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Entanglement in quantum integrable and chaotic systems with Markov dynamics

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ABSTRACT

Entanglement evolution is studied in open systems represented by rings of qubits with the Ising interaction and variously oriented external field and with Markov environments. The effect of thermal or dephasing environment is manifested as exponential decrease of the entanglement superposed on its dynamics in the isolated system.

1. Introduction

Entanglement dynamics is an important topic of current research in quantum information processing and foundations of quantum mechanics [1,2]. The relation between qualitative properties of quantum dynamics and the entanglement evolution plays the crucial role in understanding of quantum to classical transition. Systematic study of this question is difficult because: (a) a clear categorization of the qualitative properties of quantum dynamics is lacking [3], (b) suitable and easy to calculate general measure of entanglement is not available [4] and (c) the quantum dynamics must be considered as that of an open system because of the crucial and unavoidable interaction of the system with its environment [5]. This Letter presents results of our analysis of the entanglement dynamics in examples of qualitatively different open systems of qubits with Markov dynamics, that fall in the following three categories: (a) quantum integrable with completely integrable classical limit; (b) quantum nonintegrable with chaotic classical limit and (c) quantum integrable with a chaotic classical limit.

Dynamics of entanglement in quantum systems possessing qualitatively different dynamical properties has been studied from different perspectives. Two broad classes of such studies can be distinguished according to what is considered as the distinctive feature to characterize the quantum dynamics. From the point of view of the theory of Hamiltonian dynamical systems all quantum system with finite number of degrees of freedom are completely integrable and have qualitatively the same dynamical properties [6]. In order to introduce a meaningful distinction between types of quantum evolution two strategies have been employed, within the context of the problem of quantum to classical transition or of the problem of solvability of the quantum evolution. The first one relays on quantization of classical systems with qualitatively different dynamical properties [7] and the second one relays on the notion of integrability for infinite dynamical systems and its quantum (noncommutative) generalization [8]. Properties of the entanglement dynamics have been studied within both of these approaches.

Within the approach based on quantization of systems with different classical dynamics, it is found that (a) quantum systems obtained by quantization of classical Hamiltonian system with qualitatively different dynamics show different spectral properties, and qualitatively different properties of entanglement in eigenstates in different parts of spectra have been observed [9]; (b) entanglement in the wave function initially localized in qualitatively different parts of the phase space of some semi-classical approximation of the quantum system has clearly different dynamics [10–18].

There is no generally accepted notion of genuinely quantum integrability [8]. The definition of what is a quantum chaotic system is even less unique [19]. The most common approach, at least for lattice spin systems, is based on the generalization of the notion of thermodynamical integrability of classical spin systems [20]. Such classical spin system in the thermodynamical limit is called integrable (or exactly solvable) if it is possible to determine its partition function exactly. This somewhat restrictive definition requires that the exact solution is given by the Bethe ansatz or requires the existence of a nontrivial solution of the Yang–Baxter equations. Analogously, quantum systems are called integrable if they are exactly solvable by application of the generalized Bethe ansatz or by the quantum inverse scattering method [8]. This is commonly accepted definition of integrability for systems represented by spin chains [3], but is certainly not unique [19]. A quantum system is nonintegrable if it has not been integrated. In what sense a quantum nonintegrable system can be considered as quantum chaotic is a matter of a debate [3]. Some quantum systems of finite number of spins whose thermodynamical limit is quantum nonintegrable,

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show the same spectral properties as the systems obtained by quantization of classically chaotic systems, and, furthermore, display the mixing properties that lead to expected equilibrium and non-equilibrium thermodynamical behavior [3]. The dynamics of entanglement in such quantum chaotic systems has been studied and compared with the entanglement dynamics in quantum integrable systems [9,21–25]. Later in the paper we shall compare the results of these studies with out results.

It is a notorious fact that a quantum system, for all practical purposes, be that the dynamics of approach to equilibrium or the quantum computing, must be considered in interaction with its environment [5]. Dynamics of a special class of open quantum systems is described by continuous semi-groups of completely positive operators [19], and in that case the evolution of the quantum state is governed by the Lindblad equation [5,27], with the Hamiltonian and Lindblad operators that describe the corresponding internal dynamics and the environmental influences. Common examples of the environmental influences are thermal or dephasing environments. If such an environment acts locally on each of the parts of the quantum system then it is expected that the entanglement between the parts will be destroyed eventually. In what follows we shall report the results of our study of the interplay of qualitatively different types of interaction and different environments on the entanglement dynamics.

The Letter is organized as follows. In the next section we shall first describe three examples of qubit chains with qualitatively different dynamics. We shall be careful to distinguish the chain with integrable quantum dynamics and the integrable classical limit, the chain with nonintegrable quantum dynamics and chaotic classical limit and the chain which is considered quantum integrable, with some corresponding spectral properties, but with chaotic classical limit. In the same section, brief recapitulation of the Lindblad equation and thermal and dephasing environments will be given. Finally, relevant measures of entanglement will be recapitulated. In Section 3. we shall present the numerical results and discuss the entanglement dynamics for isolated systems. It might be expected, and is confirmed by our computations, that the entanglement dynamics qualitatively depends on the interaction but also on the entanglement of the initial state. In the case of isolated qubits chains, which are all the time in pure states, we shall compute and compare the dynamics of pairwise entanglement with the global entanglement. In Section 4, the influence of thermal and dephasing environment will be illustrated and discussed. Summary and our main conclusions are given in Section 5.

2. Representative models, Markov dynamics of open systems and measures of entanglement

In our analyzes we shall use finite chains of qubits with a Hamiltonian of the following form:

$$H(k_x, k_z, J) = \sum_{i=1}^{N}\bigl(k_x\sigma_i^x + k_z\sigma_i^z\bigr) + J\sum_{i=1}^{N}\sigma_i^x\sigma_{i+1}^x,$$
$$\sigma_{N+1} = \sigma_N. \tag{1}$$

The Hamiltonian (1) describes a ring lattice of N $1/2$ spins with the Ising interaction subjected to a constant magnetic field tilted in the (x, z) plane. We shall always use units in which $\hbar = 1$, fix $J = 1$, and present all the results in units of dimensionless time $\tau \equiv Jt$.

