

Nonlinear evolution of coarse-grained quantum systems with generalized purity constraints

Nikola Burić

Institute of Physics, PO Box 68, 11000 Belgrade, Serbia

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ABSTRACT

Constrained quantum dynamics is used to propose a nonlinear dynamical equation for pure states of a generalized coarse-grained system. The relevant constraint is given either by the generalized purity or by the generalized invariant fluctuation, and the coarse-grained pure states correspond to the generalized coherent, i.e. generalized nonentangled states. Open system model of the coarse-graining is discussed. It is shown that in this model and in the weak coupling limit the constrained dynamical equations coincide with an equation for pointer states, based on Hilbert–Schmidt distance, that was previously suggested in the context of the decoherence theory.

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1. Introduction

Coarse-grained description of a dynamical system is based on a separation of observables into two classes: the class of distinguished, i.e. important observables and the class of observables that are considered inaccessible. A new system, whose state can be maximally determined by measuring the preferred observables, is then defined. Evolution of the coarse-grained system should be completely described in terms of the distinguished observables only.

From an operational point of view, the choice of distinguished observables is dictated in practice by what can be measured on the given system. For example: (a) If all observables of a quantum system with the Hilbert space with N complex dimensions H^N are considered experimentally accessible then every Hermitian operator represents an observable of the system, i.e. all observables are distinguished, and the pure states of the system are by definition rays in the Hilbert space H^N ; (b) In the case of two spatially separated qubits $H^4 = H_1^2 \otimes H_2^2$ one could consider experimentally accessible only the local observables $\sigma_i^1 \otimes 1$ and $1 \otimes \sigma_i^2$. In this case the coarse-grained states are the product states; (c) In a collection of n spins $H^{2^n} = H^2 \otimes H^2 \otimes \dots \otimes H^2$ one might be able to observe only the macroscopic magnetizations $m_i = \sum \sigma_i/n$, which are then the coarse-grained distinguished observables.

The set of distinguished observables as a subalgebra of the algebra $U(N)$ is important in the definitions of notions such as quantum degrees of freedom and quantum integrability [1], generalized coherent states [2,3] and generalized entanglement [4–7], and provides a framework to study the relations between these no-

tions [8]. In particular we shall be interested in the coarse-grained states representing the generalized nonentangled (g -nonentangled) states as introduced and studied in [4–6]. Our goal is to derive an evolution equation for which the set of g -nonentangled states is invariant for arbitrary Hamiltonian, and discuss its physical interpretation.

In the next section we recapitulate the theory of generalized entanglement and generalized purity. In Section 3 we treat a quantum dynamical system on H^N as a classical Hamiltonian system on R^{2N} , which enables us to discuss constrained quantum dynamics. This is used to derive an evolution equation of the states which preserves the maximal generalized purity, that is of the g -nonentangled states. This evolution equation is nonlinear and can generate, depending on the Hamiltonian, chaotic dynamics of the coarse-grained system. Open quantum system model of the evolution of the distinguished states is discussed in Section 4. In this section we show that our constrained evolution equation coincides in the weak coupling limit with the approximate evolution equation of the robust states derived in [9] by different means and in the context of decoherence theory.

2. Generalized entanglement and generalized purity

A selected set of distinguished observables is used to define the generalized notions of nonentangled and entangled states [4–6]. The coarse-graining by the distinguished observables, understood in the traditional probabilistic sense as replacing probabilities by conditional probabilities is crucial in this definition.

Consider a subset $g \in u(N)$ of distinguished observables. A state ρ_g is called g -reduced state of the state ρ if $\text{Tr}[\rho L_i] = \text{Tr}[\rho_g L_i]$ for any $L_i \in g$. The reduced state ρ_g is the projection of the state ρ on the subspace determined by distinguished observables. Identifying

E-mail address: buric@ipb.ac.rs.

the quantum states with probabilities the standard definition of the conditional probability is recognized. Pure state $\rho = |\psi\rangle\langle\psi|$ is generalized nonentangled if the corresponding reduced state ρ_g is pure $\rho_g^2 = \rho_g$. Otherwise the pure state $|\psi\rangle$ is g -entangled. In the case that the Hilbert space has the bipartite tensor product structure and each distinguished observable act nontrivially only in one of the components, the previous definition of g -entanglement reduces to the standard definition of the bipartite entanglement for pure states.

In a large class of situations of physical interest the set of distinguished observables forms a Lie algebra. In this case a measure of the generalized entanglement of the pure state $|\psi\rangle$ is provided by the generalized purity, which is the purity of the reduced state ρ_g , and is given by:

$$P_g(\psi) = \sum_l \langle \psi | L_l | \psi \rangle^2, \quad L_l \in g \quad (1)$$

where L_l form a bases of the Lie algebra g . The state $|\psi\rangle$ is generalized nonentangled if $P_g(\psi)$ is maximal. Pure states with $P_g(\psi)$ less than maximal represent g -entangled states, i.e. the states in which the coarse grained system g is entangled with the environment, i.e. with the operators not in g . Obviously, whether a pure state $|\psi\rangle \in H^N$ is generalized entangled or not depends on the choice of the distinguished observables. Once the distinguished observables are chosen, the question if the future orbit of a g -nonentangled $|\psi\rangle$ will remain in the set of g -nonentangled states depends on the evolution equation satisfied by $|\psi\rangle$.

An equivalent measures of g -entanglement is given by the total dispersion of the algebra of distinguished observables

$$\Delta_g(\psi) = \sum_l \langle L_l^2 \rangle - \langle L_l \rangle^2 = \sum_l (\Delta L_l)^2. \quad (2)$$

$\Delta_g(\psi)$ is minimal iff $P_g(\psi)$ is maximal. Expressions for the minimal value of $\Delta_g(\psi)$ and the maximal value of $P_g(\psi)$ in terms of the simple roots of g are known [10,7] and read

$$\Delta_q(\psi) \geq \sum_l k_l \langle \alpha_l, \alpha_l \rangle \equiv \min, \\ P_q(\psi) \leq \langle C^2 \rangle - \min \equiv \max \quad (3)$$

where the highest weight vector $\lambda = \sum_l k_l \alpha_l$ in terms of simple roots α_l and C is the quadratic Casimir operator.

Generalized coherent states have been defined for an arbitrary semi-simple Lie algebra [3]. If the algebra of distinguished observables g is semi-simple the minimum of $\Delta_q(\psi)$ and the maximum of $P_q(\psi)$ is achieved on the corresponding generalized coherent states [11]. Thus, in this case the class of g -nonentangled and g -coherent states coincide.

