QUANTIZATION OF MAGNETIC TOP IN THE SPINOR REPRESENTATION

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Classical magnetic top described in terms of spinors is quantized by applying Dirac's method of quantization for systems with constraints.

1. Introduction

From various studies the Cartan spinor [1] arose as a very appropriate object for the description of top's orientation.

In his discussion of the prehistory of spin kinematics, Turrin pointed out [2] that those roots originate in Darboux's work [3] on the spinor treatment of the rigid-body motion. Euler's parameters, customarily known as the Cayley–Klein parameters [4], are considered also to be the predecessors of spinors. Later, as the orientation coordinates of a top, spinors were extensively studied and analyzed by Sudarshan and Mukunda [5], used by Hara, Goto, Tsai and Yabuki in their quantum theory of rigid body [6] and by Tisza [7] in the model of spin based on an orientable object.

We showed recently [8, 9] that the Lagrange equations of motion of a top, as well as of a magnetic top, in terms of spinors reduce to the harmonic oscillator equations, free of singularities, and different from the Lagrange equations written in terms of Euler's angles. Spinors are more convenient than Euler's angles since they transparently transform under the $SU(2)$ group elements. Spinors are global coordinates while Euler's angles are not well defined for $\vartheta = 0$ and $\vartheta = \pi$. From these features we have concluded [8, 9] that the spinor space is a natural configuration space for determining the rotational dynamics of the classical spherical top and the classical magnetic top. Quantization of a magnetic top described in terms of spinors is exposed in the present paper.

2. Spinor as the orientation coordinate of a top

By Cartan's definition [1], a two-dimensional spinor

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \tag{2.1}$$

is associated with an isotropic vector

$$c^2 = \mathbf{c}\cdot\mathbf{c} = c_1^2 + c_2^2 + c_3^2 = 0 \tag{2.2}$$

through the relations

$$c_1 = \frac{\xi_2^2 - \xi_1^2}{2}, \qquad c_2 = \frac{\xi_2^2 + \xi_1^2}{2i}, \qquad c_3 = \xi_1\xi_2. \tag{2.3}$$

These equations have two solutions given, for example, by the formulae:

$$\xi_1 = \pm\sqrt{-c_1 + ic_2}, \qquad \xi_2 = \pm\sqrt{c_1 + ic_2}. \tag{2.4}$$

Cartan pointed out that “it is necessary to keep solutions with both signs because it is not possible to give a consistent choice of sign which will hold for all isotropic vectors in such a manner that the solution varies continuously with the vector”.

By separating the isotropic vector $\mathbf{c}$ in the real and imaginary parts one has

$$\mathbf{c} = \mathbf{a} + i\mathbf{b}. \tag{2.5}$$

Brinkman found [10] that the condition $\mathbf{c}^2 = 0$ amounts to

$$\mathbf{a}^2 - \mathbf{b}^2 = 0, \tag{2.6}$$

$$\mathbf{a}\cdot\mathbf{b} = 0. \tag{2.7}$$

It implies that the introduction of an isotropic vector $\mathbf{c}$, as well as of the corresponding spinor ξ, is equivalent to that of an orthogonal vector triple of equal length (say ℓ),

$$\mathbf{a} = \ell\mathbf{e}_1, \qquad \mathbf{b} = \ell\mathbf{e}_2, \qquad \ell\mathbf{e}_3 = \ell\mathbf{e}_1 \times \mathbf{e}_2. \tag{2.8}$$

Since the orientation of a top is determined by the orientation of a frame attached to it, this correspondence shows that spinor is a suitable object for the orientation coordinate of a top.

Being interested in the internal motion of a top, we shall assume that its center of mass is situated at the common point of a body ($\mathcal{B}$) frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and of an external laboratory ($\mathcal{L}$) frame $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$.