The symmetric system with longitudinal magnetic field $k_z = 0$; $k_x \neq 0$ is integrable by the Bethe ansatz [8]. The system with transverse magnetic field $k_x = 0$; $k_z \neq 0$ is transformed with the Jordan–Wigner–Bogolybov (non-canonical) mapping [28] into a system of noninteracting fermions, and as such is exactly solvable. The system with both $k_x \neq 0$, $k_z \neq 0$ is considered quantum nonintegrable. There are several ways to introduce classical approximations of the dynamics of these three systems [29–32]. However, the classical symmetric system $H(k_x, 0, J)$ is always completely integrable, and both nonsymmetric systems $H(0, k_z, J)$ and $H(k_x, k_z, J)$ have classical counterparts which display the transition from mostly regular to dominantly chaotic dynamics [30]. On the other hand, the quantum systems with either of the fields set to zero, that is the systems that are considered quantum integrable, display the spectral properties typical of quantized classically completely integrable systems [9,24], irrespective of the lack of the additional symmetry in $H(0, k_z, J)$. In particular, the standard diagnostic tool of the quantized chaos, namely the nearest neighbor level spacing distribution (NNLS distribution) in cases of $H(k_x, 0, J)$ and $H(0, k_z, J)$ is Poisonian, whereas in the case of $H(k_x, k_z, J)$ for sufficiently large k_x, k_z, for example if $k_x = k_z = 1.4$; $J = 1$ the NNLS distribution is Gaussian. The latter system is then considered quantum chaotic and the symmetric $H(k_x, 0, J)$ and nonsymmetric $H(0, k_z, J)$ systems are considered quantum integrable [3,24]. We shall analyze if differences in qualitative properties of the entanglement dynamics with qualitatively different initial states follow this classification into quantum integrable and quantum chaotic. In particular it is interesting to see if the case of the nonsymmetric quantum integrable system $H(0, k_z, J)$ has some special status with respect to the generated entanglement dynamics, at least for some initial states.

Quantum integrability and quantum chaos in different spin chains, and in particular of the form (1), with or without time-dependent fields, have been intensively studied ([3] and references therein). Entanglement in such spin chains has also been studied, in particular the relation between phase transitions and the properties of entanglement in the ground or thermal state [33]. Dynamics of entanglement in the system (1) and in the integrable versus chaotic case has been recently studied in [9,23,24], but the properties of the nonsymmetric quantum integrable case $H(0, k_z, J)$ where not emphasized. Dependence of the entanglement dynamics on the symmetry of the spin Hamiltonian for dynamics from an initial separable state was analyzed in [31] using only a pair of interacting qubits.

2.1. Open systems

Quantum systems with a Hamiltonian like (1) with relatively small number of qubits have been recently manufactured in laboratories [34], and are considered as potentially useful for quantum simulations and maybe quantum computations. In the conditions of real experiments the quantum system is never isolated from its environment, and for some purposes like the explanation of quantum to classical transition such influence might play the crucial role [35–37]. Therefore it is important to include the effects of the environment in the studies of the entanglement dynamics. Evolution of any quantum system is given by completely positive transformations of its state or its observables. In many cases the environment is such that the state transformations at successive instants of time form a continuous Markov evolution. Then the evolution of the state is describe by the Lindblad equation for the state density matrix [5,19,26,27]

$$\frac{d\rho(t)}{dt} = -i[H, \rho] - \frac{1}{2}\sum_k\bigl[L_k\rho, L_k^\dagger\bigr] + \bigl[L_k, \rho L_k^\dagger\bigr], \tag{2}$$

where $-i[H, \rho]$ describes the unitary part and the rest is the dissipative part. The Lindblad operators L_k are interpreted and inferred from different types of the influence that the environment exerts on the system. We shall analyze the entanglement dynamics in the three examples of qualitatively different systems with the corresponding Hamiltonian $H(k_x, k_z, J)$ plugged in Eq. (2) and with the Lindblad operators that correspond to local thermal and dephasing environments. The Lindblad operators for these two types of the environment are given by the following formulas [4,5]: $L_i \equiv \mathbf{1}_1 \otimes \cdots (L_i) \cdots \otimes \mathbf{1}_N$ where for each qubit:

$$L_i = \frac{\Gamma(\bar{n}+1)}{2}\sigma_i^- + \frac{\Gamma\bar{n}}{2}\sigma_i^+ \tag{3}$$

describes the thermal environment, and

$$L_i = \Gamma\sigma_i^+\sigma_i^- \tag{4}$$

corresponds to the dephasing effect of the environment. The parameter $\bar{n}$ in (3) is proportional to the temperature and Γ in (3) or (4) (not necessarily the same small numerical value) is treated here as a small phenomenological parameter that describes the strength of the coupling between a qubit and its environment.

The main problem with numerical solutions of the Lindblad equation for finite chains of qubits is that the evolution of $\rho(t)$ requires large effective storage space. Recently, great savings of the storage space have been achieved by applications of the time dependent density matrix renormalization group method (tDMRG) [42]. However, approximations involved in tDMRG rely on the assumption that the entanglement between many different subparts of the system is small, which is the case only for some initial states and for some types of Hamiltonian interaction. In general, and in particular for complex quantum evolution, it is expected that long term dynamics leads to the significant entanglement between all subparts of the system. It has been demonstrated that QNI evolution leads to exponential growth of memory resources needed for tDMRG computation with the evolution time [3].

As is well known there is a class of alternative but equivalent descriptions of an open Markov system dynamics given entirely in terms of pure states. The evolution equation in these formulations is a stochastic Schrödinger equation (SSE) with the state space of dimension 2^N. Such SSEs are called stochastic unraveling [5,36,43] of the Lindblad master equation for the reduced density matrix $\rho(t)$. There are many different forms of nonlinear and linear SSEs that have been used in the context of open systems [5,36,43–45]. In particular SSEs have been used to study the quantum to classical transition of complex quantum dynamics [36,37]. All unravelings of the Lindblad equation are consistent with the requirement that the solutions of (2) and of the corresponding SSE satisfy

$$\rho(t) = E\big[\big|\psi(t)\big\rangle\big\langle\psi(t)\big|\big], \tag{5}$$

where $E[|\psi(t)\rangle\langle\psi(t)|]$ is the expectation with respect to the distribution of the stochastic process $|\psi(t)\rangle$.

In our computations we shall use a special form of the SSE, given by the quantum state diffusion (QSD) theory because the correspondence between QSD evolution equation and the Lindblad equation is unique. We used the QSD method only as an efficient computational tool, and did not explore the possibility to use the fact that the stochastic evolution of pure states provides valuable insides which can not be inferred from the density matrix approach [5,36,46–48].

The QSD evolution equation is given by the following formula:

$$\begin{aligned}|d\psi\rangle &= -iH|\psi\rangle\,dt \\ &\quad + \left[\sum_k 2\langle L_k^\dagger\rangle L_k - L_k^\dagger L_k - \langle L_k^\dagger\rangle\langle L_k\rangle\right]\big|\psi(t)\big\rangle\,dt \\ &\quad + \sum_l \big(L_k - \langle L_k\rangle\big)\big|\psi(t)\big\rangle\,dW_k,\end{aligned} \tag{6}$$

where L_k are the same Lindblad operators as in (2), and where $\langle\,\rangle$ denotes the quantum expectation in the state $|\psi(t)\rangle$ and dW_k are independent increments of complex Wiener c-number processes $W_k(t)$ satisfying

$$\begin{aligned}&E[dW_k] = E[dW_k\,dW_{k'}] = 0,\\ &E[dW_l\,d\bar{W}_{k'}] = \delta_{k,k'}\,dt,\\ &\quad k = 1, 2, \ldots, m.\end{aligned} \tag{7}$$

Here $E[\cdot]$ denotes the expectation with respect to the probability distribution given by the multi-dimensional process W, and $\bar{W}_k$ is the complex conjugate of W_k.