Recently, an algebraic geometric characterization of generalized $SU(N)$ coherent states for arbitrary N as algebraic varieties associated with generalized Veronese embeddings was reported in [12]. These results could be used to provide an explicit parametrization of the manifold of $SU(N)$ -coherent states.

3. Evolution equation of the g -nonentangled pure states

In general, reduction of the pure state $|\psi\rangle$ results in a mixed state ρ_g and the unitary Schroedinger evolution of $|\psi\rangle$ upon reduction becomes nonunitary, resulting in different forms (under different approximations) of master equations for $\rho_g(t)$. However, if the g -nonentangled pure state $|\psi\rangle$ evolves in the subset of g -nonentangled states the reduced state always remains pure. In order for this to occur in general the Hamiltonian linear evolution of $|\psi(t)\rangle$ is not enough and a nonlinear constrain has to be

added to ensure the preservation of the g -purity $P_g(\psi(t))$. In order to formulate such constrained evolution we shall use geometric Hamiltonian formulation of the quantum evolution.

It is well known (please see [13] or [14] and references therein) that the evolution of a quantum pure state in H^N as given by the Schroedinger equation can be equivalently described by a Hamiltonian dynamical system on R^{2N} with the evolution equations in the Hamiltonian form:

$$\dot{x}^i = \omega^{ij} \nabla_j \mathcal{H}, \quad (4)$$

where $x^i = q^i = (c_i^* + c_i)/\sqrt{2}$, $i = 1, 2, \dots, N$; $x^i = p^i = \sqrt{-1}(c_i^* - c_i)/\sqrt{2}$, $i = N + 1, 2, \dots, 2N$ is the vector of coordinates q_i and momenta p_i , and c_i are complex expansion coefficients of the pure state $|\psi\rangle$ in some basis. The Hamilton's function $\mathcal{H}(x)$ is given by the quantum expectation of the Hamiltonian H in the state $|\psi\rangle$: $\mathcal{H} = \langle \psi | H | \psi \rangle$, and the inverse of the symplectic form ω^{ij} is given by the imaginary part of the scalar product in H^N . In the canonical coordinates x_i the symplectic form ω_{ij} assumes the standard form

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (5)$$

where 0 and 1 are N -dimensional zero and unit matrices.

We shall use the Hamiltonian geometric formulation of a quantum dynamical system in order to derive an equation for the quantum evolution constrained on a submanifold of R^{2N} that corresponds to pure coarse-grained states.

Consider first the example of a pair of qubits. In this case the subspace of product states $|\psi^1\rangle \otimes |\psi^2\rangle$ is characterized by the following condition: A vector $|\psi\rangle$ represented in the computational basis by the expansion coefficients c_1, c_2, c_3, c_4 is separable iff $c_1 c_4 = c_2 c_3$. In an other basis the separability condition assumes different form. This condition characterizes the submanifold of R^{2N} of points corresponding to the product states. The characterization of product, i.e. nonentangled, states is also given by the condition of maximality of $P_g(\psi)$ where g is the algebra of local operators generated by $\sigma_{x,y,z}^1 \otimes 1, 1 \otimes \sigma_{x,y,z}^2$. In terms of the canonical coordinates (q, p) introduced just after Eq. (4) we obtained the purity function $P_g(q, p)$. In fact local purities $\langle \sigma_x^{1,2} \rangle^2 + \langle \sigma_y^{1,2} \rangle^2 + \langle \sigma_z^{1,2} \rangle^2$ as functions of the canonical coordinates are equal and are represented by half of the total purity function. It can be demonstrated that the function $P_g(p, q)$ is maximized precisely when the separability constraint $c_1 c_4 = c_2 c_3$ is satisfied.

Hamiltonian equations with the algebraic constraint $c_1 c_4 = c_2 c_3$ have been studied for the first time in [14] and in [15]. The formalism of quantum constrained dynamics developed in [14] is based on Dirac's approach to classical constrained Hamiltonian systems and requires the constraint to be given explicitly in terms of an even number of independent real functions. In our present case there is in general only one real constraint: $\Phi(x) \equiv P_g(x) - \max = 0$, and the approach with symplectic constraints of [14,15] cannot be applied. However, the formalism of the so-called metrical constraints, developed in [16], with an arbitrary number of real functions defining the constraint submanifold is applicable. For the benefit of the reader we shall rederive the constrained dynamical equations with only one real constraint which is of interest here.

The purity constraint

$$\Phi(x) = P_g(x) - \max = 0 \quad (6)$$

represents a single scalar condition that we want to impose on the evolution. In order to achieve this the component of the Hamiltonian vector field \dot{x} (4) normal to the constraint submanifold has to be removed, resulting in

$$\dot{x}^i = \omega^{ij} \nabla_j \mathcal{H} - \lambda g^{ij} \nabla_j \Phi, \quad (7)$$

where g^{ij} is the unit metric on R^{2N} and λ is a single Lagrange multiplier to be determined. Substitution of (7) in $\dot{\Phi}(x(t))$ results in

$$\omega^{ij} \nabla_i \Phi \nabla_j \mathcal{H} = \lambda g^{ij} \nabla_i \Phi \nabla_j \Phi, \quad (8)$$

from which

$$\lambda = \frac{\omega^{ij} \nabla_i \Phi \nabla_j \mathcal{H}}{g^{ij} \nabla_i \Phi \nabla_j \Phi}. \quad (9)$$

Substituting this λ in (7) results in the constrained dynamical equations

$$\dot{x}^i = \omega^{ij} \nabla_j \mathcal{H} - \frac{\omega^{ij} \nabla_i \Phi \nabla_j \mathcal{H}}{g^{ij} \nabla_i \Phi \nabla_j \Phi} g^{ij} \nabla_j \Phi. \quad (10)$$

We propose the reduction of the constrained equation (10) on the constrained manifold to represents dynamical equation of the coarse-grained pure states.

Observe that the numerator in (10) represent the Poisson bracket $\{\Phi, \mathcal{H}\} = \dot{\Phi}$ and the denominator is $\|\nabla\Phi\|^2$. Using the equalities

$$L_{ij} q_j = \delta_{ij} \frac{\partial \langle L \rangle}{\partial q_j}, \quad L_{ij} p_j = \delta_{ij} \frac{\partial \langle L \rangle}{\partial p_j}, \quad (11)$$

where L_{ij} are matrix elements of the operator L and in our case $\Phi(\psi) = P(\psi) - \max$, given by (1), the denominator can be further transformed as follows

$$g^{ij} \sum_{l,k} \nabla_i \langle L_l \rangle^2 \nabla_j \langle L_k \rangle^2 = 4 \sum_{l,k} \langle L_l L_k \rangle - \langle L_l \rangle \langle L_k \rangle = 4 \sum_l (\Delta L_l)^2 = 4 \Delta(\psi). \quad (12)$$

Before presenting few examples we would like to make some comments concerning the constrained equation (10).

