0 & [2 + (p^1)^2 + (q^1)^2][2 + (p^2)^2 + (q^2)^2]/8 \\ -[2 + (p^1)^2 + (q^1)^2][2 + (p^2)^2 + (q^2)^2]/8 & 0 \end{pmatrix}, \tag{37}$$

and is nonsingular. Thus the compatibility conditions can be solved for the Lagrange multipliers $\lambda_1(q,p), \lambda_2(q,p)$

$$\begin{aligned} \lambda_1 &= 4\mu \frac{4p^1p^2q^1q^2 + [(q^1)^2 - 2][2 + (p^2)^2 - (q^2)^2] + (p^1)^2[(q^2)^2 - (p^2)^2 - 2]}{[2 + (p^1)^2 + ((q^1)^2)^2(2 + (p^2)^2) + (q^2)^2]^2}, \\ \lambda_2 &= 8\mu \frac{(p^1)^2p^2q^2 - p^2q^2[(q^1)^2 - 2] + p^1q^1[2 + (p^2)^2 - (q^2)^2]}{[2 + (p^1)^2 + ((q^1)^2)^2(2 + (p^2)^2) + (q^2)^2]^2}. \end{aligned} \tag{38}$$

Finally, the dynamics of the constrained system is described by Eqs. (32) and (31) with $\lambda_1(q,p), \lambda_2(q,p)$ and $f_1(q,p), f_2(q,p)$ given by (38) and (35). For the Hamiltonian (27) the resulting equations of motion for q^1, q^2, p^1, p^2 are

$$
\begin{aligned}
\dot{q}^1 &= -\frac{4\mu p^1 q^1 q^2 + 2\omega p^1[2+(p^2)^2+(q^2)^2]}{2+(p^2)^2+(q^2)^2},\\
\dot{q}^2 &= -\frac{4\mu p^2 q^1 q^2 - 2\omega p^2[2+(p^1)^2+(q^1)^2]}{2+(p^1)^2+(q^1)^2},\\
\dot{p}^1 &= \frac{2\mu q^2[(q^1)^2-(p^1)^2-2] + 2\omega q^1[2+(p^2)^2+(q^2)^2]}{2+(p^2)^2+(q^2)^2},\\
\dot{p}^2 &= \frac{2\mu q^1[(q^2)^2-(p^2)^2-2] + 2\omega q^2[2+(p^1)^2+(q^1)^2]}{2+(p^1)^2+(q^1)^2}.
\end{aligned}
\tag{39}
$$

The same procedure for the symmetric hamiltonian (28) results with the following equations of motion:

$$
\begin{aligned}
\dot{q}^1 &= \frac{2\mu p^1[(p^2)^2+((q^2)^2-2)] - 2\omega p^1[2+(p^2)^2+(q^2)^2]}{2+(p^2)^2+(q^2)^2},\\
\dot{q}^2 &= \frac{2\mu p^2[(p^1)^2+(q^1)^2-2] - 2\omega p^2[(2+(p^1)^2)+(q^1)^2]}{2+(p^1)^2+(q^1)^2},\\
\dot{p}^1 &= \frac{-2\mu q^1[(q^2)^2+(p^2)^2-2] + 2\omega q^1[2+(p^2)^2+(q^2)^2]}{2+(p^2)^2+(q^2)^2},\\
\dot{p}^2 &= \frac{-2\mu q^2[(q^1)^2+(p^1)^2-2] + 2\omega q^2[2+(p^1)^2+(q^1)^2]}{2+(p^1)^2+(q^1)^2}.
\end{aligned}
\tag{40}
$$

There are also the equations expressing $\dot{q}^3$ and $\dot{p}^3$ in terms of q^1, q^2, p^1, p^2, but the solutions of these are already given by the constraints.

3.2. Qualitative properties of the constrained dynamics of two interacting qubits

In this section, we present the results of numerical analyzes of the qualitative properties of the dynamics generated by the constrained equations (40) and (39), corresponding to the quantum Hamiltonians (28) with the $SO(2)$ symmetry and (27) without such symmetry.