Using the Pauli matrices,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.9}$$

the components of the complex vector $\mathbf{c}$ and of the vector triple $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are written in terms of spinors as follows [9]

$$c_1 = \frac{\hat{\xi}\sigma_1\xi}{2}, \quad c_2 = \frac{\hat{\xi}\sigma_2\xi}{2}, \quad c_3 = \frac{\hat{\xi}\sigma_3\xi}{2}, \tag{2.10}$$

$$e_{(1)k} = \frac{c_k + c_k^*}{\varrho} = \frac{\hat{\xi}\sigma_k\xi + (\hat{\xi}\sigma_k\xi)^*}{2\varrho}, \quad e_{(2)k} = \frac{c_k - c_k^*}{i\varrho} = \frac{\hat{\xi}\sigma_k\xi - (\hat{\xi}\sigma_k\xi)^*}{2i\varrho},$$
$$e_{(3)k} = \frac{\hat{\xi}^*\sigma_k\xi}{\varrho}. \tag{2.11}$$

Here

$$\hat{\xi} \equiv i\sigma_2 \xi = \begin{pmatrix} \xi_2 \\ -\xi_1 \end{pmatrix}, \tag{2.12}$$

and

$$\varrho \equiv \varrho(\tilde{\xi}^*, \xi) \equiv \tilde{\xi}^* \xi. \tag{2.13}$$

When written on the left-hand side of a matrix, $\tilde{\xi}$ denotes the row matrix.

For all spinors associated with orthogonal vector triples of a given length ℓ, the function ϱ has constant value

$$\varrho(\tilde{\xi}^*, \xi) = |\xi_1|^2 + |\xi_2|^2 = 2\ell. \tag{2.14}$$

The direction cosines of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ relative to the axes $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$,

$$a_{ij} \equiv \cos(\mathbf{e}_i, \mathbf{E}_j) = \mathbf{e}_i \cdot \mathbf{E}_j = e_{(i)j} \tag{2.15}$$

satisfy the orthogonality relations

$$a_{ki} a_{ji} = \delta_{kj} \tag{2.16}$$

and specify completely orientation of the top.

Using the well-known expressions for the matrix elements in terms of Euler's angles [4] and the relations (2.4), (2.11) and (2.15), one finds the relations between the spinor components and Euler's angles:

$$z_1 \equiv \frac{\xi_1}{\sqrt{\varrho}} = i \cos\frac{\vartheta}{2}\, e^{-i(\varphi+\chi)/2}, \qquad z_2 \equiv \frac{\xi_2}{\sqrt{\varrho}} = \sin\frac{\vartheta}{2}\, e^{i(\varphi-\chi)/2}. \tag{2.17}$$

In order to take into account both signs in the relation (2.4), the usual region of Euler's angles is extended and defined by the inequalities:

$$0 \le \tfrac{1}{2}(\varphi + \chi) \le 2\pi, \qquad -\pi \le \tfrac{1}{2}(\varphi - \chi) \le \pi, \qquad 0 \le \vartheta \le \pi, \tag{2.18}$$

allowing for the periodicity in the directions of the $\varphi+\chi$ and $\varphi-\chi$ axes. As shown by Jonker and de Vries [11], with the above choice of the extended region for Euler's angles, every point on the hypersphere

$$(\mathrm{Re}\,\xi_1)^2 + (\mathrm{Re}\,\xi_2)^2 + (\mathrm{Im}\,\xi_1)^2 + (\mathrm{Im}\,\xi_2)^2 = \varrho^2 \tag{2.19}$$

can be reached.

After lengthly algebraic manipulations, it may be shown that under an active rotation of the frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, determined by Euler's angles β_φ, β_ϑ, β_χ (defined with respect to the $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ frame) a unit spinor z transforms in the following way

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} Z_1 & -Z_2^* \\ Z_2 & Z_1^* \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \tag{2.20}$$

where

$$Z_1 = \cos\frac{\beta_\vartheta}{2} e^{-i\frac{\beta_\varphi+\beta_\chi}{2}}, \qquad Z_2 = -i \sin\frac{\beta_\vartheta}{2} e^{i\frac{\beta_\varphi-\beta_\chi}{2}}. \tag{2.21}$$

Evidently, the transformation matrix is unitary and unimodular—an element of $SU(2)$ group.