1° In the open system picture of the distinguished system, to be discussed in the next section, and in the usual weak coupling approximation (WCA), with the distinguished observables identified with the Lindblad generators, the above equation is greatly simplified. Namely, in the WCA the Hamiltonian and the system operators L_l that couple with the environment operators satisfy

$$[H, L_l] = \lambda_l L_l \quad (13)$$

and $\dot{\Phi}$ satisfies

$$\dot{\Phi} = \dot{P}(\psi) = 2 \sum_l (\Delta L_l)^2 = 2 \Delta(\psi), \quad (14)$$

so that Eq. (10) is reduced to

$$\dot{x}^i = \omega^{ij} \nabla_j \mathcal{H} - \frac{1}{2} g^{ij} \nabla_j \Phi. \quad (15)$$

The open system interpretation of the constrained Eqs. (10) and (15) will be discussed in more details later in the next section.

2° An equivalent constrained equations, of the same form as (10) and in the special case (15), are obtained if instead of the purity constraint $P(\psi) = \max$ the constraint $\Delta(\psi) = \sum_l (\Delta L_l)^2 = \min$ is used. In particular, the special case (15), valid under the same conditions, with the use of (11) can be written in the form

$$\frac{d|\psi\rangle}{dt} = -i[H, \psi] + \sum_l (L_l^2 + \langle L_l \rangle^2 - 2\langle L_l \rangle L_l) |\psi\rangle, \quad (16)$$

to be discussed in the next section.

3° Combination of a symplectic and a gradient flows has also been obtained as a classical approximation of a quantum evolution with non-Hermitian Hamiltonian [17]. However, the evolution

equation in [17] is an approximate equation on the space of (generalized) coherent states. In our case Eq. (10) is an equation on the full Hilbert state space and the symplectic and Riemannian structures are that of the Hilbert space and not of the manifold of coherent states like in [17]. Furthermore, the classical approximation employed in [17] consist in the common replacement of the coherent state Q-symbols of combinations of the dynamical algebra generators by the same combinations of the Q-symbols of the algebra generators. This approximation is applied on the Hermitian and the non-Hermitian parts of the Hamiltonian resulting in the symplectic and gradient flows on the coherent states phase space respectively. This type of approximation is subject to the same type of objections as in the unitary case, namely the set of coherent states is not dynamically invariant unless the real and imaginary parts of the Hamiltonian are linear in the algebra generators. A special choice of the dissipative part can improve the approximations similarly as in the situation discussed in the next section.

4° Number of variables and equations in (10), and in (15), can be reduced if, prior to imposing the constraints, the normalization of $\langle \psi | \psi \rangle$ and the global phase invariance of $|\psi\rangle$ are explicitly used. The Hamiltonian Schroedinger equation (4) is then formulated on S^{2N-1}/S^1 instead of R^{2N} . The constrained equations have the same form as in (10) with the appropriate symplectic ω^{ij} and metric g^{ij} forms. An example is provided in example (b) below.

5° The geometric Hamiltonian formulation and the constrained equations can be generalized to an infinite-dimensional Hilbert space.

Before we analyze an open system physical model of the coarse-grained dynamics let us present few examples of the g-constrained systems.

3.1. Examples

(a) The first example is trivial in the sense that all pure states are g-coherent, and serves the purpose of illustrating the self-consistency of the approach. Consider a single qubit with the Hilbert space H^2 and an arbitrary Hamiltonian H . As the algebra of distinguished observables we take $g = su(2)$. The g-purity is $P_g(\psi) = \langle \sigma_x^2 \rangle + \langle \sigma_y^2 \rangle + \langle \sigma_z^2 \rangle$ and is maximal for any pure state. In this case all pure states are g-nonentangled and g-coherent.

The constraint $\Phi = P_g(\psi) - \max = 0$ in the real canonical coordinates assumes the following form

$$(p_1^2 + p_2^2 + q_1^2 + q_2^2)^2 = 2. \quad (17)$$

The gradient of the constrains is given by

$$\begin{aligned} \nabla_{q_1} \Phi &= q_1 (p_1^2 + p_2^2 + q_1^2 + q_2^2), \\ \nabla_{q_2} \Phi &= q_2 (p_1^2 + p_2^2 + q_1^2 + q_2^2), \\ \nabla_{p_1} \Phi &= p_1 (p_1^2 + p_2^2 + q_1^2 + q_2^2), \\ \nabla_{p_2} \Phi &= p_2 (p_1^2 + p_2^2 + q_1^2 + q_2^2), \end{aligned} \quad (18)$$

and the Poisson bracket $\{\Phi, \mathcal{H}\}$ is zero for arbitrary Hamiltonian $\mathcal{H} = \langle H \rangle$. Thus, the constraints are trivially satisfied, and the constraint dynamics is reduced to the linear Schroedinger part: $\dot{x}^i = \omega^{ij} \nabla_j \mathcal{H}$. This example extends to the general case of H^N with the distinguished algebra $g = u(N)$.

(b) As the second example we consider the system of two qubits with the distinguished algebra of local observables $g = su(2) \otimes su(2)$. In this case g-entanglement is the standard bipartite entanglement. g-nonentangled are the product states. Subsequent formulas are simplified if the condition $\langle \psi | \psi \rangle = 1$ and the phase invariance are explicitly used. With this the system is reduced on the projective space S^7/S^1 . Purity $P(\psi) = \langle (\sigma_x^1)^2 \rangle + \langle (\sigma_y^1)^2 \rangle +$

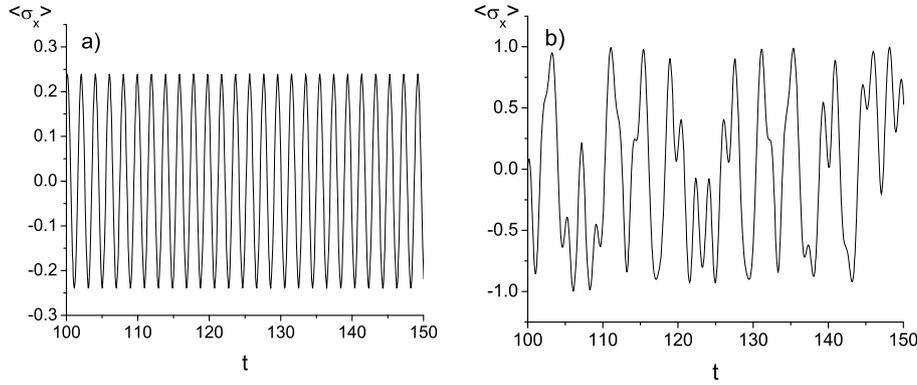


Fig. 1. Illustrated are segments of time series $q_1(t)$ for the system (24) in (a) and (25) in (b).