It is well known that any quantum system is integrable when considered as the Hamiltonian dynamical system on the symplectic space $\mathcal{H}$, and that the reduction on the symplectic manifold $P\mathcal{H}$ preserves this property. This is simply a consequence of the form of the quantum Hamiloton's function, which is always defined as the mean value of the Hamiltonian operator. Contrary to the case of classical Hamiltonian systems, the symmetry of the physical system has no relevance for the property of integrability in the Hamiltonian formulation of the Schroedinger equation. We illustrate this fact, in Figs. 1a and b, by projections on (q^1, p^1) plane of a typical orbit for the symmetric and nonsymmetric Hamiltonians of the pair of qubits. The motion on $\mathbf{C}P^3$ in the symmetric case has further degeneracy compared with the nonsymmetric case, but both cases generate integrable, regular Hamiltonian dynamics.

On the other hand, the qualitative properties of the dynamics constrained by the separability conditions, are quite different. Typical orbits in the symmetric and nonsymmetric cases are illustrated in Figs. 1c and d. Symmetric dynamics constrained by separability is

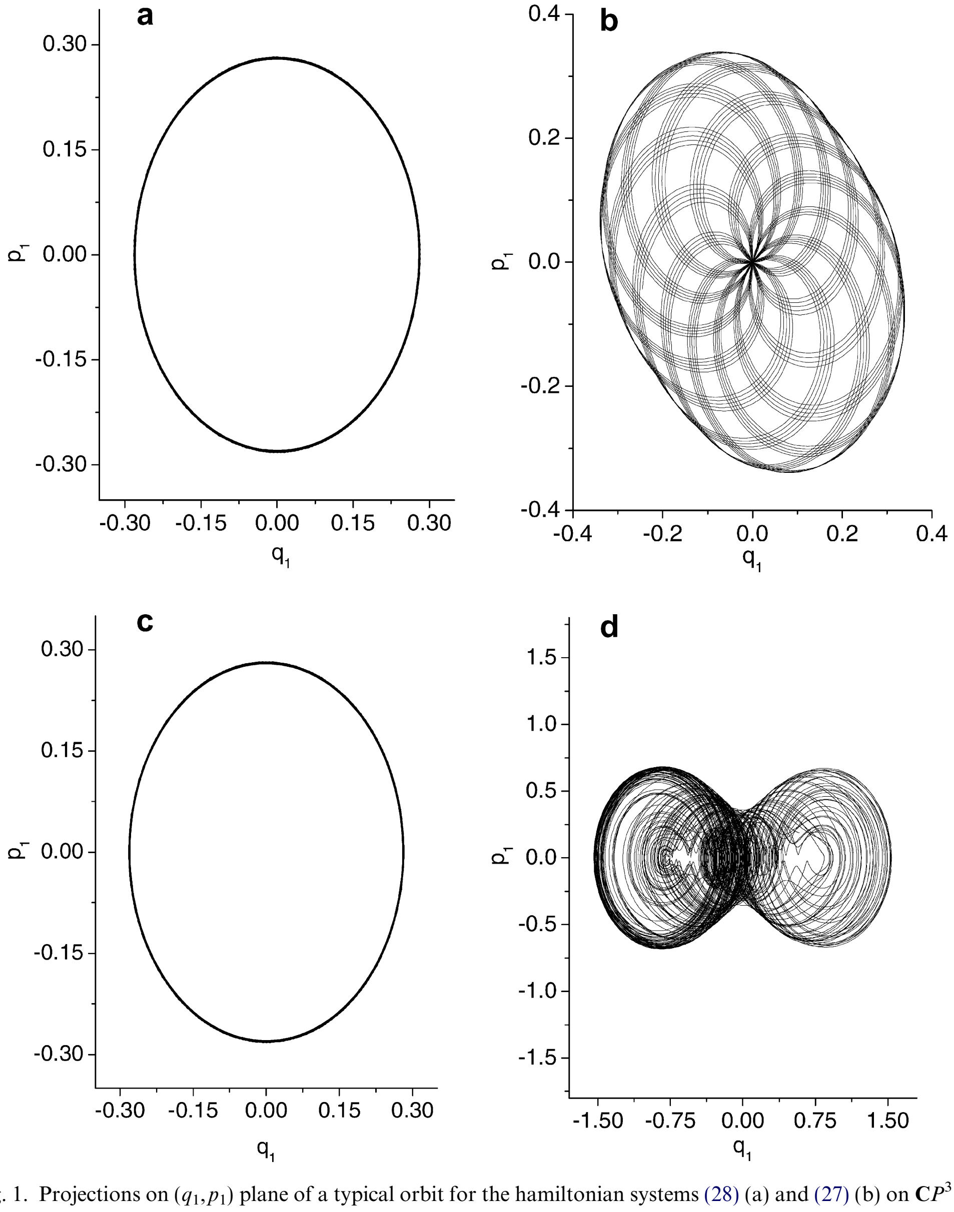

Fig. 1. Projections on (q_1,p_1) plane of a typical orbit for the hamiltonian systems (28) (a) and (27) (b) on $\mathbf{C}P^3$ and on the submanifold of separable states (c) for (40) and (d) for (39). The values of the parameters are $\omega = 1$ and $\mu = 1.7$.

still regular, while the nonsymmetric Hamiltonian generates the constrained dynamics with typical chaotic orbits. This is further illustrated in Fig. 2, where we show Poincaré surfaces of section, defined by $q^2 = 0, p^2 > 0$ and $H(p^1,q^1,p^2,q^2) = h$ for different values of the coupling μ. Obviously, the constrained system displays the transition from predominantly regular to predominantly chaotic dynamics, with all the intricate structure of the