3. Magnetic top in terms of spinors—the classical system with constraints

The magnetic top was introduced as a classical model for spin by Barut *et al.* [12]. It is a spherically symmetric top with a magnetic moment **M** proportional to the kinetic angular momentum $\mathbf{\Sigma}$,

$$\mathbf{M} = \gamma\mathbf{\Sigma} = \gamma I\boldsymbol{\omega} \tag{3.1}$$

(I is the moment of inertia and $\boldsymbol{\omega}$ is the angular velocity). Lagrangian of the top in an external magnetic field is

$$L = T - U = \frac{I\boldsymbol{\omega}^2}{2} + \gamma I\boldsymbol{\omega}\cdot\mathbf{B}. \tag{3.2}$$

In terms of the Cartan's spinor, T, U and L read as [9]:

$$T = \frac{2I}{\varrho}\dot{\tilde{\xi}}\dot{\xi} - \frac{I}{2\varrho^2}(\dot{\tilde{\xi}}\xi + \tilde{\xi}^*\dot{\xi})^2 = \frac{I}{2}\left(\frac{ds}{dt}\right)^2 = \frac{I}{2}\frac{g_{ij}d\xi_i d\xi_j}{dt^2},$$
$$i,j = 1,2,3,4, \quad \xi_3 = \xi_1^*, \quad \xi_4 = \xi_2^*, \tag{3.3}$$

$$U = -\gamma BI\frac{i}{\varrho}(\tilde{\xi}^*\sigma_3\dot{\xi} - \dot{\tilde{\xi}}\sigma_3\xi), \tag{3.4}$$

$$L = \frac{2I}{\varrho}\dot{\tilde{\xi}}\dot{\xi} - \frac{I}{2\varrho^2}(\dot{\tilde{\xi}}\xi + \tilde{\xi}^*\dot{\xi})^2 + \gamma BI\frac{i}{\varrho}(\tilde{\xi}^*\sigma_3\dot{\xi} - \dot{\tilde{\xi}}\sigma_3\xi). \tag{3.5}$$

The magnetic top is a classical dynamical system with three degrees of freedom. But, using the Cartan's spinor ξ for the description of the orientation of a top, we have substituted the three-dimensional configuration space by the four-dimensional configuration space with one holonomic constraint

$$h_1 \equiv \varrho(\tilde{\xi}^*, \xi) - 2\ell = 0, \quad 2\ell \text{ is a time-independent constant.} \tag{3.6}$$

The Lagrangian L is singular, i.e. $\det\frac{\partial^2 L}{\partial\dot{\xi}_j\partial\dot{\xi}_k} = 0$.

The expression (3.3) for the kinetic energy T implies the following form of the metric tensor

$$(g_{ij}) = \frac{1}{\varrho^2}\begin{array}{c|cccc} & \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ \hline \xi_1 & -\xi_1^{*2} & -\xi_1^*\xi_2^* & \xi_1\xi_1^* + 2\xi_2\xi_2^* & -\xi_2\xi_1^* \\ \xi_2 & -\xi_1^*\xi_2^* & -\xi_2^{*2} & -\xi_1\xi_2^* & 2\xi_1\xi_1^* + \xi_2\xi_2^* \\ \xi_3 & \xi_1\xi_1^* + 2\xi_2\xi_2^* & -\xi_1\xi_2^* & -\xi_1^2 & -\xi_1\xi_2 \\ \xi_4 & -\xi_2\xi_1^* & 2\xi_1\xi_1^* + \xi_2\xi_2^* & -\xi_1\xi_2 & -\xi_2^2 \end{array}. \tag{3.7}$$

It follows that $D \equiv \sqrt{\det(g_{ij})} = 0$.

The conjugate momenta are:

$$\pi_j = \frac{\partial L}{\partial\dot{\xi}_j} = \frac{2I\dot{\xi}_j^*}{\varrho} - \frac{I\dot{\varrho}\xi_j^*}{\varrho^2} + (-1)^{j-1}\gamma BI\frac{i}{\varrho}\xi_j^*,$$
$$\pi_j' = \frac{\partial L}{\partial\dot{\xi}_j^*} = \frac{2I\dot{\xi}_j}{\varrho} - \frac{I\dot{\varrho}\xi_j}{\varrho^2} - (-1)^{j-1}\gamma BI\frac{i}{\varrho}\xi_j, \qquad \pi = \begin{pmatrix}\pi_1\\ \pi_2\end{pmatrix}, \tag{3.8}$$

where we find a primary constraint

$$h_2 = \pi\xi + \pi^*\xi^* = 0, \tag{3.9}$$

which is of the first class at the moment.

