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Excitable systems with internal and coupling delays

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Abstract

Two delayed coupled excitable systems with internal delays are studied. For different parametric values each of the isolated units displays excitable, bi-stable or oscillatory dynamics. Bifurcational relations among coupling time-lag and coupling constant for different values of the internal time-lags are obtained. Possible types of synchronization between the units in typical dynamical regimes are studied.

1. Introduction

Possibility to model oscillatory behavior of complex dynamical systems in Physics and Biology using delayed differential equations (DDE‘s) is well established (a sample of references is [1–7]). Often, in models with more than one variable, several and independent time-lags are justified. However, questions related to stability and bifurcations for systems of DDE‘s with more than one fixed and discrete time-lags are comparatively more difficult to analyze than the same questions for systems with one time-lag [8–11]. Furthermore, complex dynamical units, like for example neurons, appear as constitutive elements of more complex systems, and must transmit excitations among them. The transmission of excitations is certainly not instantaneous, and the representation by non-local and instantaneous interactions should be considered only as a very crude approximation. Thus, it is of some interest to study the collective behavior of systems composed of several units which are coupled by time-delayed interaction, and such that each unit if decoupled from the system would have an attractor determined by an intrinsic time-lag.

In particular, we shall be interested in the interplay of oscillations, produced by delayed coupling and by internal time-lags, in a collection of the so called excitable systems. Excitability is a common property of many complex systems [12]. Although there is no precise definition [13] the intuitive meaning is clear: A small perturbation from the single stable stationary state can result in a large and long lasting excursion away from the stationary state before the system is returned back asymptotically to equilibrium. For example, the excitability is found as the typical behavior of isolated or coupled neurons. Transition from excitable to oscillatory dynamics in common neuronal models, i.e the bifurcation of the stationary point into a stable periodic orbit, is usually achieved by varying an external parameter. On the other

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hand, it is known [14] that the same effect, can be produced by varying an internal time-lag. However, the bifurcation sequence which leads from the single fixed point attractor to the single limit cycle attractor is more complicated in the case of the transition caused by the internal time-delay. In this case, there are values of internal time-lags when the isolated single unit has two coexisting attractors, the stable fixed point and the stable limit cycle, i.e. the system is bi-stable. For such values of the internal parameters and time-lags the delayed coupling produces interesting effects, which we shall analyze in detail.

As the model of the excitable dynamics we shall use the FitzHugh–Nagumo system in the following form [15]:

$$\begin{aligned}\dot{x} &= -x^3 + (a+1)x^2 - ax - y + I_a \equiv X(x,y)\\ \dot{y} &= bx - \gamma y \equiv Y(x,y),\end{aligned} \tag{1}$$

where a, b, γ are positive parameters and I_a is an external variable that is dynamically independent of x and y. When $I_a = 0$ and in an appropriate range of a, b and γ the system displays typical excitable behavior (see the next section). The parameter I_a is commonly used to induce the Hopf bifurcation, which replaces the stable fixed point by a stable limit cycle, and thus turns the excitable into the oscillatory behavior. Let us stress that in this paper I_a is always equal to zero, so that the oscillatory behavior can be induced only either by the interaction among neurons or by the internal time delays.

A pair of delayed coupled system (1) is given by:

$$\begin{aligned}\dot{x}_1(t) &= X(x_1(t), y_1(t)) + cf(x_1(t), x_2(t-\tau))\\ \dot{y}_1(t) &= Y(x_1(t), y_1(t)),\\ \dot{x}_2(t) &= X(x_2(t), y_2(t)) + cf(x_2(t), x_1(t-\tau))\\ \dot{y}_2(t) &= Y(x_2(t), y_2(t)),\end{aligned} \tag{2}$$

where the field $X(x,y)$, $Y(x,y)$ describes the single isolated unit, and the function $f(x_i(t), x_j(t-\tau))$ describes the time-delayed coupling between the two excitable units with the coupling constant c. The coupled system (2), for such a, b, γ that the single units are excitable, displays transition from excitable to oscillatory behavior as the coupling strength and time-lag τ are varied, but the transition could be through an intermediate regime when the system (2) is bi-stable, with the coexisting stable fixed point and the limit cycle. This sequence of bifurcations was studied in Refs. [16,17], and the analyzes was extended to the case of a chain of N units in [18]. These papers provide an extended list of references to the relevant related results, which we shall not repeat hear.

Qualitatively similar transition from excitability to oscillatory dynamics occurs also in the system (1) if we assume that there is a possibility of an internal time-delay between the variables x and y pertaining to the single unit. This phenomenon was studied in [14]. Such generalized single unit is described by the following equations:

$$\begin{aligned}\dot{x}(t) &= -x^3(t) + (a+1)x^2(t) - ax(t)\\ &\quad -[a_1 y(t) + (1-a_1)y(t-\tau_1)],\\ \dot{y}(t) &= b[a_2 x(t) + (1-a_2)x(t-\tau_2)] - \gamma y(t),\end{aligned} \tag{3}$$

with the same limits on the parameters as in (1) with $I_a = 0$.