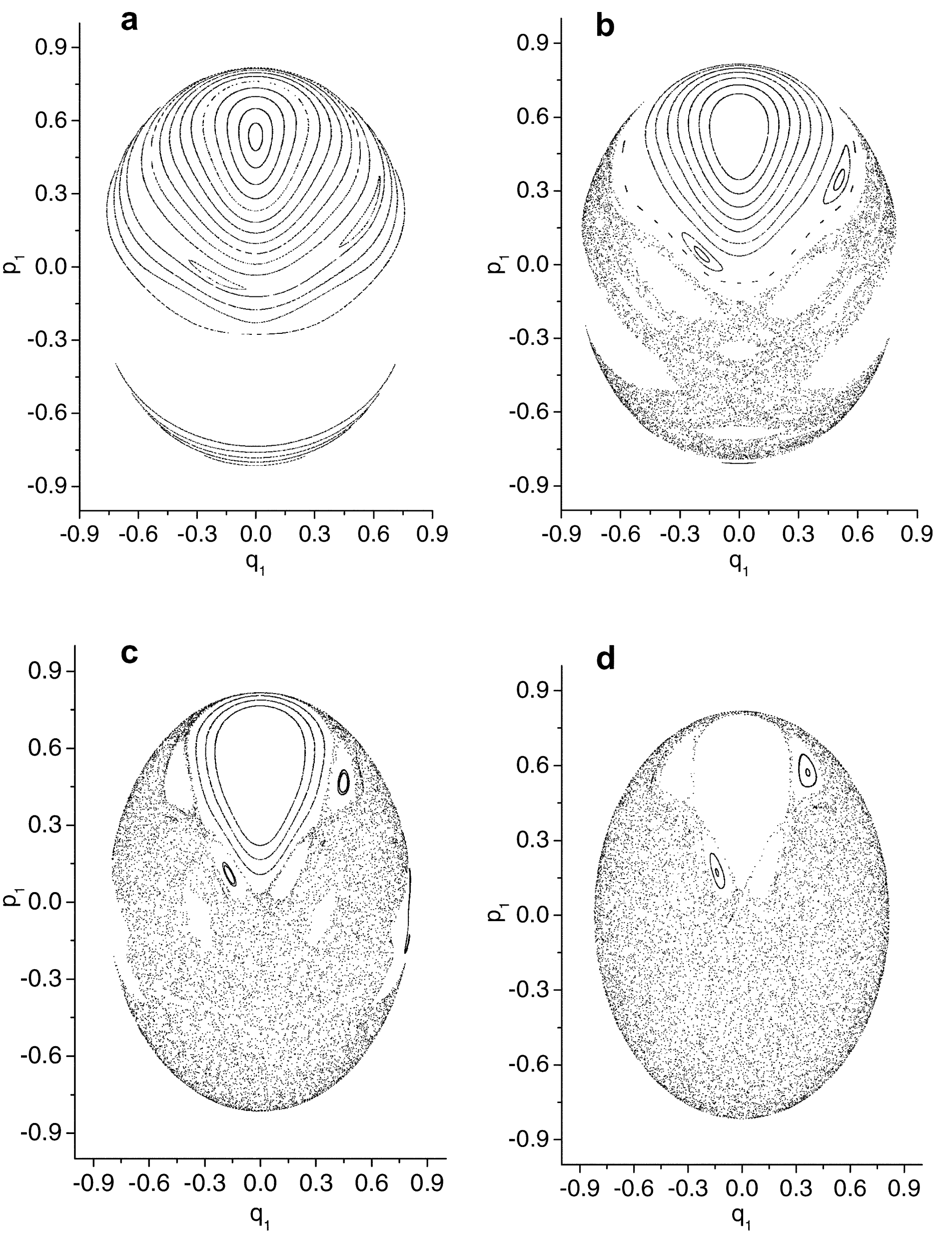

Fig. 2. Poincaré sections for the separability constrained nonsymmetric quantum dynamics (39). The parameters are $\omega = 1, h = 1.5$ and (a) $\mu = 1.1$, (b) $\mu = 1.3$, (c) $\mu = 1.5$ and (d) $\mu = 1.7$.

phase portrait, characteristic for typical Hamiltonian dynamical systems. Thus, we can conclude that the quantum system constrained on the manifold of separable state behaves as typical classical Hamiltonian systems. If there is enough symmetry, i.e. enough integrals of motion, the constrained dynamics is integrable, otherwise the constrained quantum dynamics is that of typical chaotic Hamiltonian system.

4. Summary and discussion

We have studied Hamiltonian formulation of quantum dynamics of two interacting qubits. Hamiltonian dynamical system on the state space $\mathbf{C}P^3$ as the phase space, is integrable irrespective of the different symmetries of the quantum system. We have then studied the dynamics of the quantum Hamiltonian system constrained on the manifold of separable states. The main result of this analyzes, and of the paper, is that the quantum Hamiltonian system without symmetry generates nonintegrable chaotic dynamics on the set of separable states, while the