The vector quantity

$$\mathbf{s} = \boldsymbol{\Sigma} + I\gamma\mathbf{B} = I(\boldsymbol{\omega} + \gamma\mathbf{B}), \tag{3.10}$$

called the canonical angular momentum or spin, is the characteristic quantity of the magnetic top [12–15] since s^2, s_3 and s'_i are constants of motion. The components of s relative to the $\mathcal{L}$ and $\mathcal{B}$ frame, respectively, expressed in terms of spinors read as:

$$s_i = -\frac{i}{2}(\tilde{\pi}\sigma_i\xi - \tilde{\xi}^*\sigma_i\pi^*), \tag{3.11}$$

$$s'_1 = \frac{1}{2}(\tilde{\pi}\sigma_2\xi^* - \tilde{\pi}^*\sigma_2\xi), \quad s'_2 = \frac{i}{2}(\tilde{\pi}\sigma_2\xi^* + \tilde{\pi}^*\sigma_2\xi), \quad s'_3 = \frac{i}{2}(\tilde{\pi}^*\xi^* - \pi\xi). \tag{3.12}$$

From the above it follows the expression for the square of the vector s in the spinor form

$$\mathbf{s}^2 = s^2 = (\tilde{\xi}^*\xi)(\tilde{\pi}^*\pi) - \tfrac{1}{4}(\tilde{\pi}\xi + \tilde{\pi}^*\xi^*)^2. \tag{3.13}$$

The Poisson brackets between the spin components form the same algebra as in the case of Euler's angles:

$$\{s_i, s_j\} = \varepsilon_{ijk}s_k, \qquad \{s'_i, s'_j\} = -\varepsilon_{ijk}s'_k, \qquad \{s_i, s'_j\} = 0 \tag{3.14}$$

($\varepsilon_{123} = 1$). Note the minus sign in the second equation.

The canonical Hamiltonian takes the form (for $\mathbf{B} = B\mathbf{E}_3$)

$$H_c = \frac{1}{2I}\left\{(\tilde{\xi}^*\xi)(\tilde{\pi}^*\pi) - \tfrac{1}{4}(\tilde{\pi}\xi + \tilde{\xi}^*\pi^*)^2 + \gamma BIi(\tilde{\pi}\sigma_3\xi - \tilde{\xi}^*\sigma_3\pi^*) + \gamma^2B^2I^2\right\} =$$

$$= \frac{1}{2I}[\mathbf{s}^2 - 2\gamma IBs_3 + \gamma^2I^2B^2]. \tag{3.15}$$

But, for a system with the primary constraint (3.9) it is necessary to construct the effective Hamiltonian [16, 17]

$$H_{\text{eff}} = H_c + v_2h_2. \tag{3.16}$$

We can fix the arbitrary function v_2 by imposing the gauge condition $h_1 = \tilde{\xi}\xi^* - 2\ell = 0$. h_1 and h_2 now turn to be the second-class constraints. The Poisson bracket $\{\cdot,\cdot\}$ should be replaced by the Dirac bracket $\{\cdot,\cdot\}^*$,

$$\{A, B\}^* = \{A, B\} - \{A, h_i\}c_{ij}^{-1}\{h_j, B\}, \tag{3.17}$$

where $c_{ij} = \{h_i, h_j\}$. However, it is not altered since $\{H, h_i\} = 0$, and we will keep writing $\{\cdot,\cdot\}$ instead of $\{\cdot,\cdot\}^*$. Similarly, the corrected Hamiltonian $H' = H_c - \{H_c, h_i\}c_{ij}^{-1}h_j$ is the same as the canonical one, $H' = H_c$.

4. Quantization in terms of spinors

In the case of Euler's angles as the orientation coordinates, the quantization is realized in the Hilbert space $\mathcal{H}^E$ of wave functions $\psi(\varphi, \vartheta, \chi)$ with the scalar product

$$\langle \psi_1(\varphi, \vartheta, \chi) | \psi_2(\varphi, \vartheta, \chi) \rangle = \int\limits_V \sin\vartheta\, \psi_1^*(\varphi, \vartheta, \chi) \psi_2(\varphi, \vartheta, \chi) d\vartheta\, d\varphi\, d\chi. \tag{4.1}$$