In this paper we shall study the system (2) but with the fields $X(x,y)$, $Y(x,y)$ given by (3). In what follows the time-lags τ_1 and τ_2 will be called internal and τ will be called the coupling time-lag. Such system could be interpreted as a collection of delayed coupled complex excitable systems where the internal delays could produce the dynamics of the single unit which is akin to that of coupled simple excitable systems.

The paper is organized as follows. In the next section we analyze the bifurcations of the fixed point due to varying coupling delay τ and constant c, for fixed internal parameters a, b and γ that correspond to the excitable behavior of (1), and for qualitatively different values of the internal delays τ_1 and τ_2. Bifurcation curves in (c, τ) plane for values of τ_1 and τ_2 that correspond to qualitatively different dynamics of (3) are illustrated, and serve as a guide to the numerical analyzes, presented in Section 3, of the global dynamics of the full system. We shall be particularly interested in the types of synchronization that can occur for different values of c and τ in the domains of different dynamics of the isolated single unit. The paper ends with a summary and discussion given in Section 4.

2. Local bifurcations of the fixed point

In this section we study bifurcations of the zero stationary point $(0,0,0,0)$ of two diffusively coupled FitzHugh–Nagumo excitable systems with internal delay's and the delay in the diffusive coupling given by the following general form:

$$\begin{aligned}\dot{x}_1 &= -x_1^3 + (a+1)x_1^2 - ax_1 - [a_1y_1 + (1-a_1)y_1(t-\tau_1)] + c(x_1 - x_2(t-\tau)),\\ \dot{y}_1 &= b[a_2x_1 + (1-a_2)x_1(t-\tau_2)] - \gamma y_1,\\ \dot{x}_2 &= -x_2^3 + (a+1)x_2^2 - ax_2 - [a_1y_2 + (1-a_1)y_2(t-\tau_1)] + c(x_2 - x_1(t-\tau)),\\ \dot{y}_2 &= b[a_2x_2 + (1-a_2)x_2(t-\tau_2)] - \gamma y_2.\end{aligned} \tag{4}$$

In the general form (4) we allow for the possibility that the variables x_i, y_i depend on the instantaneous as well as the delayed values of y_i, x_i, respectively.

Linearization of the system (4) and substitution $x_i(t) = A_i e^{\lambda t}$, $y_i(t) = B_i e^{\lambda t}$, $x_i(t-\tau) = A_i e^{\lambda(t-\tau)}$, $x_i(t-\tau_2) = A_i e^{\lambda(t-\tau_2)}$, $y_i(t-\tau_1) = B_i e^{\lambda(t-\tau_1)}$ results in a system of equations for the constants A_i and B_i. This system has a nontrivial solution if the following is satisfied:

$$\Delta_1(\lambda)\cdot\Delta_2(\lambda) = 0, \tag{5}$$

where

$$\begin{aligned}\Delta_1(\lambda) &= \lambda^2 + (a+\gamma-c)\lambda + (a-c)\gamma + a_1a_2b + a_1(1-a_2)be^{-\lambda\tau_2} + (1-a_1)a_2be^{-\lambda\tau_1}\\ &\quad + (1-a_1)(1-a_2)be^{-\lambda(\tau_1+\tau_2)} - (\lambda+\gamma)ce^{-\lambda\tau},\end{aligned} \tag{6}$$

$$\begin{aligned}\Delta_2(\lambda) &= \lambda^2 + (a+\gamma-c)\lambda + (a-c)\gamma + a_1a_2b + a_1(1-a_2)be^{-\lambda\tau_2} + (1-a_1)a_2be^{-\lambda\tau_1}\\ &\quad + (1-a_1)(1-a_2)be^{-\lambda(\tau_1+\tau_2)} + (\lambda+\gamma)ce^{-\lambda\tau}.\end{aligned} \tag{7}$$

Eq. (5) is the characteristic equation of the system (4). Infinite dimensionality of the system is reflected in the transcendental character of (5). However, the spectrum of the linearization of Eq. (4) is discrete and can be divided into infinite dimensional hyperbolic and finite dimensional non-hyperbolic parts [19]. As in the finite dimensional case, the stability of the stationary solution $(x_1, y_1, x_2, y_2) = (0, 0, 0, 0)$ is typically, i.e. in the hyperbolic case, determined by the signs of the roots of (5). Exceptional roots equal to zero or with zero real part, correspond to the finite dimensional center manifold where the qualitative features of the dynamics, such as local stability, depend on the nonlinear terms.