2.2. Measures of entanglement

There are different types of entanglement that can be of interest in the system of qubits like (1), and some of them can be efficiently calculated [38]. If the total system is in a pure state $|\psi\rangle$ one can calculate the bipartite entanglement of formation between a subsystem consisting of $k < N$ qubits and the rest of the chain. In particular, entanglement of formation between the ith qubit and the rest of the chain is given by the Von Neuman entropy

$$E_i(\rho_i) = \rho_i \operatorname{Log}(\rho_i), \tag{8}$$

where $\rho_i = \operatorname{Tr}_{\neq i}|\psi\rangle\langle\psi|$, and $\operatorname{Tr}_{\neq i}$ denotes the partial trace over the spaces of all except the ith qubit. We shall always use Log with the base 2. Analogously, the bipartite entanglement of formation between a pair of qubits, say the pair at sites 1 and 2, and the rest of the chain is given by the Von Neumann entropy of the reduced density matrix $\rho_{ij} = \operatorname{Tr}_{\neq i,j}|\psi\rangle\langle\psi|$ of the considered pair:

$$E_{12,3\ldots N} = \rho_{12}\operatorname{Log}(\rho_{12}). \tag{9}$$

Single qubit reduced matrix (ρ_i) can be used to define and effectively compute a global measure of entanglement for the system (1) in a pure state, which measures also some form of genuine multi-partite entanglement. The measure is known as Meyer–Wallach geometric measure [39], is given by

$$Q\big(|\psi\rangle\big) = 2\left(1 - \frac{1}{N}\sum_{i=1}^{N}\operatorname{Tr}\big(\rho_i^2\big)\right), \tag{10}$$

and is not applicable if the entire system is in a mixed state. In the case of open system of qubits, which are in a mixed state, Q would also include the entanglement between the qubits and the environment. Theory of multi-partite entanglement for systems in mixed state is far from completed, and there are only estimates of upper bounds that are relatively easy to compute [4].

Entanglement of formation between two qubits in the chain can also be efficiently calculated for the particular pair of qubits using the reduced density matrix ρ_{ij}, i.e. all the bi-partite correlation functions $\langle\sigma_i^{x,y,z}\sigma_j^{x,y,z}\rangle$ for that particular pair [40,41]. First, a concurrence is calculated by the following formula

$$C(\rho_{i,j}) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}, \tag{11}$$

where $\lambda_1 > \cdots > \lambda_4$ are the eigenvalues of the matrix $\rho_{ij}(\sigma_i^y \otimes \sigma_j^y)\bar{\rho}_{ij}(\sigma_i^y \otimes \sigma_j^y)$, where $\bar{\rho}_{ij}$ is the complex conjugate of ρ_{ij} calculated in the standard bases. The entanglement of formation is then given via the function

$$h(x) = -x\log x - (1-x)\log(1-x)$$

by the following formula

$$E(\rho_{ij}) = h\left(\frac{1 + \sqrt{1 - C(\rho_{i,j})^2}}{2}\right). \tag{12}$$

The total amount of local pair-wise entanglement of formation $E(\rho_{ij})$ is given by the sum over all (i, j) pairs. Instead, we shall use the sum of all pair-wise squared concurrences (8).