In the case of spinors as the orientation coordinates we work with the Hilbert space $\mathcal{H}^\xi$ of wave functions $\Psi(\xi_1, \xi_2, \xi_1^*, \xi_2^*) = \Psi(\xi, \xi^*)$. Spinors introduce a new variable ϱ which enters the volume element as $dV = \frac{\varrho \sin\vartheta}{4} d\chi d\varphi d\vartheta d\varrho$. Turning to ξ's we obtain $dV = d\xi_1 d\xi_2 d\xi_1^* d\xi_2^*$, i.e.

$$\langle \Psi_1(\xi, \xi^*) | \Psi_2(\xi, \xi^*) \rangle = \int\limits_V \Psi_1^*(\xi, \xi^*) \Psi_2(\xi, \xi^*)\, d\xi_1\, d\xi_2\, d\xi_1^*\, d\xi_2^*. \tag{4.2}$$

From the general expression for the momentum operators in the framework of Schrödinger's quantization method

$$\hat{p}_i \psi = \frac{\hbar}{i} \frac{1}{\sqrt{D}} \frac{\partial \sqrt{D} \psi}{\partial q_i}, \tag{4.3}$$

where $D = \sqrt{\det |g_{ij}|}$ and q_i are coordinates, the following expressions for the operators of canonical momenta were obtained [13] in the case of Euler's angles ($D = \sin\vartheta$):

$$\hat{p}_\varphi \to -i\hbar \frac{\partial}{\partial \varphi}, \quad \hat{p}_\chi \to -i\hbar \frac{\partial}{\partial \chi}, \quad \hat{p}_\vartheta \to -i\hbar \left(\frac{\partial}{\partial \vartheta} + \frac{\operatorname{ctg}\vartheta}{2} \right). \tag{4.4}$$

The corresponding Hamilton operator reads:

$$\hat{H}_E = -\frac{\hbar^2}{2I} \left[\frac{\partial^2}{\partial \vartheta^2} + \operatorname{ctg}\vartheta \frac{\partial}{\partial \vartheta} + \frac{1}{\sin^2\vartheta} \left(\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial \chi^2} \right) - 2 \frac{\operatorname{ctg}\vartheta}{\sin\vartheta} \frac{\partial^2}{\partial \chi \partial \varphi} \right]$$
$$+ \gamma B i\hbar \frac{\partial}{\partial \varphi} + \frac{\gamma^2 I B^2}{2}. \tag{4.5}$$

In the case of spinors, the determinant $D = \det g_{ij} = 0$, and is therefore independent of ξ. This means that we may write the general expression (4.3) in the form

$$\hat{p}_i \psi = \frac{\hbar}{i} \frac{\partial}{\partial q_i} \psi. \tag{4.6}$$

Therefore, the coordinate and momentum operators in the Hilbert space $\mathcal{H}^\xi$ are:

$$\hat{\xi} \Psi(\xi) = \xi \Psi(\xi), \qquad \hat{\pi} \Psi(\xi) = -i\hbar \frac{\partial \Psi(\xi)}{\partial \xi}. \tag{4.7}$$

The spin operators $\hat{s}_1, \hat{s}_2, \hat{s}_1', \hat{s}_2'$ can be written directly from (3.11) and (3.12). In the components $\hat{s}_3$ and $\hat{s}_3'$ we have an ordering problem with the factors $\xi_1 \pi_1$, $\xi_2 \pi_2$, $\xi_1^* \pi_1^*$ and

$\xi_2^*\pi_2^*$. We can calculate them using $i\hbar\hat{s}_3 = [\hat{s}_1, \hat{s}_2]$ and $-i\hbar\hat{s}'_3 = [\hat{s}'_1, \hat{s}'_2]$, in accordance with (3.14). The result is:

$$\hat{s}_i = -\tfrac{1}{2}i(\hat{\tilde{\pi}}\sigma_i\hat{\xi} - \hat{\tilde{\xi}}^*\sigma_i\hat{\pi}^*), \tag{4.8}$$

$$\hat{s}'_1 = \tfrac{1}{2}(\hat{\tilde{\pi}}\sigma_2\hat{\xi}^* - \hat{\tilde{\pi}}^*\sigma_2\hat{\xi}), \quad \hat{s}'_2 = \tfrac{1}{2}i(\hat{\tilde{\pi}}\sigma_2\hat{\xi}^* + \hat{\tilde{\pi}}^*\sigma_2\hat{\xi}), \quad \hat{s}'_3 = \tfrac{1}{2}i(\hat{\tilde{\pi}}^*\hat{\xi}^* - \hat{\tilde{\pi}}\hat{\xi}). \tag{4.9}$$