The general system (4) apparently has three independent time-lags. The two internal time-lags τ_1 and τ_2 appear independently in the characteristic equation only if a_1 and a_2 are different from zero and unity. However possible types of dynamics in this most general case are qualitatively similar to the situation when there is effectively only one internal time lag, which are the cases that we shall study further. We shall analyze the roots of (5) in the following two special cases: 1^0 pure delays $a_1 = a_2 = 0$, and 2^0 one internal delay $a_2 = 1$, $a_1 \neq 0; 1$. In both of these cases Eq. (4) have two independent time-lags: one for the internal delay and one for the coupling time delay.

The case of pure delays

In this case the system of DDE (4) is reduced to:

$$\begin{aligned}\dot{x}_1 &= -x_1^3 + (a+1)x_1^2 - ax_1 - y_1(t-\tau_1) + c(x_1 - x_2(t-\tau)),\\ \dot{y}_1 &= bx_1(t-\tau_2) - \gamma y_1,\\ \dot{x}_2 &= -x_2^3 + (a+1)x_2^2 - ax_2 - y_2(t-\tau_1) + c(x_2 - x_1(t-\tau)),\\ \dot{y}_2 &= bx_2(t-\tau_2) - \gamma y_2,\end{aligned} \tag{8}$$

and the two factors of the characteristic equation (6) and (7) become now:

$$\Delta_1(\lambda) = \lambda^2 + (a+\gamma-c)\lambda + (a-c)\gamma + be^{-\lambda(\tau_1+\tau_2)} - (\lambda+\gamma)ce^{-\lambda\tau} \tag{9}$$

$$\Delta_2(\lambda) = \lambda^2 + (a+\gamma-c)\lambda + (a-c)\gamma + be^{-\lambda(\tau_1+\tau_2)} + (\lambda+\gamma)ce^{-\lambda\tau} \tag{10}$$

As we see the internal delays appear only in the combination $\tau_1 + \tau_2$, which will be denoted by $\tau_1 + \tau_2 = \tau_{12}$.

We shell seek for the relations $\tau = f(a, b, \gamma, a_1, a_2, \tau_{12}, c)$ such that some solutions of the characteristic equation given by (9) and (10) are pure imaginary $\lambda = \pm\imath\omega$ with real and positive ω. Under some additional conditions [19], these relations correspond to the Hopf bifurcation.

Substituting $\lambda = \imath\omega$, where ω is real and positive, into first factor, multiplying with $(-\imath\omega + \gamma)$ and separating real and imaginary part gives

$$\begin{aligned}c(\gamma^2+\omega^2)\cos(\omega\tau) &= (a-c)(\gamma^2+\omega^2) + F\\ c(\gamma^2+\omega^2)\sin(\omega\tau) &= -\omega(\gamma^2+\omega^2) + G,\end{aligned} \tag{11}$$

where

$$F = b(\gamma\cos(\omega\tau_{12}) - \omega\sin(\omega\tau_{12})),$$
$$G = b(\omega\cos(\omega\tau_{12}) + \gamma\sin(\omega\tau_{12})).$$

The same manipulations applied with the second factor result in

$$\begin{aligned} c(\gamma^2+\omega^2)\cos(\omega\tau) &= -(a-c)(\gamma^2+\omega^2) - F,\\ c(\gamma^2+\omega^2)\sin(\omega\tau) &= +\omega(\gamma^2+\omega^2) - G, \end{aligned} \tag{12}$$

with F and G given as before.

Squaring and adding the previous two pairs of Eqs. (11) or (12) results in the same parametric equation for the coupling strength

$$c = \frac{(a^2+\omega^2)(\gamma^2+\omega^2) - 2G\omega + 2aF + b^2}{2a(\gamma^2+\omega^2) + 2F}. \tag{13}$$

The corresponding critical time lag follows by dividing the pair of Eq. (11) for the first factor:

$$\tau^1_{c,k} = \frac{1}{\omega}\left[\arctan\left(\frac{-\omega(\gamma^2+\omega^2)+G}{(a-c)(\gamma^2+\omega^2)+F}\right) + k\pi\right], \tag{14}$$

and the critical time lag from the second factor is obtained by dividing the pair of Eq. (12):

$$\tau^2_{c,k} = \frac{1}{\omega}\left[\arctan\left(\frac{\omega(\gamma^2+\omega^2)-G}{-(a-c)(\gamma^2+\omega^2)-F}\right) + k\pi\right]. \tag{15}$$

The bifurcation curves are illustrated in Fig. 1a–c. The figures correspond to three typical situations that occur in a single isolated unit for different values of $\tau_1 + \tau_2$. The internal delays are chosen such that in the case α (Fig. 1a) the internal delays are small and the isolated unit has the stable fixed point as the only attractor; in the case β for medium internal time-lags (Fig. 1b) the isolated unit has the stable fixed point and the stable limit cycle as the only two attractor and in the case γ (Fig. 1c) for large τ_{12} the isolated unit has the stable limit cycle as the only attractor.