$\langle(\sigma_z^1)\rangle^2 + \langle(\sigma_x^2)\rangle^2 + \langle(\sigma_y^2)\rangle^2 + \langle(\sigma_z^2)\rangle^2$ in the computational basis and in canonical coordinates $\{q_1, q_2, q_3, p_1, p_2, p_3\}$ of S^7/S^1 is represented by

$$P(q, p) = 1 + 4(2\sqrt{2}p_1(p_2q_3 - p_3q_2) + 2q_3^2 + p_1^2(p_2^2 + q_2^2)) + 4(2\sqrt{2}q_1(p_2p_3 + q_2q_3) - q_1^2(p_2^2 + q_2^2) - 2p_3^2). \quad (19)$$

$P(\psi) = \max$ is equivalent to $c_1c_4 = c_2c_3$ where c_1, c_2, c_3, c_4 are coefficients of $|\psi\rangle$ in the computational basis, and the equation of this constraint is equivalent to two real equations (please see [14])

$$\sqrt{2}p_3 = p_2q_1 + p_1q_2, \quad \sqrt{2}q_3 = q_1q_2 - p_1p_2. \quad (20)$$

As for the Hamiltonian we consider two typical examples

$$H_s = \sigma_z^1 + \sigma_z^2 + \mu\sigma_z^1\sigma_z^2, \quad (21)$$

$$H_{ns} = \sigma_z^1 + \sigma_z^2 + \mu\sigma_x^1\sigma_x^2. \quad (22)$$

The reduced g -constrained dynamics in $\{q_1, q_2, p_1, p_2\}$ coordinates is equivalently described by Eqs. (10) and the metrical constraints (19), or by the symplectic constrained equations with the constraints (20). The constrained equations with constraints (20) turn out to be of a simpler form and are reproduced here. The details of the derivation have been presented in [14].

For the Hamiltonian H_s the constrained equations read

$$\begin{aligned} \dot{q}_1 &= -\frac{4\mu p_1 q_1 q_2 + 2\omega p_1 [2 + (p_2)^2 + (q_2)^2]}{2 + (p_2)^2 + (q_2)^2}, \\ \dot{q}_2 &= -\frac{4\mu p_2 q_1 q_2 - 2\omega p_2 [2 + (p_1)^2 + (q_1)^2]}{2 + (p_1)^2 + (q_1)^2}, \\ \dot{p}_1 &= \frac{2\mu q_2 [(q_1)^2 - (p_1)^2 - 2] + 2\omega q_1 [2 + (p_2)^2 + (q_2)^2]}{2 + (p_2)^2 + (q_2)^2}, \\ \dot{p}_2 &= \frac{2\mu q_1 [(q_2)^2 - (p_2)^2 - 2] + 2\omega q_2 [2 + (p_1)^2 + (q_1)^2]}{2 + (p_1)^2 + (q_1)^2}, \end{aligned} \quad (23)$$

and for the Hamiltonian H_{ns}

$$\begin{aligned} \dot{q}_1 &= \frac{2\mu p_1 [(p_2)^2 + (q_2)^2 - 2] - 2\omega p_1 [2 + (p_2)^2 + (q_2)^2]}{2 + (p_2)^2 + (q_2)^2}, \\ \dot{q}_2 &= \frac{2\mu p_2 [(p_1)^2 + (q_1)^2 - 2] - 2\omega p_2 [2 + (p_1)^2 + (q_1)^2]}{2 + (p_1)^2 + (q_1)^2}, \\ \dot{p}_1 &= \frac{-2\mu q_1 [(q_2)^2 + (p_2)^2 - 2] + 2\omega q_1 [2 + (p_2)^2 + (q_2)^2]}{2 + (p_2)^2 + (q_2)^2}, \\ \dot{p}_2 &= \frac{-2\mu q_2 [(q_1)^2 + (p_1)^2 - 2] + 2\omega q_2 [2 + (p_1)^2 + (q_1)^2]}{2 + (p_1)^2 + (q_1)^2}. \end{aligned} \quad (24)$$

There are also the equations expressing \dot{q}_3 and \dot{p}_3 in terms of q_1, q_2, p_1, p_2 , but the solutions of these are already given by the constraints.

The dynamics generated by (23) and (24) is illustrated in Figs. 1 and 2. In Fig. 1 we illustrate the time series $q_1(t)$ for single typical orbit of (23) (Fig. 1a) and of (24) (Fig. 1b). In Figs. 2a, b, c, d the Poincaré sections $q_2 = 0, p_2 > 0$ for H_{ns} (24) are shown. It should be observed that g -constrained dynamics of the symmetric Hamiltonian H_s is regular, while that of the Hamiltonian H_{ns} with no such symmetry displays typical properties of the Hamiltonian chaos. Thus, although the linear Schroedinger equation always generates an integrable Hamiltonian system, the coarse-grained quantum system evolving according to the constrained equations can display all complexities of typical chaotic dynamics.

(c) In this example we again consider a system with $g = su(2)$ distinguished algebra but with the spin $s = 1$, i.e. with H^3 Hilbert space. As for the Hamiltonian we take a nonlinear expression of $su(2)$ generators

$$H = J_z - 2J_x + \mu J_z^2. \quad (25)$$

When $\mu \neq 0$ the Schroedinger evolution with the Hamiltonian (25) does not preserve the $su(2)$ -coherent states. The set of $su(2)$ -coherent states is preserved when $\mu = 0$.