A straightforward calculation leads to

$$\begin{aligned}\hat{s}^2 &= \hat{s}_1^2 + \hat{s}_2^2 + \hat{s}_3^2 = \hat{s}_1'^2 + \hat{s}_2'^2 + \hat{s}_3'^2 \\ &= \tfrac{1}{4}[4(\hat{\tilde{\pi}}^*\hat{\pi})(\hat{\tilde{\xi}}^*\hat{\xi}) + 2i\hbar(\hat{\tilde{\pi}}\hat{\tilde{\xi}} + \hat{\tilde{\xi}}^*\hat{\tilde{\pi}}^*) - (\hat{\tilde{\pi}}\hat{\tilde{\xi}} + \hat{\tilde{\xi}}^*\hat{\tilde{\pi}}^*)^2].\end{aligned} \tag{4.10}$$

The Hamiltonian is equal to

$$\hat{H} = \frac{1}{2I}\left\{\frac{1}{4}[4(\hat{\tilde{\pi}}^*\hat{\pi})(\hat{\tilde{\xi}}^*\hat{\xi}) + 2i\hbar(\hat{\tilde{\pi}}\hat{\tilde{\xi}} + \hat{\tilde{\xi}}^*\hat{\tilde{\pi}}^*) - (\hat{\tilde{\pi}}\hat{\tilde{\xi}} + \hat{\tilde{\xi}}^*\hat{\tilde{\pi}}^*)^2] \right.$$
$$\left. + \gamma BIi(\hat{\tilde{\pi}}\sigma_3\hat{\xi} - \hat{\tilde{\xi}}^*\sigma_3\hat{\pi}^*) + \gamma^2B^2I^2\right\}. \tag{4.11}$$

A comparison of this Hamiltonian with (4.5) is helpful. We can express the differential operators $\frac{\partial}{\partial\xi}$ in terms of $\frac{\partial}{\partial\varphi}$, $\frac{\partial}{\partial\vartheta}$, $\frac{\partial}{\partial\chi}$ and $\frac{\partial}{\partial\varrho}$ using (2.17) and *vice versa*. It turns out that $\hat{s}_i$ when transformed from ξ's to φ, ϑ, χ and ϱ *contains neither* ϱ *nor* $\frac{\partial}{\partial\varrho}$. The Hamiltonians (4.5) and (4.11) are equal:

$$\hat{H}_E = \frac{1}{2I}(\hat{s}_1^2 + \hat{s}_2^2 + \hat{s}_3^2 - 2\hat{s}_3\gamma BI + \gamma^2B^2I^2) = \hat{H}. \tag{4.12}$$

Independently of which configuration space variables are used, the Hamilton operator commutes with the operators $\hat{s}^2$, $\hat{s}_3$ and $\hat{s}'_3$. Therefore, the energy eigenstates are simultaneously eigenstates of $\hat{s}^2$, $\hat{s}_3$ and $\hat{s}'_3$. It is therefore convenient to label the energy eigenstates of $\hat{H}_E$, $\psi_{s\mu\nu}(\vartheta, \varphi, \chi)$, with quantum numbers associated with the eigenvalue equations:

$$\begin{aligned}\hat{s}^2\psi_{s\mu\nu}(\vartheta, \varphi, \chi) &= \hbar^2 s(s+1)\psi_{s\mu\nu}(\vartheta, \varphi, \chi), \\ \hat{s}_3\psi_{s\mu\nu}(\vartheta, \varphi, \chi) &= \mu\hbar\psi_{s\mu\nu}(\vartheta, \varphi, \chi), \\ \hat{s}'_3\psi_{s\mu\nu}(\vartheta, \varphi, \chi) &= \nu\hbar\psi_{s\mu\nu}(\vartheta, \varphi, \chi).\end{aligned} \tag{4.13}$$