One internal time lag

The general equation (4) in the case of one internal time-lag with $a_2 = 1$, $a_1 \neq 0;1$ reduces to

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + (a+1)x_1^2 - ax_1 - [a_1y_1 + (1-a_1)y_1(t-\tau_1)] + c(x_1 - x_2(t-\tau)),\\ \dot{y}_1 &= bx_1 - \gamma y_1\\ \dot{x}_2 &= -x_2^3 + (a+1)x_2^2 - ax_2 - [a_1y_2 + (1-a_1)y_2(t-\tau_1)] + c(x_2 - x_1(t-\tau)),\\ \dot{y}_2 &= bx_2 - \gamma y_2, \end{aligned} \tag{16}$$

with corresponding factors of the characteristic equation:

$$\Delta_1(\lambda) = \lambda^2 + (a+\gamma-c)\lambda + (a-c)\gamma + a_1b + (1-a_1)be^{-\lambda\tau_1} - (\lambda+\gamma)ce^{-\lambda\tau}, \tag{17}$$
$$\Delta_2(\lambda) = \lambda^2 + (a+\gamma-c)\lambda + (a-c)\gamma + a_1b + (1-a_1)be^{-\lambda\tau_1} + (\lambda+\gamma)ce^{-\lambda\tau}. \tag{18}$$

Pure imaginary solutions of (17) and (18) are analyzed is the same way as in the first case. The real and imaginary parts for the first factor (17) are:

$$\begin{aligned} c(\gamma^2+\omega^2)\cos(\omega\tau) &= (a-c)(\gamma^2+\omega^2) + ba_1\gamma + FF,\\ c(\gamma^2+\omega^2)\sin(\omega\tau) &= -\omega(\gamma^2+\omega^2) + ba_1\omega + GG, \end{aligned} \tag{19}$$

and for the second factor

$$\begin{aligned} c(\gamma^2+\omega^2)\cos(\omega\tau) &= -(a-c)(\gamma^2+\omega^2) - ba_1\gamma - FF,\\ c(\gamma^2+\omega^2)\sin(\omega\tau) &= +\omega(\gamma^2+\omega^2) - ba_1\omega - GG, \end{aligned} \tag{20}$$

where

$$FF = (1-a_1)b(\gamma\cos(\omega\tau_1) - \omega\sin(\omega\tau_1)),$$
$$GG = (1-a_1)b(\omega\cos(\omega\tau_1) + \gamma\sin(\omega\tau_1)).$$

From these equations, in the same manner as in the firs case, we obtain equations for the coupling strength:

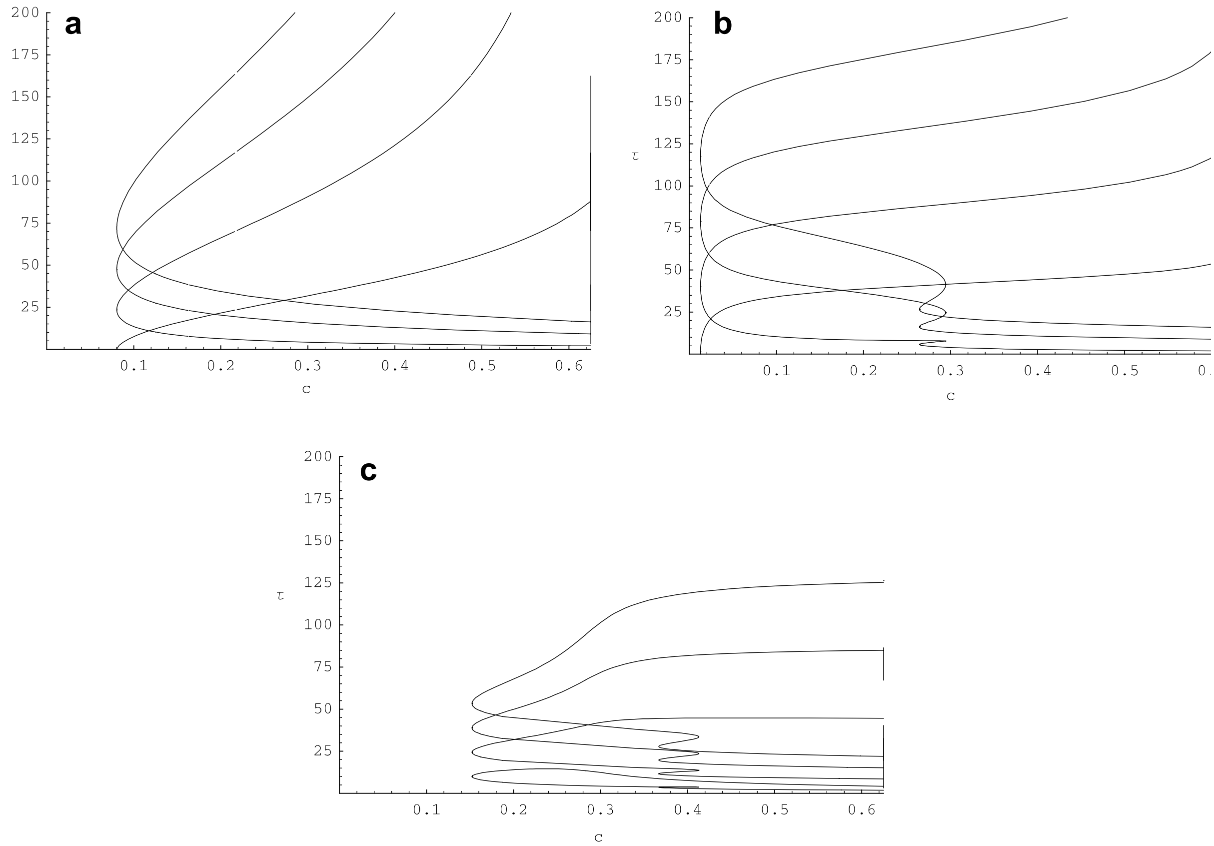

Fig. 1. Hopf bifurcation curves $\tau^{c}_{j,\pm}(c)$ for the fixed values of the local parameters $a = 0.25$, $b = \gamma = 0.02$, and for (a) $\tau_1 = \tau_2 = 3$ corresponding to the case α; (b) $\tau_1 = \tau_2 = 9$ corresponding to the case β; (c) $\tau_1 = \tau_2 = 15$ corresponding to the case γ. (see the main text).

$$c = \frac{(a^2+\omega^2)(\gamma^2+\omega^2) + 2H + b^2 I}{2a(\gamma^2+\omega^2) + 2a_1 b\gamma + 2FF}, \tag{21}$$

where

$$H = aa_1 b\gamma + aFF - \omega^2 a_1 b - \omega GG,$$
$$I = a_1^2 + 2a_1(1-a_1)\cos(\omega\tau_1) + (1-a_1)^2,$$

The critical time lags from the first factor are given by

$$\tau^1_{c,k} = \frac{1}{\omega}\left[\arctan\left(\frac{-\omega(\gamma^2+\omega^2) + ba_1\omega + GG}{(a-c)(\gamma^2+\omega^2) + ba_1\gamma + FF}\right) + k\pi\right], \tag{22}$$

and from the second factor by

$$\tau^2_{c,k} = \frac{1}{\omega}\left[\arctan\left(\frac{\omega(\gamma^2+\omega^2) - ba_1\omega - GG}{-(a-c)(\gamma^2+\omega^2) - ba_1\gamma - FF}\right) + k\pi\right]. \tag{23}$$

Previous formulas ()(13)–(15) for the case of pure delay and for the case with one delayed argument ()(21)–(23)give parametric representations of the bifurcation curves in the plane (c, τ) for fixed values of internal parameters.

The type of the Hopf bifurcation can be seen by calculation of the variations of the real parts $\mathbf{Re}\lambda$ as the time lag is changed true the critical values. This is given by the sign of $\mathrm{d}\mathbf{Re}\lambda/d\tau$ for $\tau = \tau^c$. Using the factorized characteristic equations, we obtain

$$\left[\frac{\mathrm{d}\mathbf{Re}\lambda}{\mathrm{d}\tau}\right] = \frac{2\omega^3 + (\gamma^2 + a^2 - 2ac)\omega - A_1 - B_1}{c^2\omega(\gamma^2+\omega^2)}, \tag{24}$$

where

$$A_1 = b\omega\cos(\omega\tau_{12}) + G + b(a-c)\sin(\omega\tau_{12}),$$
$$B_1 = \tau_{12}((a-c)G + \omega F),$$

for the first case, and for the second case:

$$A_2 = b\omega\cos(\omega\tau_1) + G + b(a-c)\sin(\omega\tau_1),$$
$$B_2 = \tau_1((a-c)G + b^2a_1\sin(\omega\tau_1) + \omega F).$$

Substitution of particular values in this formulas gives the sign of $d\mathbf{Re}\lambda/d\tau$ and the determines the type of the considered Hopf bifurcation.

3. Global dynamics and synchronization

In this section we summarize the main qualitative global dynamical properties of (4). The results have been obtained by extensive numerical computations, for the time-lags smaller then the refractory time of the single isolated unit. For example, for $a = 0.25$, $b = \gamma = 0.02$ the refractory time is about 100. We shall be particularly interested in the possible types of attractors and the properties of synchronization on these attractors. Let us point out that, as we shall see, when the whole system (4) has only one attractor, corresponding to either stable fixed point or to the stable limit cycle with synchronized dynamics, then the whole system behaves as a single excitable system or as a single relaxation oscillator respectively. However, there are more complicated cases when the whole system is bistable with stable fixed point and stable limit cycle, or with two stable limit cycles corresponding to oscillations with different synchronization properties.