$$\sum_{i<j} C^2(\rho_{ij}). \tag{13}$$

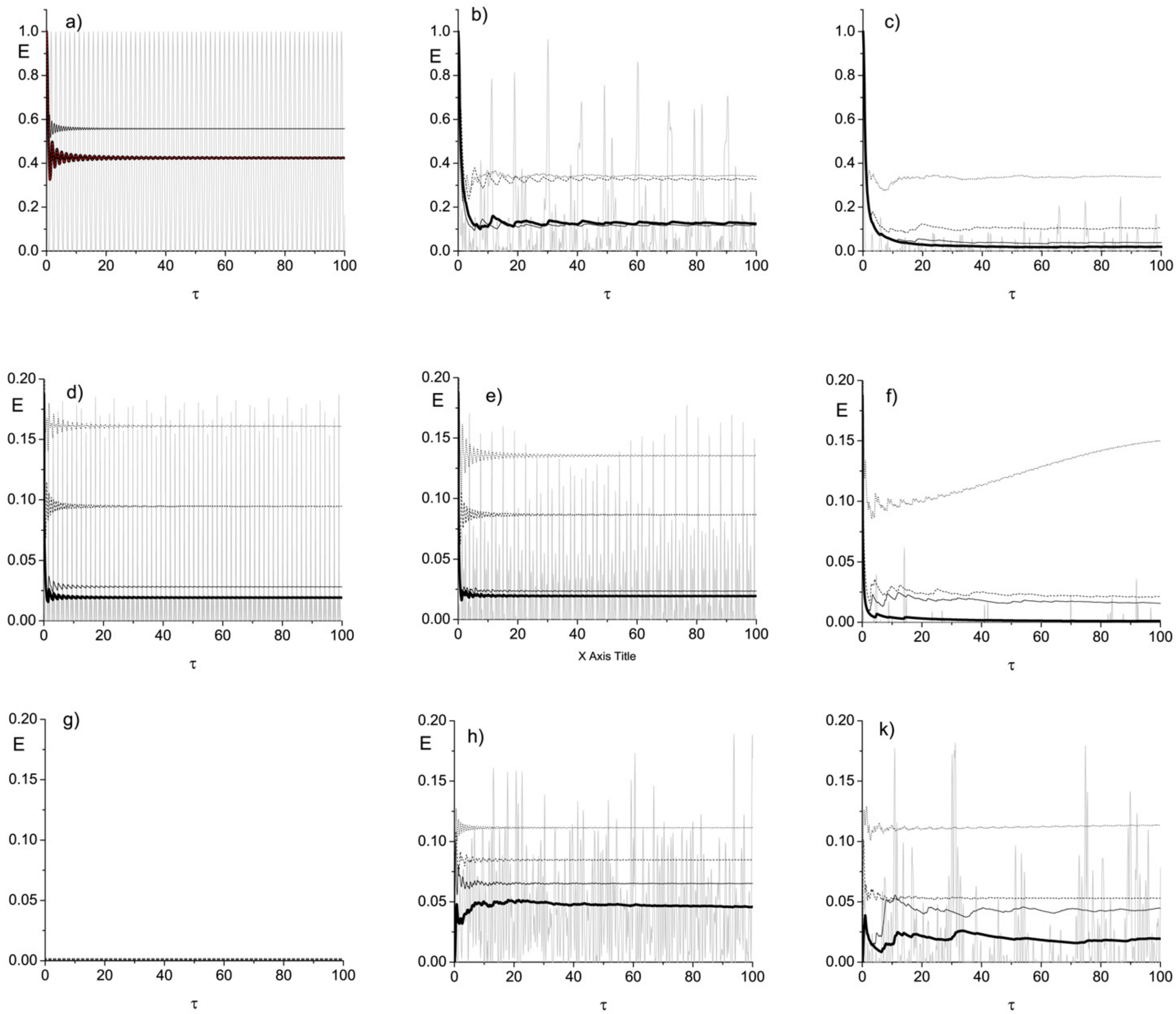

Fig. 1. Illustrates bipartite entanglement dynamics in isolated systems. E_{12} for $N=6$ (gray), $\bar{E}_{12}$ for $N=3$ (dotted), for $N=4$ (dashed) for $N=5$ (thin full) and $N=5$ (thick full) are shown. Figures (a), (b), (c) correspond to $|\max\rangle$, (d), (e), (f) to $|W\rangle$ and (g), (h), (k) to $|\text{sep}\rangle$ initial states. Figures (a), (d), (g) correspond to $H(k_x=1.4, k_z=0, J=1)$, (b), (e), (h) to $H(k_x=0, k_z=1.4, J=1)$ and (c), (f), (k) to $H(k_x=1.4, k_z=1.4, J=1)$ systems.

Notice that $E_{ij} \equiv E(\rho_{ij})$ characterizes the entanglement between ith and jth qubit even when the entire system is entangled to the environment, i.e. when the entire system is in a mixed state. We shall calculate the time dependence of the pair-wise entanglement of formation E_{ij} in the cases of isolated and open systems and the Von Neumann entropy $E_i(\rho_i)$ and $Q(|\psi\rangle)$ only for the isolated systems.

3. Entanglement dynamics

We shall systematically illustrate our computations of the entanglement dynamics using the rings of $N=6$ qubits. However, our main conclusions have been confirmed with larger rings up to $N=12$. In all results that are presented here we use the same value of the interaction parameter $J=1$. The parameters k_x, k_z are set to typical values as follows. Symmetric quantum integrable (QI) case is represented by $H(k_x=1.4, k_z=0, J=1)$, nonsymmetric quatum integrable by $H(k_x=0, k_z=1.4, J=1)$ and the quantum nonintegrable (QNI) case by $H(k_x=1.4, k_z=1.4, J=1)$. As the initial states we have considered pure N qubit states with three different distributions of entanglement: the separable states

$$|\text{sep}\rangle \equiv |\rightarrow_1, \uparrow_2, \uparrow_3, \ldots, \uparrow_N\rangle, \tag{14}$$

the states with only one $(i, i+1)$-pair maximally entangled and the rest in product form

$$|\max\rangle \equiv \left[\left(|\uparrow_1, \downarrow_2\rangle + |\downarrow_1, \uparrow_2\rangle\right) \otimes \left(|\downarrow_3, \ldots, \downarrow_N\rangle\right)\right]/\sqrt{2} \tag{15}$$

and an example of a state with distributed entanglement

$$|W\rangle \equiv \big(|\uparrow_1, \downarrow_2, \downarrow_3, \ldots, \downarrow_N\rangle + |\downarrow_1, \uparrow_2, \downarrow_3, \ldots, \downarrow_N\rangle + \cdots + |\downarrow_1, \downarrow_2, \ldots, \uparrow_N\rangle\big)/\sqrt{N}. \tag{16}$$

The states $|\text{sep}\rangle$ and $|\max\rangle$ are typical in the sense that the entanglement dynamics from other separable states or from other states with initially localized entanglement that we have studied are qualitatively the same as the entanglement dynamics from $|\text{sep}\rangle$ and $|\max\rangle$ respectively. The entanglement dynamics from $|W\rangle$ represents an example of an initial state with distributed