The g -constraint $\Phi(\psi) = P_{su(2)}(\psi) - 1 = \langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle - 1$ in the eigenbases of J_z and in the real canonical coordinates of R^6 assumes the form

$$\begin{aligned} 4P_{su(2)}(q, p) &= -4 + p_1^4 + p_3^4 - 2p_3^2 q_1^2 + q_1^4 + 8p_2 p_3 q_1 q_2 \\ &\quad + 2p_3^2 q_2^2 + 2q_1^2 q_2^2 + 2p_2^2 (p_3^2 + (q_1 - q_3)^2) \\ &\quad + 4q_1 q_2^2 q_3 + 2(p_3^2 - q_1^2 + q_2^2) q_3^2 \\ &\quad + q_3^4 + 4p_1 (p_2^2 p_3 - p_3 q_2^2 + 2p_2 q_2 q_3) \\ &\quad + 2p_1^2 (p_2^2 - p_3^2 + q_1^2 + q_2^2 - q_3^2). \end{aligned} \quad (26)$$

The Poisson bracket of the constraint and the Hamiltonian, that is needed for the constraint equations (10), reads

$$\begin{aligned} \omega^{ij} \nabla_i \Phi \nabla_j \mathcal{H} &= 2\mu [(p_3 q_1 + p_1 q_3) (q_2^2 - p_2^2) \\ &\quad + 2p_2 q_2 (p_1 p_3 - q_1 q_3)]. \end{aligned} \quad (27)$$

We see that, when $\mu = 0$, the Poisson bracket (27) is zero and the g -constrained equations reduce to the Schroedinger equation. The squared norm of the gradient $\nabla\psi$ is given by somewhat complicated function of the canonical coordinates (q, p) and will not be reproduced here. We illustrate the form of simplified constrained equations (15) by the formula for \dot{p}_1

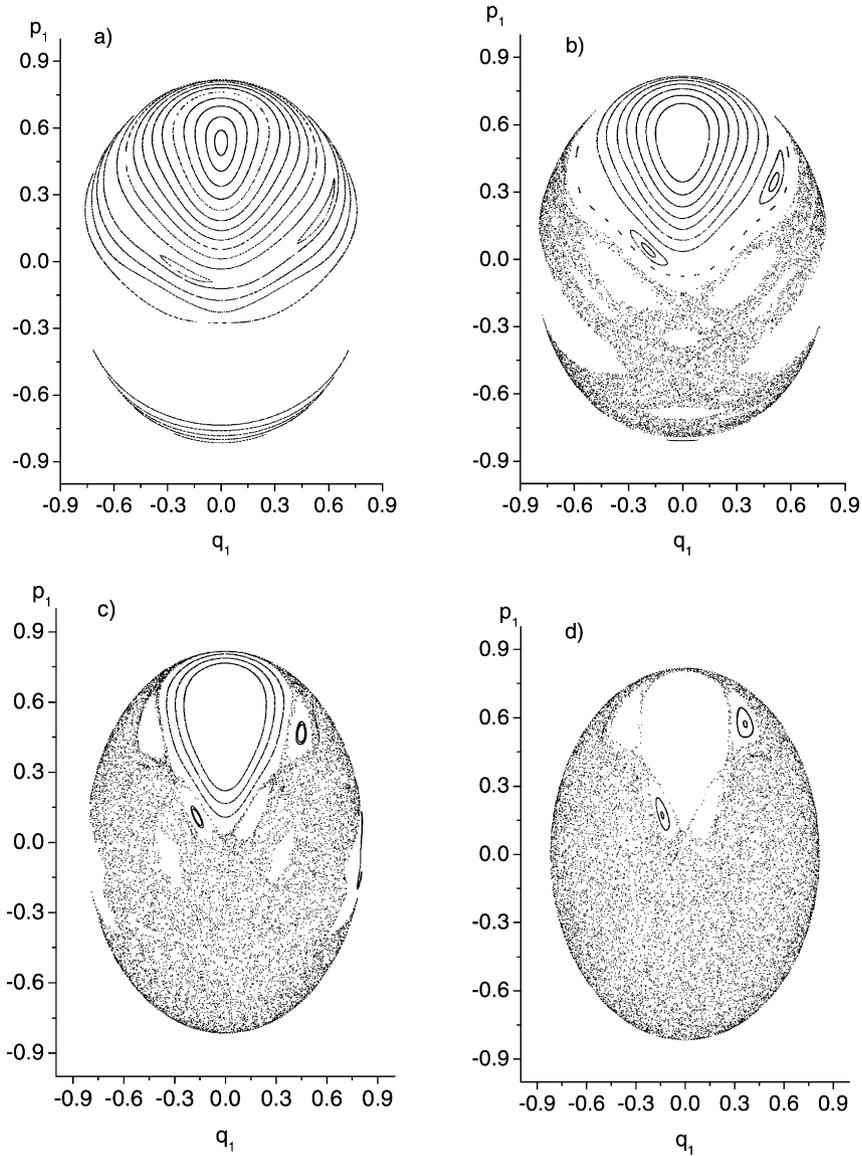


Fig. 2. Poincaré sections $q_2 = 0$, $p_2 > 0$ and $\mathcal{H} = 1.5$ for the system (25). The parameters are (a) $\mu = 1.1$, (b) $\mu = 1.3$, (c) $\mu = 1.5$ and (d) $\mu = 1.7$.

$$\begin{aligned} \dot{p}_1 = & -\sqrt{2}[(1 + \mu)q_1 - q_2] + p_1(p_2^2 - p_3^2 + q_1^2 + q_2^2 - q_3^2) \\ & - 2p_2q_2q_3 + p_3q_2^2 - p_2^2p_3 - p_1^3, \end{aligned} \quad (28)$$

where the first line is the Hamiltonian term and the second line is from the gradient of the constraint.

We shall come back to this example in the next section.

4. Open system model

The coarse-grained system specified by the distinguished variables can be considered as an open system with the larger closed system characterized by the full algebra $u(N)$. In the case when the Hilbert space can be split into the tensor product with one component corresponding to the distinguished reduced system the standard open system model of decoherence applies. This theory singles out a distinguished set of states, the pointer or the robust states by one of the following two equivalent properties: (a) as pure states of the reduced open system that remain pure under evolution, or (b) as states in which the reduced open system is not and does not get entangled with the environment during the full system evolution. Reduced states of the general coarse-grained system, with the evolution equation (10) satisfy the same proper-

ties as the robust states of an open system if the interaction of the open system and the environment is mediated by all of the distinguished observables. It has been demonstrated that the robust states in this case coincide with the g -coherent states [10] in the weak coupling limit. Thus, the pointer states are identified with reduction of the g -nonentangled or g -coherent states. One can conclude that the coarse-graining physically occurs due to decoherence of the distinguished system induced by specific interaction with the environment. The specific interaction in fact models generalized simultaneous measurement of all distinguished observables.