As was pointed out before, there are three qualitatively different situations depending on the properties of the single isolated unit: (α) Internal time-lags τ_1, τ_2 are such that the isolated unit has only one attractor that is the fixed point at the origin; (β)τ_1, τ_2 are such that the isolated unit is bi-stable. There is the stable fixed point and the stable limit cycle; (γ)τ_1, τ_2 are such that the stable limit cycle is the only attractor. The three cases differ mostly when the coupling c is small. Larger values of c lead to qualitatively similar types of dynamical behavior in all three cases.

Typical dynamical situations in cases β and γ are illustrated in Figs. 2 and 3. The case α is qualitatively the same as for $\tau_1 = \tau_2 = 0$, and have been analyzed and illustrated in [16,18]. Let us briefly describe qualitatively different situations that occur in the case α, with special emphasis on the domain of small c, since as we shall see, typical dynamics for large c occurs also in the cases β and γ. Instantaneous diffusive coupling can induce oscillations via Hopf bifurcation for the coupling parameter $c = c_0$. The resulting oscillations, for $c > c_0$, of each unit are coherent. Let us point out that if the parameter c is such that the units are excitable, i.e. the only attractor is the stable stationary solution, and c is far away from c_0, then the time-lag has no effect on the qualitative properties of the dynamics. The stationary solution remains stable for any τ. There are two distinct domains of the coupling c for which the dynamics is oscillatory. Firstly, there is a small interval near c_0 for which the instantaneously coupled units are excitable but the time-lag does introduce stable oscillations. The other situations occurs for $c > c_0$, i.e. when the instantaneously coupled units oscillate, and when further Hopf bifurcations are possible due to non-zero time-lag. For the coupling c larger than c_0 but very close, the two units oscillate out of phase. This pattern persists for any time-lag τ. On the other hand, if c is larger, then for most values of τ there is only one attractor corresponding to the oscillatory dynamics. Then, the intervals of τ when the attractor is such that the two units are asynchronous or exactly synchronous alternate. However, there are small intervals of τ such that there are two attractors, one with asynchronous and one with exactly synchronous dynamics.

Qualitatively different situations occurring in case β are illustrated in Fig. 2. For small coupling c and small coupling time-lag τ there are only two attractors (Fig. 2a), the stable fixed point and the stable limit cycle with asynchronous dynamics (Fig. 2b). Increasing τ for fixed small c does not change the stability of the fixed point. The oscillatory dynamics on the large stable limit cycle changes from asynchronous to exactly synchronous and for larger τ back to asynchronous. However, due to very small coupling and non-zero coupling time-lag there is a domain of initial conditions such that one of the units is attracted towards the stable cycle and the other towards the fixed point. Typical dynamics for larger values of the coupling c and small or larger τ is illustrated in Fig. 2c–f respectively. The stable fixed point has disappeared through the Hopf bifurcation due to sufficiently large c. For small τ, there is only one attractor, the large limit cycle created by sufficiently large $\tau_1 + \tau_2$. The dynamics on this cycle is asynchronous. Larger τ lead to further Hopf bifurcations of the (already unstable) fixed point. The interplay of relatively large c and different values of the coupling time-lag can produce three qualitatively different situation. Either there is only one limit cycle attractor, and this one caries asynchronous (Fig. 2c and d) or exactly synchronous dynamics (not illustrated), or there could