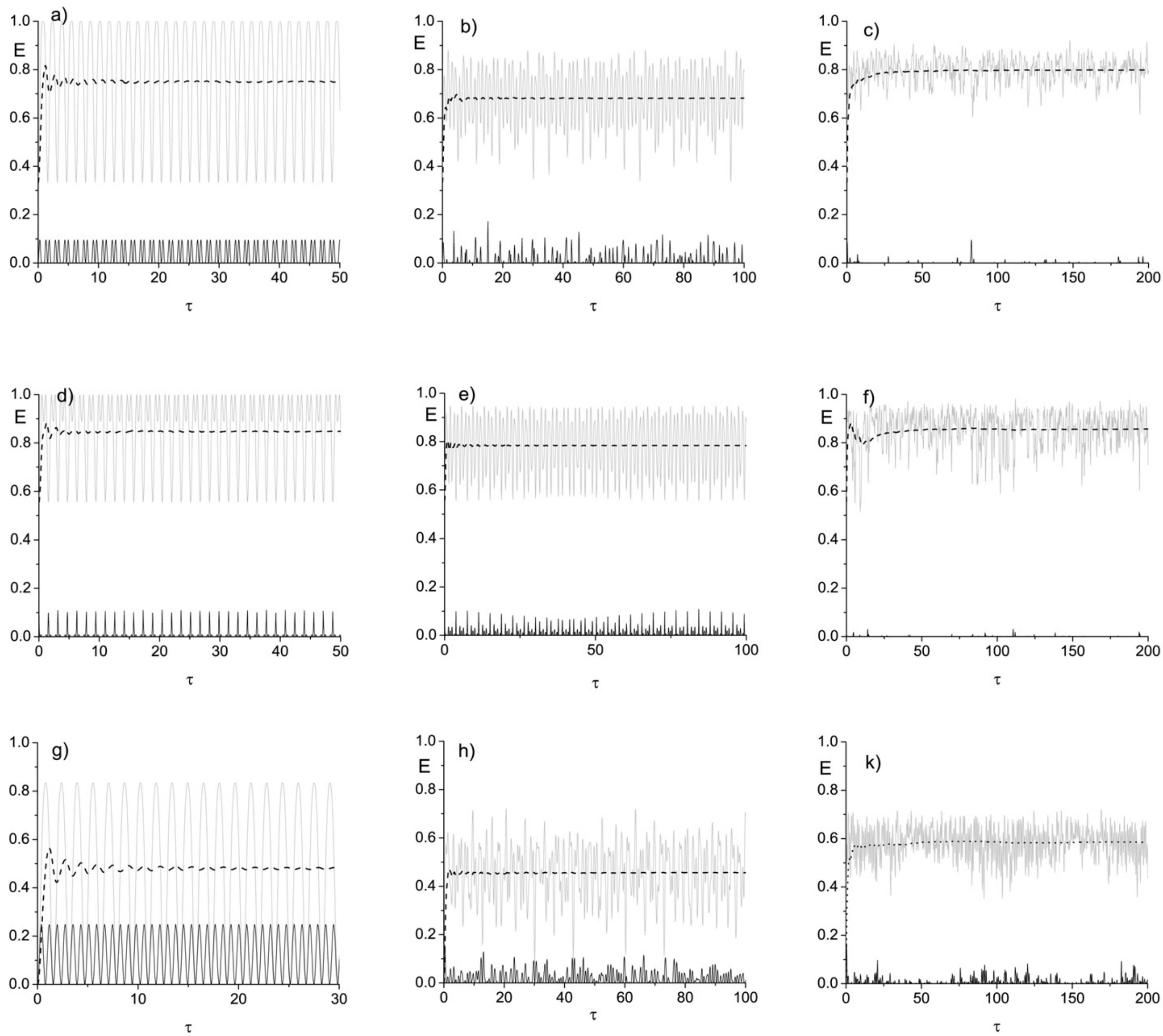

Fig. 2. Illustrates total pair-wise squared concurrence (10) (black) and the global entanglement $Q(\psi)$ (gray) dynamics in isolated systems for $N = 6$. Time averaged $\bar{Q}$ (black dotted) is also shown. Figures (a), (b), (c) correspond to $|\max\rangle$, (d), (e), (f) to $|W\rangle$ and (g), (h), (k) to $|\mathrm{sep}\rangle$ initial states. Figures (a), (d), (g) correspond to $H(k_x = 1.4, k_z = 0, J = 1)$, (b), (e), (h) to $H(k_x = 0, k_z = 1.4, J = 1)$ and (c), (f), (k) to $H(k_x = 1.4, k_z = 1.4, J = 1)$ systems.

entanglement as opposed to $|\mathrm{sep}\rangle$ and $|\max\rangle$. Other states with nonequivalent types of distributed entanglement should also be analyzed. This becomes more relevant for chains with larger number of qubits.

Local thermal or dephasing environment exerts on all considered systems long-term effects of exponential dumping of the entanglement in the similar way. The major factor that determines the differences in the entanglement evolution are the qualitative properties of the isolated systems dynamics. Therefor, we shall first illustrate and discuss the dynamics of entanglement for isolated systems.

3.1. Isolated systems

Isolated systems of the form (1) can always be considered to be in a pure state. The initial state can be separable or entangled and the subsequent states in the evolution are most often entangled due to the interaction between the qubits. Each qubit interacts only with two other qubits, its nearest neighbors, but the quantum correlations, as expressed by different types of entanglement, are established among all the qubits in the ring. Our task in this subsection is to describe, for isolated systems of the form (1), the evolution of the pair-wise entanglement between different pairs of qubits and the global entanglement as described by entanglement of each of the qubits with all $N - 1$ others and by the geometric measure $Q(|\psi\rangle)$. We shall stress the differences in the entanglement evolution due to different qualitative properties of the Hamiltonian. Entanglement evolution also qualitatively depends on the entanglement in the initial state. As we shall see, this dependence of the qualitative properties of the entanglement evolution on the initial state is most obvious in the case of the integrable nonsymmetric Hamiltonian $H(0, k_z, J)$.

Figs. 1 and 2 illustrate the entanglement dynamics for isolated systems using the ring with $N = 6$ qubits. Fig. 1 shows the long

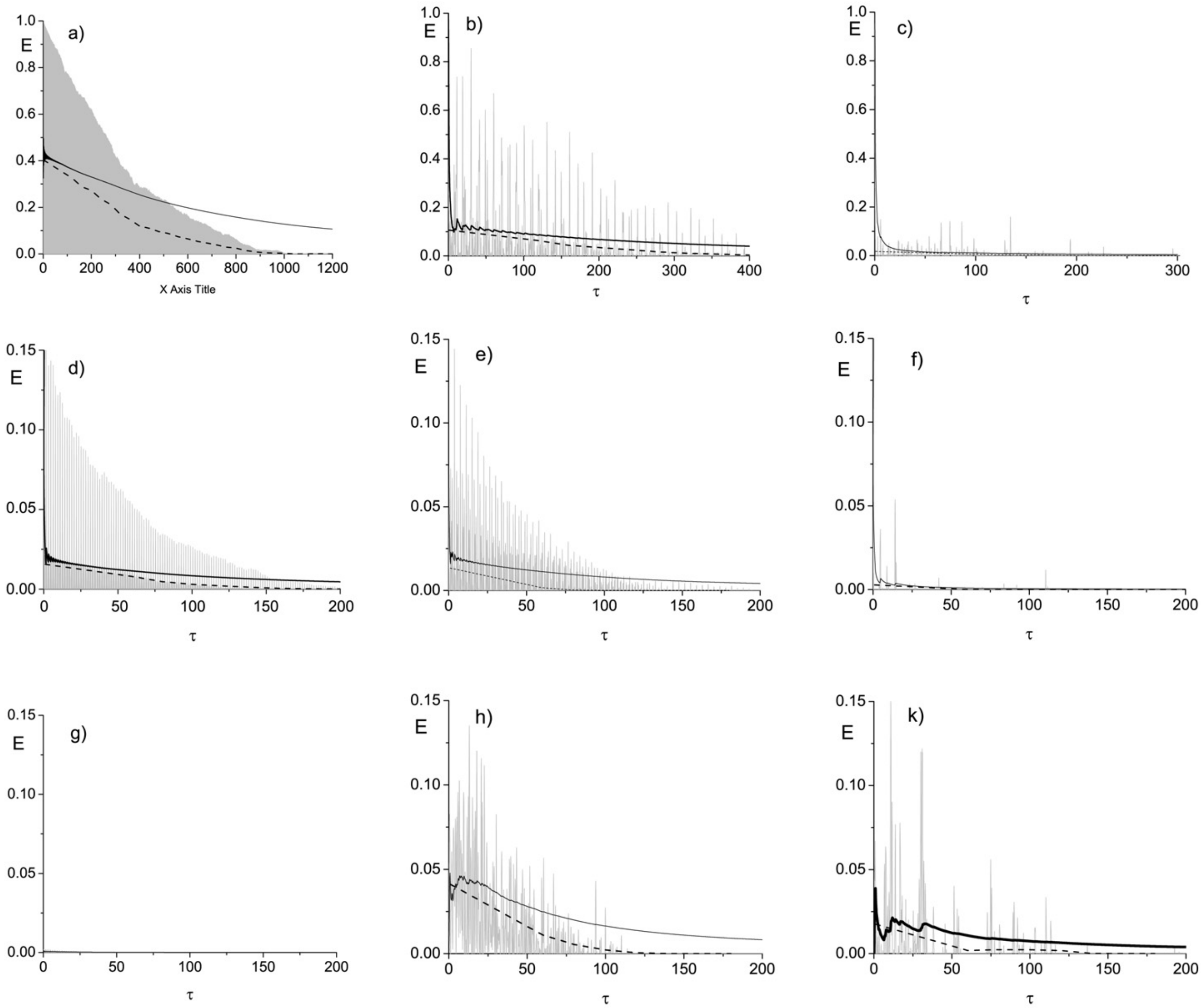

Fig. 3. Illustrates bipartite entanglement dynamics with Lindblad corresponding to the thermal environment for $N = 6$. E_{12} (gray), $\bar{E}_{12}(T)$ (black full) and $E_{12}(t, \Delta T)$ (dotted) are shown. Figures (a), (b), (c) correspond to $|\max\rangle$, (d), (e), (f) to $|W\rangle$ and (g), (h), (k) to $|\text{sep}\rangle$ initial states. Figures (a), (d), (g) correspond to $H(k_x = 1.4, k_z = 0, J = 1)$, (b), (e), (h) to $H(k_x = 0, k_z = 1.4, J = 1)$ and (c), (f), (k) to $H(k_x = 1.4, k_z = 1.4, J = 1)$ systems.