constrained symmetric dynamics gives an integrable system. It is important to bare on mind that neither the system nor the separable states that lie on an orbit of the constrained system have an underlining classical mechanical model. Thus, forcing a non-degenerate quantum system to remain on the manifold of separable states is enough to generate a dynamical system with typical properties of Hamiltonian chaos.

Our analyzes of the separability constrained quantum dynamics has been rather formal. In order to inquire into possible interpretation of our results we need a model of a physical realization of the separability constraints. To this end we consider an open quantum system of two interacting qubits, whose dynamics satisfies the Markov assumption [36], and we choose a Hermitian Lindblad operator of the following form:

$$\begin{aligned} L =& l_{11}\sigma^1_+\sigma^1_- \otimes \sigma^2_+\sigma^2_- + l_{12}\sigma^1_+\sigma^1_- \otimes \sigma^2_-\sigma^2_+ + l_{21}\sigma^1_-\sigma^1_+ \otimes \sigma^2_+\sigma^2_- + l_{22}\sigma^1_-\sigma^1_+ \otimes \sigma^2_-\sigma^2_+ \\ =& \sum_{i,j=1}^{2} l_{i,j}|i><j|^1 \otimes |i><j|^2, \end{aligned} \tag{41}$$

where $|1\rangle \equiv |\uparrow\rangle$ and $|2\rangle \equiv |\downarrow\rangle$.