We would like to identify the distinguished observables with observables that are simultaneously measured on the open system. In the weak coupling limit (WCL) the Born–Markov and rotating wave approximations result in the Lindblad master equation of the open system dynamics [18]

$$\dot{\rho}(t) = -i[H, \rho] + \frac{1}{2} \sum_l ([L_l \rho, L_l^\dagger] + [L_l, \rho L_l^\dagger]), \quad (29)$$

where H is the open system Hamiltonian and L_l are the so-called Lindblad operators. L_l are the open system operators that are coupled with the operators representing observables of an environ-

ment. If Eq. (29) corresponds to the measurement of certain observables than L_l are the Hermitian operators that represent the measured observables. In our open system model of the coarse-graining we shall suppose that the distinguished algebra is precisely the algebra formed by the Hermitian Lindblad operators in (29).

As pointed out, the pointer or robust states in the open system model of decoherence are the pure states of the open system that remain pure in the course of evolution. It has been suggested [9] that an approximate evolution equation of the pure robust states can be obtained by minimizing the Hilbert–Schmidt (HS) distance from $\rho(t)$ to the set of pure states. In the case $\rho(t)$ is given by the Lindblad equation with Hermitian Lindblad operators the equation of the HS closest pure state is [9]

$$\frac{d|\psi\rangle}{dt} = -i[H, \psi] + \sum_l (L_l^2 + \langle L_l \rangle^2 - 2\langle L_l \rangle L_l) |\psi\rangle. \quad (30)$$

This is precisely our constrained equation (15) when g -entanglement measure $P(\psi) = \sum_l \langle L_l \rangle^2$ is replaced by the equivalent measure $\Delta(\psi) = \sum_l \langle \Delta L_l \rangle^2$ in the WCL with the distinguished observables being the Lindblad generators.

Eq. (30) (or (15)) represent the deterministic part of the stochastic Schroedinger equation derived in the quantum state diffusion theory [19] for arbitrary random pure state, which we reproduce here because it will be used for numerical computations in the next example. The Ito form of the QSD equation corresponding to (29) reads

$$\begin{aligned} |d\psi\rangle &= -iH|\psi\rangle dt \\ &+ \left[\sum_l 2\langle L_l^\dagger \rangle L_l - L_l^\dagger L_l - \langle L_l^\dagger \rangle \langle L_l \rangle \right] |\psi(t)\rangle dt \\ &+ \sum_l (L_l - \langle L_l \rangle) |\psi(t)\rangle dW_l \end{aligned} \quad (31)$$

where dW_l are independent increments (indexed by l) of complex Wiener c -number processes $W_l(t)$ satisfying

$$\begin{aligned} E[dW_l] &= E[dW_l dW_{l'}] = 0, \\ dW_l d\bar{W}_{l'} &= \delta_{l,l'} dt, \\ l &= 1, 2, \dots, m, \end{aligned} \quad (32)$$

where $E[\cdot]$ denotes the expectation with respect to the probability distribution given by the (m -dimensional) process W , and \bar{W}_l is the complex conjugate of W_l .

The random vector $|\psi(t)\rangle$ which satisfies (31) is related to the density matrix $\rho(t)$ which satisfies the Lindblad equation (29) by averaging over the realizations of the process (31)

$$\rho(t) = E[|\psi(t)\rangle \langle \psi(t)|]. \quad (33)$$

Let us stress that the HS approximate robust state Eq. (30) assumes validity of WCL and coincides with the constrained equation (15) simplified from (10) under this assumption. On the other hand, the general constrained evolution given by (10) is valid, in the sense that it preserves $P(\psi)$ and $\Delta(\psi)$, with no assumption about special evolution of $\dot{P}(\psi)$ obtained under the WCL.

4.1. An example

In the case of an open system that satisfies the conditions for the weak coupling approximation (13) the dynamics of the system is described well by the Lindblad equation and the pointer states are exactly the g -coherent states [10]. Using particular examples, it has been demonstrated [9] that Eq. (30), which coincides with the

simplified form of the constrained equation (15), describes well the evolution of the pointer, i.e. g -coherent states. We shall analyze here an example that does not satisfy the condition (13) of the WCA.

Let us consider, as an example, the two mode Bose–Hubbard model (see for example [20]), given by the following Hamiltonian with $\hbar = 1$,

$$\begin{aligned} H &= \epsilon_1 a_1^\dagger a_1 + \epsilon_2 a_2^\dagger a_2 + \alpha (a_1^\dagger a_2 + a_2^\dagger a_1) \\ &+ \mu (a_1^{\dagger 2} a_1^2 + a_2^{\dagger 2} a_2^2), \end{aligned} \quad (34)$$

where $a_i, a_i^\dagger, i = 1, 2$ are bosonic annihilation and creation operators of the two modes. The dynamics preserves total particle number $N = a_1^\dagger a_1 + a_2^\dagger a_2$. Introducing operators

$$q_j = (a_j^\dagger + a_j)/\sqrt{2}, \quad p_j = i(a_j^\dagger - a_j)/\sqrt{2}, \quad j = 1, 2, \quad (35)$$

or the operators

$$\begin{aligned} J_x &= \frac{1}{2} (a_1^\dagger a_2 + a_2^\dagger a_1), \\ J_y &= \frac{i}{2} (a_1^\dagger a_2 - a_2^\dagger a_1), \\ J_z &= \frac{1}{2} (a_2^\dagger a_2 - a_1^\dagger a_1), \end{aligned} \quad (36)$$

the Hamiltonian assumes the following forms respectively in coordinates (35)

$$\begin{aligned} H &= \epsilon_1 (p_1^2 + q_1^2)/2 + \mu (p_1^2 + q_1^2)^2/4 \\ &+ \epsilon_2 (p_2^2 + q_2^2)/2 + \mu (p_2^2 + q_2^2)^2/4 \\ &+ \alpha (p_1 p_2 + q_1 q_2) \end{aligned} \quad (37)$$

and in terms of (36)

$$H = -2\alpha J_x + 2(\epsilon_2 - \epsilon_1) J_z + \mu J_z^2. \quad (38)$$

In what follows we shall always set $\alpha = 1, \epsilon_2 - \epsilon_1 = 1$.

The preserved total number of particles is related to J^2 by $J^2 = N/2(N/2 + 1)$. Thus, the effective Hilbert space of the system carries an irreducible representation of $SU(2)$, which is the dynamical group of the model. This suggest that the $SU(2)$ coherent states have a special status in the model (34). This however is not true, because the nonlinear term μJ_z^2 makes the set of $SU(2)$ coherent states noninvariant.