term dynamics of only E_{12} and in Fig. 2 we have illustrated the relation between global entanglement as expressed by $Q(|\psi\rangle)$ (7) and the total pair-wise squared concurrence $C_{i,j}^2$ given by (10). In Fig. 1 we show also the time average of $E_{12}(t)$ defined as

$$\bar{E}_{1,2}(T) = \frac{1}{T}\int_0^T E_{12}(t)\,dt. \tag{17}$$

$\bar{E}_{12}(T)$ gives expected value of E_{12} in a unit of time that is present in the system during the time interval $(0, T)$. Time averaged entanglement $\bar{E}_{12}(T)$ is displayed for $N = 3, 4, 5, 6$. We shall use $\bar{E}_{12}$ also to compare E_{12} entanglement dynamics in rings of different size N in the thermal and dephasing environments.

Let us make some remarks concerning the typical behavior of the entanglement dynamics with different Hamiltonians and different initial states, suggested by all our computations and illustrated with the ring $N = 6$.

(a) QI symmetric Hamiltonian always generates simple periodic entanglement dynamics. In the case of $|\max\rangle$ initial state E_{12} entanglement and $E_{12,3\ldots N}$ entanglement are complementary. In the case of $|\text{sep}\rangle$ initial states $E_{12} \approx 0$ and $E_{12,3\ldots N}$ regularly oscillates between zero and unity. For $|W\rangle$ initial state E_{12} and E_{13} have qualitatively the same dynamics and this is the case with the QI nonsymmetric Hamiltonian as well. It is also interesting to notice that $|\max\rangle$ initial state and symmetric QI Hamiltonian (Fig. 1(a)) lead to the same time averaged E_{12} for $N = 4, 5, 6$. In the nonsymmetric QI case time averaged E_{12} from $|\max\rangle$ initial state is similar for $N = 3, 4$ and for $5, 6$ (Fig. 4(b)). In other cases the time averaged E_{12} decreases as N is increased.

(b) Entanglement dynamics with the QNI Hamiltonian is that of qualitatively complicated aperiodic oscillations for all three types of the initial states. Global entanglement Q is clearly larger than total pair-wise entanglement for the considered initial states (Fig. 2).

(c) Consider now the details of the entanglement dynamics with nonsymmetric QI Hamiltonian. Qualitatively, the entanglement dynamics from $|\max\rangle$ and $|W\rangle$ is more similar to that with symmetric QI, but from generic $|\text{sep}\rangle$ initial states it is more similar to that with the QNI Hamiltonian. E_{12}, E_{13} and the total pair-

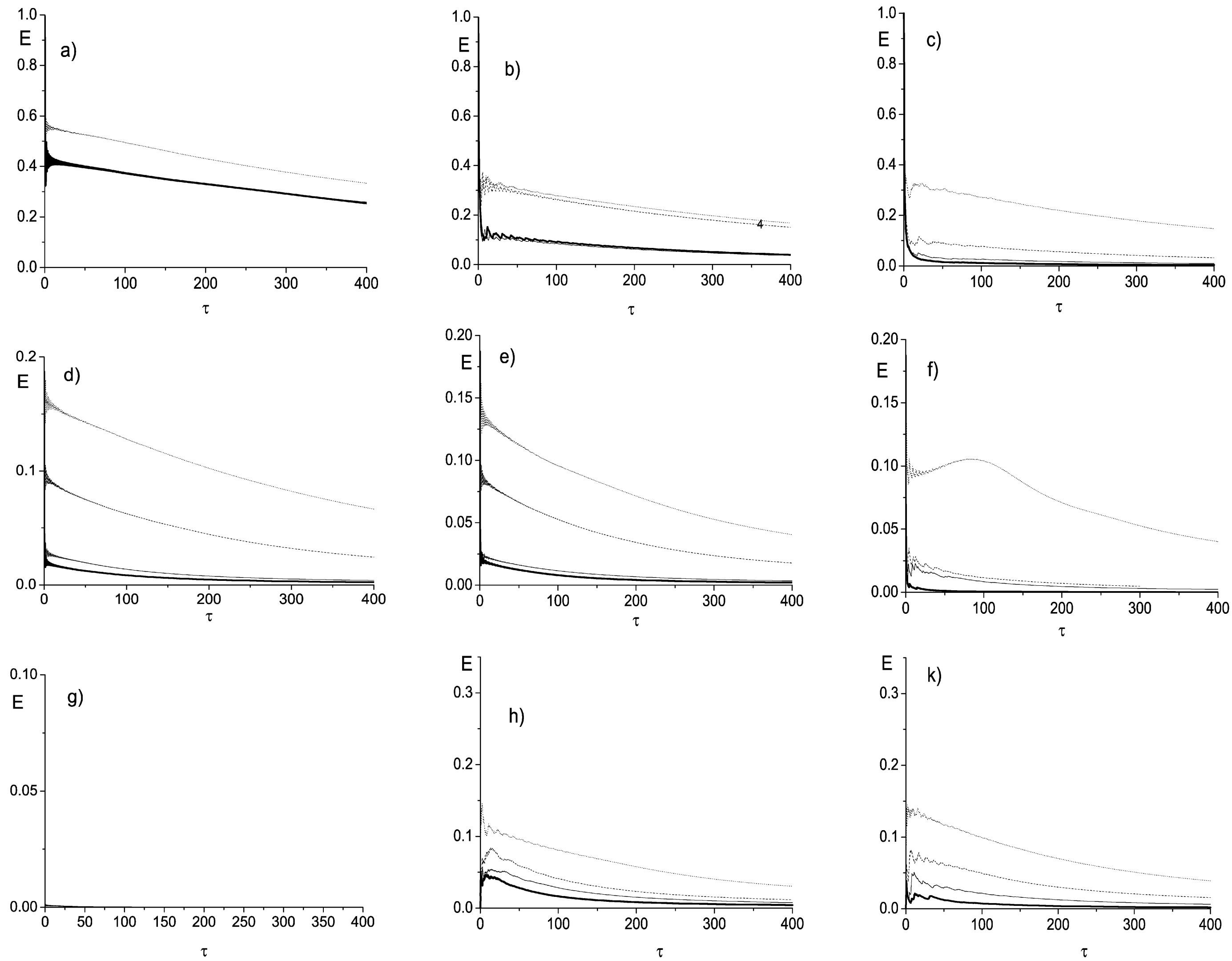

Fig. 4. Illustrates bipartite entanglement dynamics with Lindblad corresponding to the thermal environment. E_{12} for $N = 6$ (gray), $\bar{E}_{12}$ for $N = 3$ (dotted), for $N = 4$ (dashed) for $N = 5$ (thin full) and $N = 5$ (thick full) are shown. Figures (a), (b), (c) correspond to $|\max\rangle$, (d), (e), (f) to $|W\rangle$ and (g), (h), (k) to $|\text{sep}\rangle$ initial states. Figures (a), (d), (g) correspond to $H(k_x = 1.4, k_z = 0, J = 1)$, (b), (e), (h) to $H(k_x = 0, k_z = 1.4, J = 1)$ and (c), (f), (k) to $H(k_x = 1.4, k_z = 1.4, J = 1)$ systems.