The dynamics of a pure state of the open system under the action of a Hamiltonian H and the Linblad γL is described by the following stochastic nonlinear Schroedinger equation [36,37]

$$\begin{aligned} |d\psi> =& -iH|\psi> dt + \frac{\gamma^2}{4}(L- <\psi|L|\psi>)^2|\psi> dt \\ &+\gamma(L- <\psi|L|\psi>)|\psi> dW, \end{aligned} \tag{42}$$

where dW is the increment of complex Wiener c-number process $W(t)$.

Eq. (42) represent a diffusion process on a complex Hilbert space, and is central in the "Quantum State Diffusion" (QSD) theory of open quantum systems [37]. It has been used to study the systems of interacting qubits in various environments, for example, in [32,38], and the effect of the Linblad operator (41) on the entanglement between two qubits was considered in [16]. The influence of the nonHamiltonian terms of drift (proportional to γ^2) and the diffusion (proportional to γ), with the Linblad operator of the form (41), is to drive an entangled state towards one of the separable states with the corresponding probability. This process occurs on the time scale proportional to γ^{-1}. So, for large γ there occurs an almost instantaneous collapse of an entangled state into a separable one. We believe that with a proper choice of the parameters $l_{i,j}$ the long term dynamics of a pure state described by (42) can have the same qualitative properties as the separability constrained quantum dynamics. In particular, the difference between the qualitative properties of symmetric and nonsymmetric systems, reflected in the constrained Hamiltonian system, should also manifest in the dynamics of (42) for a proper choice of $l_{i,j}$. This expectations are supported by Fig. 3, which illustrate the dynamics of $(\langle\sigma_x^1\rangle, \langle\sigma_y^2\rangle)$ for the Hamiltonian operators (5) and (8) as calculated using the constrained Hamiltonian Eqs. (39) and (40) (Figs. 3b and a), or the QSD Eq. (42) (Figs. 3d and c) for a particular choice of $l_{i,j}$ and