The operator $\hat{H}$ acts on wave functions $\Psi(\xi, \xi^*)$. Using the transformation (2.17), Ψ may be written as a function of φ, ϑ, χ and ϱ

$$\Psi(\xi, \xi^*) = \Psi'(\varphi, \vartheta, \chi, \varrho). \tag{4.14}$$

Since $\hat{H}$, when expressed in terms of ϑ, φ and χ, does not depend on ϱ and $\frac{\partial}{\partial\varrho}$, we can guess the form of the most general eigenstate of $\hat{H}$:

$$\Psi(\xi, \xi^*) = \Phi_{s\mu\nu}(z, z^*)F(\varrho), \tag{4.15}$$

where

$$\Phi_{s\mu\nu}(z, z^*) = \psi_{s\mu\nu}(\varphi, \vartheta, \chi) \tag{4.16}$$

and $F(\varrho)$ is at this stage an arbitrary function.

However, the magnetic top in the spinor form is a classical system with constraints $h_1 = 0$ and $h_2 = 0$. The quantization of h_1 leads to

$$\hat{h}_1 = \hat{\tilde{\xi}}^* \hat{\xi} - 2\ell. \tag{4.17}$$

The symmetrization of terms $\xi_i \pi_i$ and $\xi_i^* \pi_i^*$ in h_2 gives

$$\hat{h}_2 = \hat{\tilde{\pi}} \hat{\xi} + \hat{\tilde{\xi}}^* \hat{\pi}^* = \hat{\tilde{\xi}} \hat{\pi} + \hat{\tilde{\pi}}^* \hat{\xi}^*. \tag{4.18}$$

Expressed in terms of φ, ϑ, χ and ϱ they read:

$$\hat{h}_1 = \hat{\varrho} - 2\ell, \qquad \hat{h}_2 = \hat{\varrho}\hat{p}_\varrho + \hat{p}_\varrho \hat{\varrho} = -i\hbar 2\varrho \frac{\partial}{\partial \varrho} - 2i\hbar. \tag{4.19}$$

The eigenstates of $\hat{H}$ have to satisfy one of the constrained equations:

$$\hat{h}_1 \Psi = 0 \ \Rightarrow \ \Psi = \Phi_{s\mu\nu}(z, z^*) \cdot \delta(\varrho - 2\ell), \tag{4.20}$$

$$\hat{h}_2 \Psi = 0 \ \Rightarrow \ \Psi = \Phi_{s\mu\nu}(z, z^*) \cdot \frac{1}{\varrho}, \tag{4.21}$$

so that the function $F(\varrho)$ is no more arbitrary.

By choosing the constraint (4.21) we conclude that the corresponding eigenstate cannot be normalized to unity unless we limit the integration in (4.2) from the whole region $(\xi_1, \xi_2, \xi_1^*, \xi_2^*)$ to the one where $a < \xi_1 \xi_1^* + \xi_2 \xi_2^* < b$, where now a and b are arbitrary, but have to be fixed.

5. Conclusion

The classical magnetic top described in terms of spinors is quantized by applying Dirac's method of quantization for systems with constraints. The spin and Hamilton operators, given by (4.8) and (4.11), operate in the Hilbert space $\mathcal{H}^\xi$ of functions $\Psi(\xi, \xi^*)$ in which the scalar product is given by

$$\langle \Psi_1 | \Psi_2 \rangle = \int\limits_{\substack{V \\ a<\varrho<b}} \Psi_1^*(\xi, \xi^*) \Psi_2(\xi, \xi^*)\, d\xi_1\, d\xi_2\, d\xi_1^*\, d\xi_2^*, \tag{5.1}$$

where a and b are positive real numbers.

It is shown that the eigenstates $\Psi_{s\mu\nu}(\xi, \xi^*)$ of $\hat{H}$ have the form

$$\Psi_{s\mu\nu}(\xi, \xi^*) = \frac{A}{\tilde{\xi}^* \xi} \Phi_{s\mu\nu}(z, z^*), \tag{5.2}$$

where A is a constant (determined by constants a and b) and $\Phi_{s\mu\nu}(z, z^*) = \psi_{s\mu\nu}(\varphi, \vartheta, \chi)$ are eigenstates of $\hat{H}_E$. In this way a one-to-one correspondence is established between states in the Hilbert spaces $\mathcal{H}^\xi$ and $\mathcal{H}^E$.

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