wise entanglement with $|\max\rangle$ and $|W\rangle$ initial states are on the average smaller than with the QI symmetric Hamiltonian but larger than with the QNI Hamiltonian. On the other hand $E_{12,3\ldots N}$ and Q are on the average larger than with QI symmetric and smaller than with QNI Hamiltonian. In the case of generic $|\text{sep}\rangle$ initial state E_{12}, E_{13} and the total pair-wise entanglement are the largest for nonsymmetric QI Hamiltonian and the smallest with symmetric QI Hamiltonian with the nonintegrable case in the middle. $E_{12,3\ldots N}$ and Q are larger with the QNI Hamiltonian than with either of the other two systems.

All these observations support the following general conclusions. Entanglement dynamics with the QNI Hamiltonian is qualitatively more complicated than with symmetric QI Hamiltonian irrespective of the initial state. Also, independently of the initial state if there is some initial entanglement it becomes more distributed between all the qubits, but the local pair-wise entanglement is much smaller, with the QNI Hamiltonian than in the, symmetric QI case. The entanglement in nonsymmetric QI system shows qualitatively different dynamics for different initial states. Separable initial conditions make the properties of the entanglement dynamics with the nonsymmetric QI Hamiltonian similar to the case of QNI system. On the other hand, globally entangled initial state renders the entanglement dynamics with the QI nonsymmetric system more similar to that of the symmetric QI system.

4. Open systems

The study of entanglement dynamics in realistic systems must take into the account the interactions, and entanglement, between the system and its environment. The environmental influence usually leads to dissipation of the entanglement from the system, i.e. to decoherence, but could also increase the entanglement between subparts of the system if the environments acting on the different parts are in some correlation.

We have used the QSD equation (16) with the Hamiltonian operators corresponding to the three qualitatively different systems (1) and the Lindblad operators corresponding to the local thermal (3) or local dephasing (4) environments. In our case there are N Lindblad operators, one of the form (3) or (4) for each qubit, so $m = N$ in Eq. (16). A single realization of the stochastic process is used to calculate, for example, $\langle\sigma_i^j\rangle$ or $\langle\sigma_i^j\sigma_{i+1}^k\rangle$; $j, k = x, y, z$ and then averaging over many sample paths gives the average values $E\langle\sigma_i^j\rangle = \mathrm{Tr}[\rho\sigma_i^j]$ and correlation functions $E[\langle\sigma_i^j\sigma_{i+1}^k\rangle] = \mathrm{Tr}[\rho\sigma_i^j i\sigma_{i+1}^k]$ which are needed for the calculation of the entangle-