We would like to analyze system (35) interacting with an environment via operators (36) or (37). The Hamiltonian (34) and operators (35) or (36) used as the Lindblad operators do not quite satisfy condition (13) for the WCA. Nevertheless, we shall suppose that the open system evolution is described by the Lindblad equation with Lindblad operators given either by (35) or by (36). Notice that the result $\Delta_g(\psi) \rightarrow \min$ obtained in [10] does not apply necessarily since the system does not satisfy the WCA condition. We shall demonstrate that the asymptotic states of the Lindblad equation of an open BH system interacting with an environment via the Lindblad operators L_l satisfy the corresponding constraint condition $\Delta_g(\psi) = \min$ almost exactly.

Let us first consider the open system evolution in terms of random pure states $|\psi(t)\rangle$ and the QSD equation (31). We first choose $L_1 = J_x, L_2 = J_y, L_3 = J_z$ and compute $\Delta_{su(2)}(\psi(t))$ and $P_{su(2)}(\psi(t))$ from an initial state equal to the number state given by $(a_1^\dagger)^2 (a_2^\dagger)^2 |0, 0\rangle$. The results are shown in Fig. 3. The state quickly converges to those with a minimal $\Delta_{su(2)}(\psi(t))$, i.e. to the $su(2)$ -coherent states. On the other hand, $\Delta_{H_4} = \Delta^2 p_1 + \Delta^2 p_2 + \Delta^2 q_1 + \Delta^2 q_2$ remains constant and large. $su(2)$ -purity is less than maximal at the beginning but quickly converges to the maximal

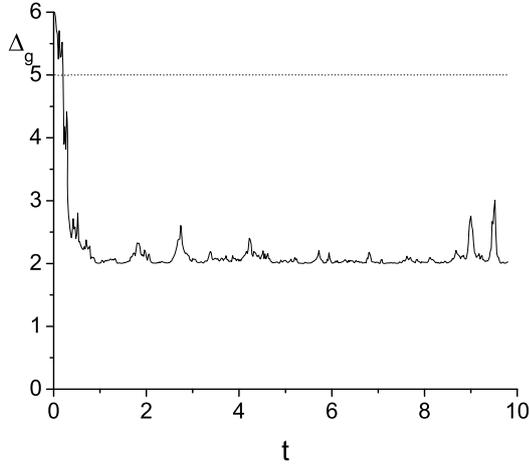


Fig. 3. Illustrates the invariant fluctuation $\Delta_g(\psi)$ (2) in the cases $g = su(2)$ (full line) and $g = H_4$ (dotted line) for the QSD evolution with the Hamiltonian (35) and Lindblads $L_1 = J_x, L_2 = J_y, L_3 = J_z$. The initial state is the number state $|2, 2\rangle = (a_1^\dagger)^2 (a_2^\dagger)^2 |0, 0\rangle$. The parameters are $\mu = 0.1, \alpha = 1, \epsilon = 0, \gamma = 0.9$.

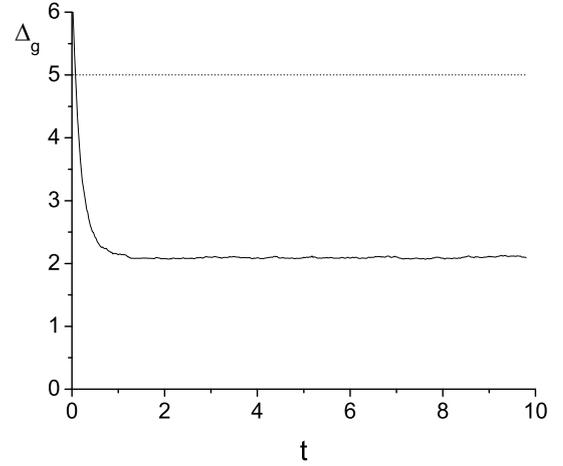


Fig. 5. Illustrates the invariant fluctuation $\Delta_g(\rho)$ (2) in the cases $g = su(2)$ (full line) and $g = H_4$ (dotted line) for the evolution by the Lindblad equation with the Hamiltonian (35) and Lindblads $L_1 = J_x, L_2 = J_y, L_3 = J_z$. The initial state is the number state $|2, 2\rangle = (a_1^\dagger)^2 (a_2^\dagger)^2 |0, 0\rangle$. The parameters are $\mu = 0.1, \alpha = 1, \epsilon = 0, \gamma = 0.9$.

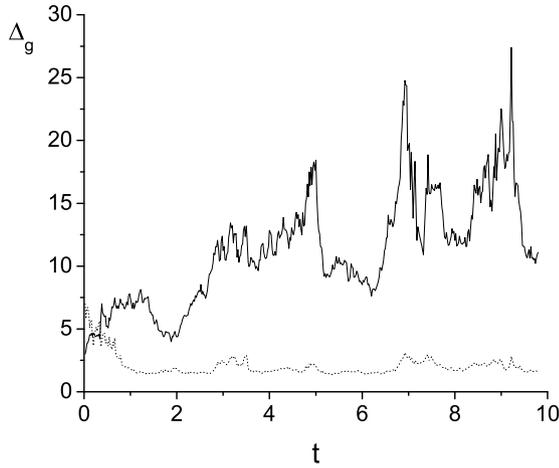


Fig. 4. Illustrates the invariant fluctuation $\Delta_g(\psi)$ (2) in the cases $g = su(2)$ (full line) and $g = H_4$ (dotted line) for the QSD evolution with the Hamiltonian (35) and Lindblads $L_1 = q_1, L_2 = q_2, L_3 = p_1, L_4 = p_2$. The initial states is an $su(2)$ coherent state. The parameters are $\mu = 0.1, \alpha = 1, \epsilon = 0, \gamma = 0.9$.