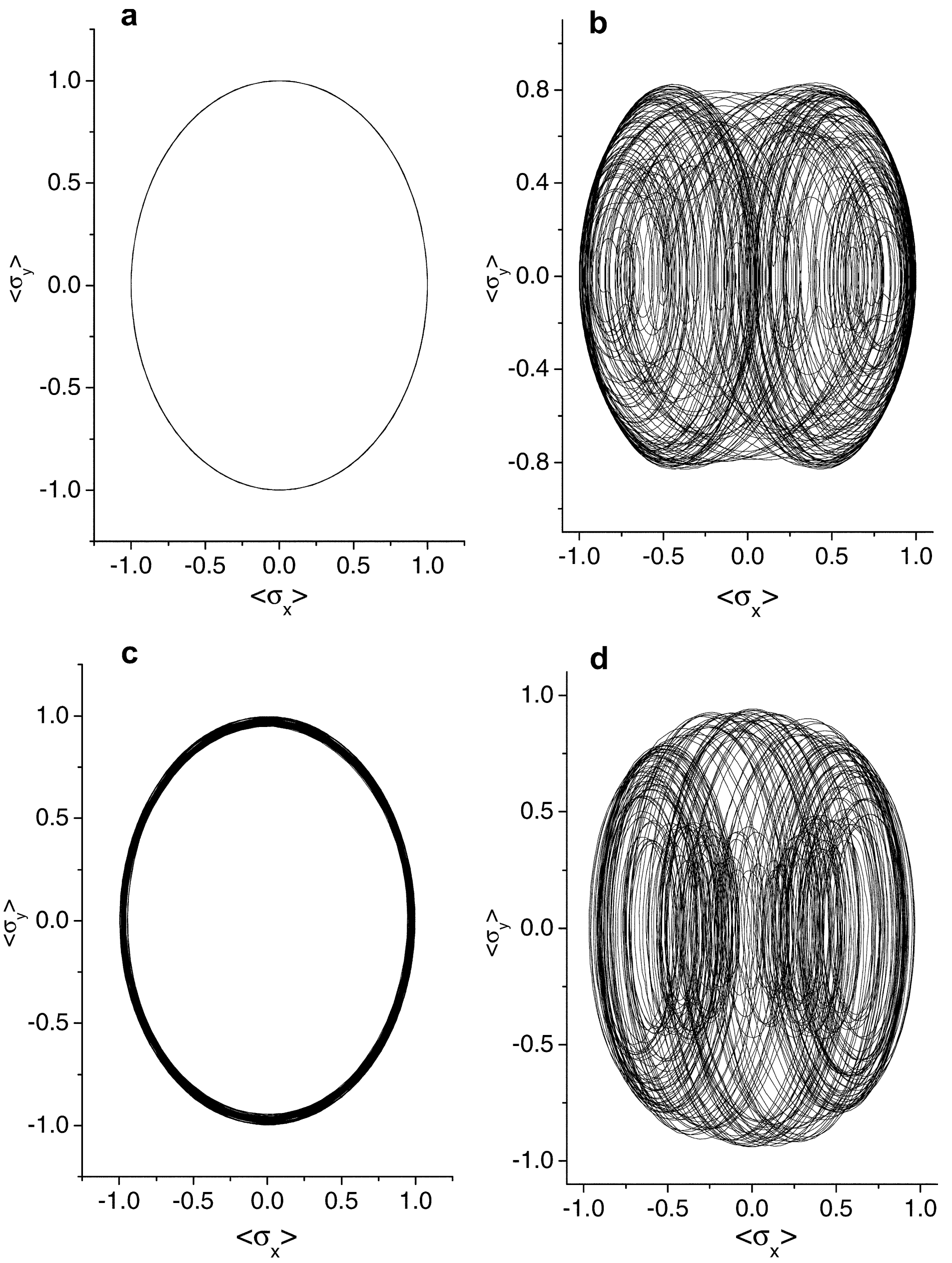

Fig. 3. Illustrate the dynamics of $(\langle\sigma_x\rangle,\langle\sigma_y\rangle)$ for the constrained Hamiltonian systems (40) (a) and (39) (b) and for the stochastic Schroedinger equation (42) with the Linblad (41) and the hamiltonians (8) (c) and (5) (d). The parameters are $\omega = 1, \mu = 1.7, \gamma = 5$ and $l_{1,1} = 0.21, l1,2 = 0.21, l_{2,1} = 0.215, l_{2,2} = 0.205$.

large $\gamma = 5$. Of course, the choice of optimal values for $l_{i,j}$ should be according to some criterion, which is the problem we are currently investigating.

The pair of coupled qubits, analyzed in this paper, is the simplest quantum system exhibiting dynamical entanglement. We intend to investigate the effects of suppression of the dynamical entanglement in systems with spacial degrees of freedom, obtained by quantization of classically chaotic systems, for example a pair of coupled nonlinear oscillators. In this case, the Hamiltonian formulation of the quantum dynamics requires an infinite-dimensional phase space, and the analyzes of the separability constrained dynamics is more complicated. However, it wold be interesting to compare the dynamics obtained by separability constraints with that of some more standard semi-classical approximation.