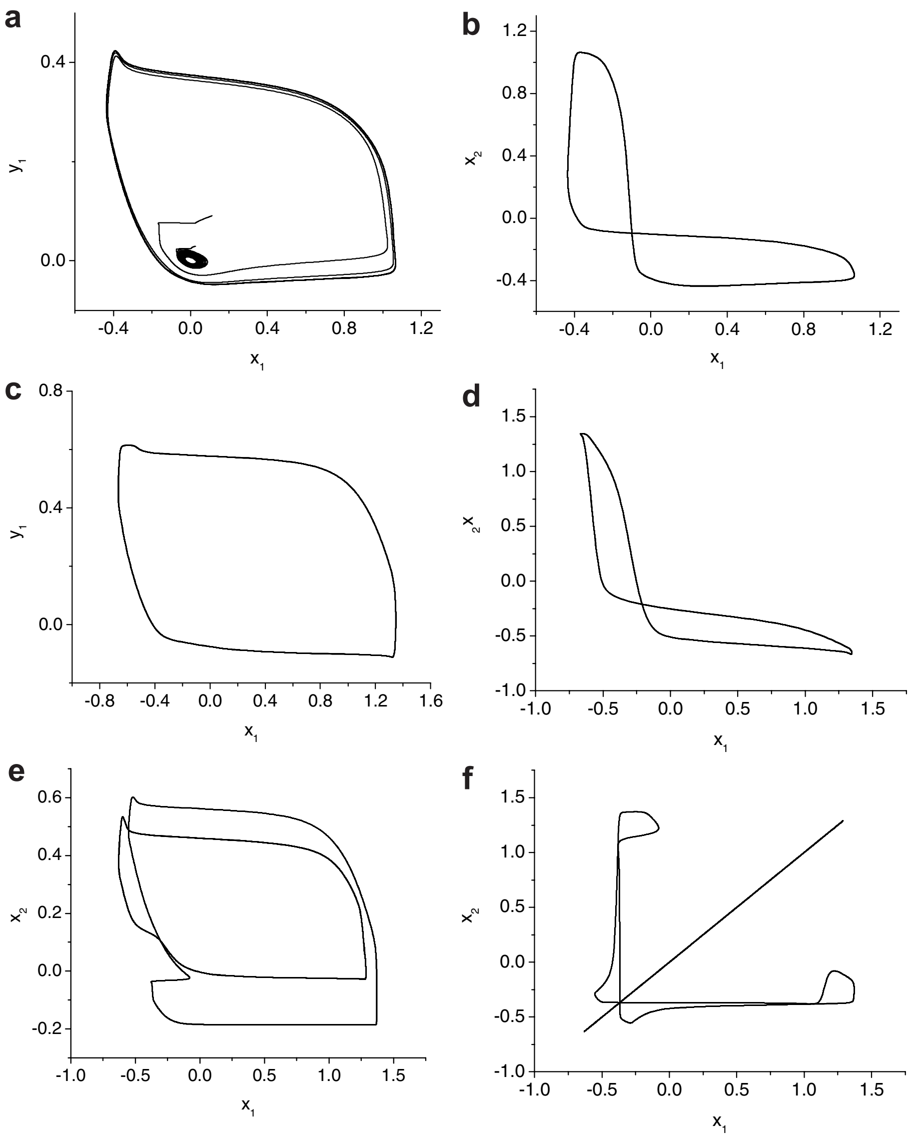

Fig. 2. Illustrates different types of synchronization and possible attractors for $\tau_1 = \tau_2 = 9$ corresponding to the case (β). The parameters (c,τ) are: (a), (b) $(c,\tau) = (0.01,0)$; (c), (d) $(c,\tau) = (0.2,0)$; (e), (f) $(c,\tau) = (0.2,30)$.

be two attractors one with asynchronous and one with exactly synchronous dynamics (Fig. 2e and f), The later situation occurs in smaller domains of (c,τ). The values of (c,τ) that correspond to these three situations are illustrated in Fig. 4a.

Dynamics in the case (γ) is similar to the case (β) when c is not to small so that the fixed point is unstable. The dynamics for small τ is characterized by only one attractor in the form of the limit cycle with asynchronous dynamics, illustrated in Fig. 3a and b. There is a large interval of intermediate τ when there is also only one limit cycle attractor but the dynamics on it is exactly synchronous. The transition as τ is increased, and for fixed c, between these two situations with one but different attractor is not sharp. There are small domains with two coexisting attractors with different synchronization properties. Let us denote by $(\tau_{\min}(c), \tau_{\max}(c))$ the interval of values of τ which, for fixed c, corresponds to the existence of the single attractor with synchronous dynamics. Then there are relatively small intervals of τ just below $\tau_{\min}(c)$ and just above $\tau_{\max}(c)$ where there are two attractors, one with the asynchronous and one with exactly synchronous dynamics. This is illustrated in Fig. 3c and d for some τ smaller then $\tau_{\min}$ but near it. The interval $(\tau_{\min}(c), \tau_{\max}(c))$ for different c corresponds to the area between the bold curves depicted in Fig. 4b.