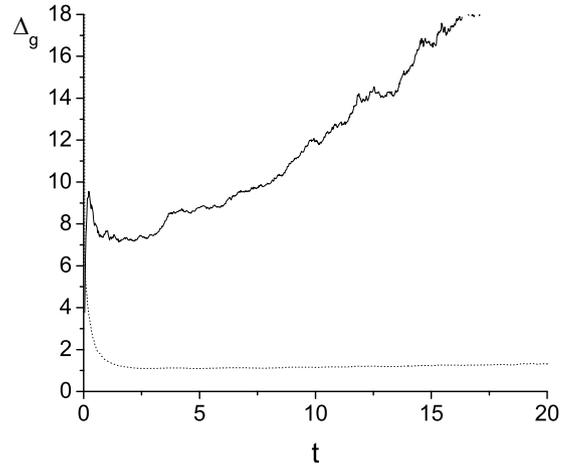


Fig. 6. Illustrates the invariant fluctuation $\Delta_g(\rho)$ (2) in the cases $g = su(2)$ (full line) and $g = H_4$ (dotted line) for the evolution by the Lindblad equation with the Hamiltonian (35) and Lindblads $L_1 = q_1, L_2 = q_2, L_3 = p_1, L_4 = p_2$. The initial states is an $su(2)$ coherent state. The parameters are $\mu = 0.1, \alpha = 1, \epsilon = 0, \gamma = 0.9$.

value. Although the state $|\psi(t)\rangle$ is always a pure state of the Hilbert space, its $su(2)$ -purity is maximal only when $|\psi\rangle$ is an $su(2)$ -coherent, i.e. an $su(2)$ -nontangled state. Analogously, assuming the Lindblad operators to be $L_1 = q_1, L_2 = q_2, L_3 = p_1, L_4 = p_2$ implies an evolution such that $\Delta_{su(2)}$ is far away from its minimum, but Δ_{H_4} converges to values close to the minimal and remains such for almost all times (please see Fig. 4).

The equivalent conclusions are obtained when the evolution is described in terms of $\rho(t) = E(|\psi(t)\rangle\langle\psi(t)|)$, i.e. by the Lindblad equation. This is illustrated in Figs. 5 and 6, with Δ_g for $g = su(2)$; $g = H_4$ and $L_1 = J_x, L_2 = J_y, L_3 = J_z$ with the number initial state (Fig. 5), and $L_1 = q_1, L_2 = q_2, L_3 = p_1, L_4 = p_2$ with an $su(2)$ coherent initial state in Fig. 6. Only two hundred QSD sample paths are used to compute $\rho(t)$ and then the corresponding $\Delta_g(\rho)$.

Furthermore, consider evolution from an $su(2)$ coherent initial state with $J = 1$ and with the Lindblads being $J_{x,y,z}$. If the WCL approximation is valid the Lindblad equation can be applied, the asymptotic states satisfy $\Delta_{su(2)} \approx \min$, and the simplified constrained equation (15) also applies. Indeed, the Lindblad evolution is well approximated by the simple form of the constrained equation (15), as is illustrated in Fig. 7a, b.

In summary, we see that an arbitrary initial state evolves very quickly into a mixture of g -coherent states, and then each of these evolves in a way that is well approximated by the nonlinear equation (10) or in the WCL by (15).

Let us stress that the coarse-graining by distinguished observables, discussed here, is specially appropriate in a description of macroscopic features of a quantum system, with the distinguished observables identified with the macroscopic quantities. In this case the Hilbert space of the quantum system does not have the bipartite tensor product structure, with one party being characterized by the macroscopic observables and the other party being the environment. The usual models of decoherence [21] with the initial separation $|\psi\rangle = |\psi_s\rangle \otimes |\psi_{env}\rangle$ do not apply. However, the picture of coarse-graining by distinguished observables with the corresponding nonlinear evolution equations can be applied.

5. Summary

We have analyzed the coarse-graining introduced by a chosen set of distinguished observables. The algebra of distinguished

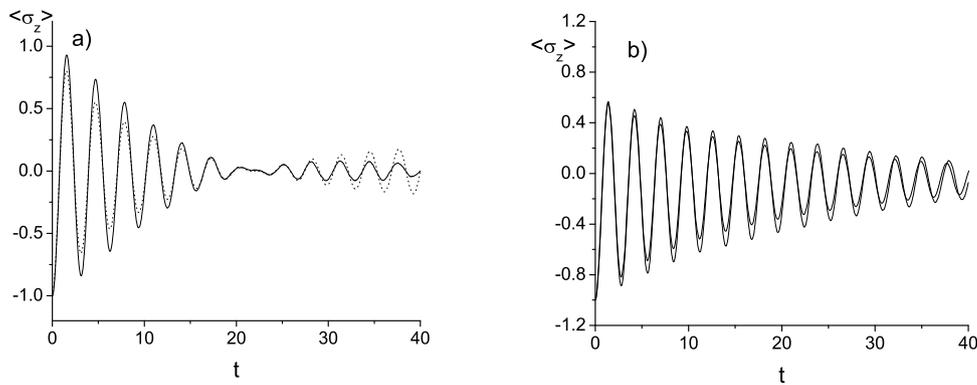


Fig. 7. Evolution of $\text{Tr}[\rho\sigma_z]$ according to the Lindblad equation (30) (full line) and of $\langle\sigma_z\rangle$ according to the simplified constrained equation (15) (dotted line) with the Hamiltonian (35) and Lindblads $L_1 = J_x$, $L_2 = J_y$, $L_3 = J_z$. The initial states is $su(2)$ coherent state $|j, j_z\rangle = |1, -1\rangle$. The parameters are $\mu = 0.1$, $\alpha = 1$, $\gamma = 0.2$ and (a) $\epsilon = 0$ and (b) $\epsilon = 1$.

observables defines the corresponding generalized nonentangled states which coincide by definition with the generalized coherent states. The states obtained by reduction on the distinguished observables of the g -nonentangled states are pure. We have propose to consider the coarse-grained evolution as constrained Schroedinger dynamics, where the constraint guaranties that the state is always pure g -nonentangled. In order to formulate the constrained evolution equations we used Hamiltonian formulation with the metrical form of the constrained dynamics as developed in [16].

Further on we discussed an open system model of the coarse-graining and of the reduced constrained equation. In the weak coupling limit the open system dynamics is given by the Lindblad master equation. In this limit, and if the Lindblad operators are taken to represent the distinguished observables then the constrained equations for the g -coherent states developed here coincide with previously suggested [9] evolution equation for the pointer states of the open system.

Our simplified constrained evolution equation (15) for the g -nonentangled states coincides with the deterministic part of the Ito stochastic Schroedinger equation developed in the quantum state diffusion theory (QSD) of open system dynamics. The stochastic Schroedinger equation describes dynamics of any random pure state. Our constrained equations describe dynamics of deterministic pure states which are in the subset of all pure states that remain pure during the evolution. From a formal point of view, it would be interesting to derive the QSD stochastic equations using the formalism of constraints, where the constraint would be given by random variables representing the obtained results of measurements with Gaussian distribution.

The formalism of coarse-graining as the constrained evolution can be used to study coarse-grained macroscopic observables of a quantum system and derive their classical behavior.

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