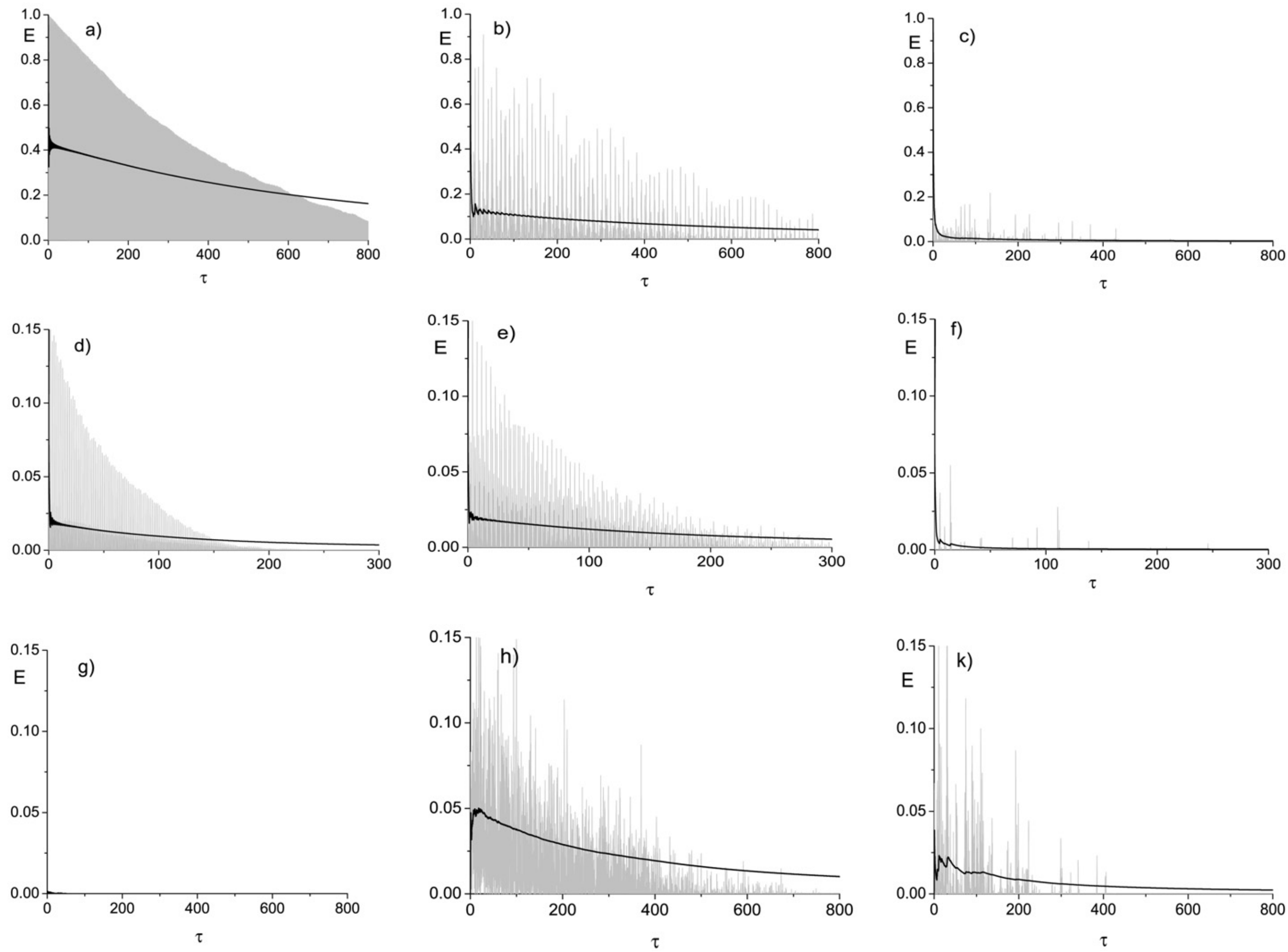

Fig. 5. Illustrates bipartite entanglement dynamics with Lindblad corresponding to the dephasing environment for $N = 6$. E_{12} (gray), $\bar{E}_{12}(T)$ (black full) are shown. Figures (a), (b), (c) correspond to $|\max\rangle$, (d), (e), (f) to $|W\rangle$ and (g), (h), (k) to $|\text{sep}\rangle$ initial states. Figures (a), (d), (g) correspond to $H(k_x = 1.4, k_z = 0, J = 1)$, (b), (e), (h) to $H(k_x = 0, k_z = 1.4, J = 1)$ and (c), (f), (k) to $H(k_x = 1.4, k_z = 1.4, J = 1)$ systems.

ment measures. Notice that the memory storage space is in general 2^N times smaller than in the direct numerical solution for the density matrix master equation. The computational time depends on the number of paths needed for the required accuracy of the averages and correlations. In order to infer clear definitive conclusions about the qualitative differences of the entanglement dynamics in our computations it was enough to use only about a couple of hundred sample paths.

There is no efficient way to compute any of the proposed measures of multi-partite or global entanglement for system in mixed states [4], so we have analyzed only the pair-wise entanglement $E_{ij}(\rho(t))$ for different pairs. Results of numerical computations using the QSD method are illustrated in Figs. 3 and 4 for the thermal and in Fig. 5 for the dephasing environments. Only $E_{12}(t)$ is illustrated, since E_{ij} for other pairs follow similar patterns. In Fig. 3 we have illustrated the long term dynamics of $E_{12}(t)$ for the ring of $N = 6$ qubits. Together with $E_{12}(t)$ plotted are the time average $\bar{E}_{12}(t)$ and the short time average $\bar{E}_{12}(t, \Delta T)$. The latter is defined as the average over a short time interval ΔT from time t up to $t + \Delta T$, i.e.

$$\bar{E}_{12}(t; \Delta T) = \frac{1}{\Delta T} \int_{t}^{t+\Delta T} E_{12}(t)\, dt. \tag{18}$$

All our computations support the conclusion that the long term dynamics of pair-wise entanglement is that of exponential decay, while the dynamics over short intervals of time displays the same qualitative properties as for the isolated systems. In fact $\bar{E}_{12}(t)$ quickly converges onto the curve of exponential decay:

$$\lim_{t\to\infty} \frac{\bar{E}_{12}(t)}{A \exp(-Bt) + C} = 1. \tag{19}$$

The coefficient B seems to be dependent on the Hamiltonian, initial state and of course on the system-environment coupling.

In Fig. 4 we have illustrated the dependence of $\bar{E}_{12}(t)$ on the number of qubits N for different Hamiltonian and different initial states. Again the exponential decay is just superposed on the type of the dependence on N that is displayed by the isolated systems, as was illustrated in Fig. 1.

Our results suggest the conclusion that the qualitative properties of the entanglement dynamics for open Markov systems with thermal or dephasing Lindblad operators are determined by the Hamiltonian and the initial state. The influence of the local environment is qualitatively the same irrespective of the Hamiltonian or the initial state.

5. Summary

We have studied dynamics of entanglement in open systems of qubits whose evolution is described by a Lindblad master equation, with the Lindblad operators of the form corresponding to local thermal or dephasing environments. As examples of qubits systems we have chosen periodic chains of spins with Ising interaction in variously oriented external fields. In the Hamiltonian limit the qualitative properties of the dynamics of the three considered systems are quite different. One of them is quantum integrable with an additional symmetry and with the integrable classical limit; the other is quantum nonintegrable with the chaotic classical limit and the third is considered quantum integrable but has the chaotic classical limit. We have been interested in manifestations of qualitatively different dynamical properties of the Hamiltonian in the entanglement dynamics of the systems influenced by the local Markov environment.

In order to solve numerically the Lindblad master equation we have used the quantum state diffusion method, which relies on a stochastic Schrödinger equation for pure states, equivalent to the Lindblad equation for mixtures, on averaging over many sample paths. In the case of the isolated systems we have computed the entanglement of formation for different pairs of qubits and also the global multi-partite entanglement as measured by Meyer–Wallach $Q(\psi)$ measure. In the case of open systems only the pair-wise entanglement dynamics was analyzed.

Results of the numerical computations can be summarized as follows. Local Markov environments exert on all considered systems long-term effects of exponential dumping of the entanglement in the similar way. The major factor that determines the entanglement evolution are the qualitative properties of the isolated systems dynamics. The later depend on the Hamiltonian, but also, and specially for the nonsymmetric quantum integrable system, on the initial state. In general entanglement of the quantum nonintegrable system is quickly distributed over the chain and remains more in the form of global multi-partite rather than local pair-wise entanglement. On the other hand, dynamics of the symmetric quantum integrable system favours various forms of bipartite entanglement between different qubits. The entanglement dynamics of the nonsymmetric quantum integrable system strongly depends on the entanglement in the initial state. In the case of globally entangled initial state, like the considered $|W\rangle$ state, the entanglement dynamics with such Hamiltonian is qualitatively and quantitatively similar to that of the quantum integrable symmetric system. On the other hand, the entanglement dynamics from the separable initial states is like that of the quantum nonintegrable system.

We have studied entanglement dynamics in different systems with the same quite strong inter-qubit interaction, and the qualitative differences of the dynamics are introduced by different orientation of the external field. The strong interaction lead to very fast entanglement dynamics and the effects of the dissipation and decoherence are seen only over time intervals that cover many characteristic oscillations of the entanglement. This is quite different from the situation recently studied for example in [49,50]. There the inter-qubit interaction is weak and of the same order as the environment-qubit interaction. In this case only few characteristic oscillations of the entanglement occur before it reaches a stationary value. On the other hand the systems and the problems analyzed in this work are similar to those reported in [23,24] and [9]. In these papers only the isolated systems have been studied and the dependence on the initial state was not emphasized. The conclusion about faster distribution of the entanglement from $|\max\rangle$ initial state with the quantum nonintegrable system is the same as in our case. The qualitative differences of the entanglement dynamics with the nonsymmetric quantum integrable system (with chaotic classical limit) for different types of initial states have not been reported before, and we believe deserve further study. It would also be interesting to study the effects on the entanglement dynamics of Lindblad operators that contain products of operators pertaining to different qubits for qualitatively different Hamiltonian systems [51].

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