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References

[1] T.W.B. Kibble, Commun. Math. Phys. 64 (1978) 73.
[2] T.W.B. Kibble, Commun. Math. Phys. 65 (1979) 189.
[3] T.W.B. Kibble, S. Randjbar-Daemi, J. Phys. A 13 (1980) 141.
[4] A. Heslot, Phys. Rev. D 31 (1985) 1341.
[5] S. Weinberg, Phys. Rev. Lett. 62 (1989) 485.
[6] S. Weinberg, Phys. Rev. Lett. 63 (1989) 1115.
[7] S. Weinberg, Ann. Phys. 194 (1989) 336.
[8] D.C. Brody, L.P. Hughston, Phys. Rev. Lett. 77 (1996) 2851.
[9] D.C. Brody, L.P. Hughston, Phys. Lett. A 236 (1997) 257.
[10] D.C. Brody, L.P. Hughston, Proc. R. Soc. Lond. 455 (1999) 1683.
[11] J.E. Marsden, T.S. Ratiu, Introduction to Mechanics and Symmetry, second ed., Springer, Berlin, 1999.
[12] I. Bengtsson, K. Życzkowski, Geometry of Quantum States, Cambridge University Press, Cambridge UK, 2006.
[13] D.C. Brody, L.P. Hughston, J. Geo. Phys. 38 (2001) 19.
[14] L.P. Hughston, Proc. R. Soc. Lond. A 452 (1996) 953.
[15] S.L. Adler, T.A. Brun, J. Phys. A: Math. Gen. 34 (2001) 4797.
[16] A. Belenkiy, S. Shnider, L. Horwitz, arXiv:quant-ph/0609142v1.
[17] D. Deutch, A. Barenco, A. Ekert, Proc. R. Soc. Lond. A 449 (1995) 669.
[18] M.A. Nielsen, I.L. Chuang, Quantum Computation and Quantum Information, Cambridge University Press, Cambridge, UK, 2001.
[19] F. Haake, Quantum Signatures of Chaos, Springer-Verlag, Berlin, 2000.
[20] W.H. Zurek, Rev. Mod. Phys. 73 (2003) 715.
[21] K. Funruya, M.C. Nemes, G.O. Pellegrino, Phys. Rev. Lett. 80 (1998) 5524.
[22] P.A. Miller, S. Sarkar, Phys. Rev. E 60 (1999) 1542.
[23] B. Georgeot, D.L. Shepelyansky, Phys. Rev. E 62 (2000) 6366.
[24] B. Georgeot, D.L. Shepelyansky, Phys. Rev. E 62 (2000) 3504.
[25] P. Zanardi, C. Zalka, L. Faoro, Phys. Rev. A 62 (2000) 030301.
[26] A. Lakshminarayan, Phys. Rev. E 64 (2001) 036207.
[27] H. Fujisaki, T. Miyadera, A. Tanaka, Phys. Rev. E 67 (2003) 066201.
[28] J.N. Bandyopathyay, A. Lakshminarayan, Phys. Rev. E 69 (2004) 016201.
[29] X. Wang, S. Ghose, B.C. Sanders, B. Hu, Phys. Rev. E 70 (2004) 016217.
[30] F. Mintert, A.R.R. Carvalho, M. Kuś, A. Buchleitner, Phys. Rep. 415 (2005) 207.
[31] M. Novaes, Ann. Phys. (NY) 318 (2005) 308.

[32] N. Burić, Phys. Rev. A 73 (2006) 052111.
[33] S. Kobajayashi, K. Nomizu, Foundations of Differential Geometry, Wiley, New York, 1969.
[34] P.A.M. Dirac, Can. J. Math. 2 (1950) 129.
[35] V.I. Arnold, V.V. Kozlov, A.I. Neisthadt, Dynamical Systems III, Springer, Berlin, 1988.
[36] H.-P. Breuer, F. Petruccione, The Theory of Open Quantum Systems, Oxford University Press, Oxford, 2001.
[37] I.C. Percival, Quantum State Diffussion, Cambridge University Press, Cambridge, UK, 1999, 1999.
[38] N. Buric, Phys. Rev. A 72 (2005) 042322.