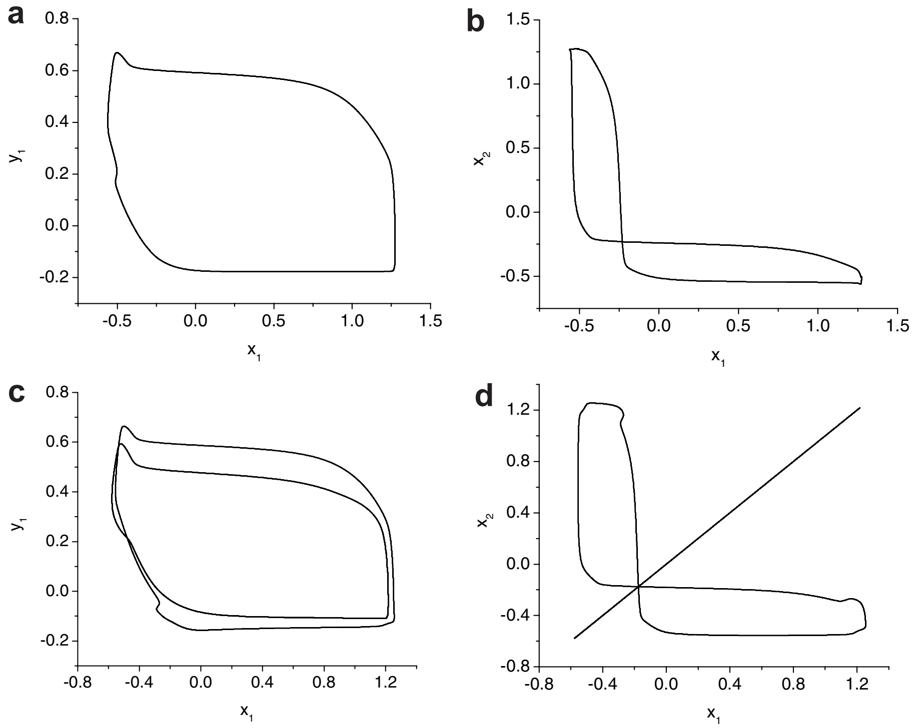

Fig. 3. Illustrates different types of synchronization and possible attractors for $\tau_1 = \tau_2 = 15$ corresponding to the case (γ). The parameters (c, τ) are: (a), (b) $(c, \tau) = (0.2, 15)$; (c), (d) $(c, \tau) = (0.2, 25)$.

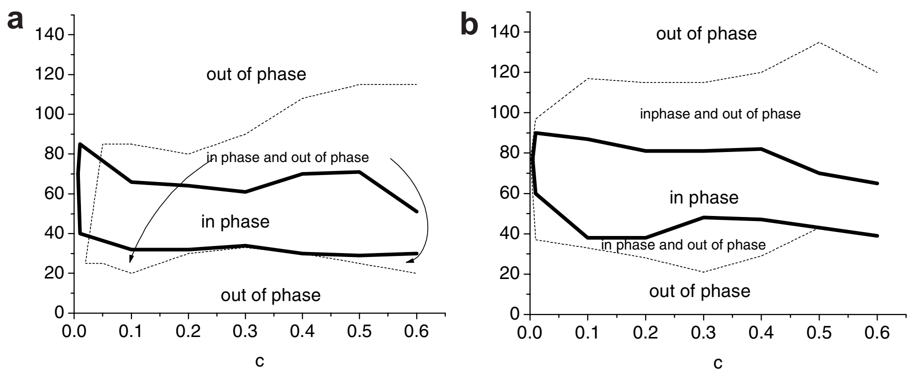

Fig. 4. Illustrates the domains in the (τ, c) that correspond to situation when there is only one attractor corresponding to the exact synchronization for: (a) the case b, and (b) the case c. The domain is between the two curves, below the lower curve and above the upper curve there are either (regions near the two curves) two attractors with synchronous and asynchronous dynamics or only one attractor with the asynchronous dynamics.

4. Summary and discussion

We have studied a pair of identical FitzHugh–Nagumo systems with internal and coupling time delays. The parameters of each of the isolated units are such that for the zero values of the internal delays the units are excitable, i.e. each has only one attractor in the form of the stable fixed point. However, there is a large interval of values of the internal delays such that each unit is bi-stable, with the stable fixed point and the stable limit cycle. In this case, each of the units could be considered as modelling dynamics that could occur in a collection of delayed coupled simple excitable systems without internal delays.

We have analyzed the stability of the fixed point of the coupled system. Bifurcation curves in the plain of coupling constant and the coupling delay, (c, τ) plain, are obtained for various fixed values of the internal delays. This indicates that three cases should be distinguished, depending on the values of the internal delays. The case when units are excitable, the case when units are bi-stable and the case when units are oscillatory. Dependance of the global dynamics on the coupling and coupling delay in these three case is studied numerically.

The following picture emerged from our analyzes. For sufficiently strong coupling the dynamics in the three cases is qualitatively similar. Depending on the coupling time lag the system has only one or two attractors that correspond to oscillatory dynamics. The situation with only one attractors occurs for much more values of the coupling time-lag. The only one attractor could correspond to exactly synchronized or asynchronous dynamics, depending on the coupling time-lag. In the synchronous case the whole system resembles a single relaxation oscillator. The situation with the two coexistent attractors is possible only in a relatively small domains of values of the coupling time-lag near the transition between the dynamics with only the asynchronous or only the synchronous single attractors. On the other hand, for weak coupling the possible types of dynamics in the three cases could be quite different. This is analyzed in details in Section 3.

There are several directions in which our analyzes should be extended. It would be interesting to study systems like (4) but with more that two, possibly nonidentical, units and with local or global coupling. Furthermore, the influence of multiplicative or additive noise on the synchronization properties should be studied.

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