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A Numerical Analysis of Chaotic Behaviour in Bianchi IX Models

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In this paper we investigate the chaotic behaviour of the Bianchi IX cosmological models using techniques developed in the study of dynamical systems and chaotic behaviour. We numerically calculate the Lyapunov exponent, λ, and show that instead of converging to a constant value, it decreases steadily. We study this effect further by studying the Lyapunov exponent using short-time averages. We show that the usual method of calculating λ is invalid in the case of a cosmological model.

1. INTRODUCTION

There has long been interest in the behaviour of cosmological models near the initial singularity [1–4]. These studies reveal that most anisotropic, homogeneous vacuum models tend asymptotically to a Kasner vacuum model. An exception to this is the so-called Mixmaster model which continually changes from one Kasner model to another as one goes back in time towards the initial singularity. The evolution of these models can be approximately described in terms of a return map: a sequence of Kasner models, each characterised by a Kasner parameter u, occuring in Kasner epochs made up of successive Kasner eras. During a Kasner epoch, two of the scale factors oscillate as the universe alternates between different Kasner models while the third decreases and the parameter u decreases by

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one through successive oscillations. When $u < 1$, the next Kasner parameter is given by $1/u$ and a new Kasner era is started. The transformation between various Kasner eras, given by the return map, can be shown to be chaotic [4] with a value of the Lyapunov number given by

$$\lambda = \frac{\pi^2}{6(\ln 2)^2}. \tag{1}$$

The system of equations for the Mixmaster model were integrated numerically by Zardecki [5]. It has been noted by Ma [6] that some of Zardecki's results appear to be incompatible with the results of Belinski et al. Zardecki also plots the Lyapunov exponents and finds them to be positive for the perfect fluid Bianchi IX and VIII models but negative for the VII_0 model. This is consistent with the fact that the vacuum Bianchi IX and VIII models exhibit chaotic behaviour but the Bianchi VII_0 model does not. Recently, Halpern [7] has examined the occurrence of chaos in Bianchi VIII models with matter. He has observed that the energy of the matter can act as a parameter heralding the onset of chaos. Very recently, Francisco and Matsas [8] have calculated the Lyapunov exponents numerically for the vacuum Bianchi IX models. Their calculations show that λ declines steadily rather than converging to a constant, non-zero, value. These results are confirmed in our calculations presented here.

In this paper we shall use a numerical approach to the study of chaotic behaviour in the Bianchi IX model. In Section 1 we shall describe the dynamical system derived by Wainwright [9] and discuss the behaviour of the variables. In Section 2 we shall proceed to calculate the principal Lyapunov exponent in the usual way and in Section 4 we shall employ the short-time-average approximation. In Section 5 we shall give some discussion of the results and suggestions for further work.

2. THE DYNAMICAL SYSTEM

We have found it convenient to follow Wainwright [9] and write the field equations for Class A orthogonal Bianchi models with a perfect fluid matter source as a dynamical system involving variables which are dimensionless. The reason for doing this is that such variables do not blow up near the singularity. The system can be derived starting from the following equations [10]

$$\dot{\theta} = -\frac{1}{3}\theta^2 - 2\sigma^2 - \frac{1}{2}(3\gamma - 2)\mu \tag{2}$$

$$\dot{\sigma}_{\alpha\beta} = -\theta\sigma_{\alpha\beta} - S^{(3)}_{\alpha\beta} \tag{3}$$

where $S^{(3)}_{\alpha\beta}$ is the trace free Ricci tensor, θ is the trace of the expansion tensor and $\sigma_{\alpha\beta}$ is the symmetric and trace-free shear tensor. There is a first integral of the system given by

$$\mu - \frac{1}{3}\theta^2 + \sigma^2 = \frac{1}{2}R^{(3)}, \tag{4}$$

where μ is the matter density and $\sigma^2 = 1/2(\sigma_{\alpha\beta}\sigma^{\alpha\beta})$. The curvature of the group orbits is determined by the quantity $n_{\alpha\beta}$ which obeys an equation of the form

$$\dot{n}_{\alpha\beta} = -\frac{1}{3}\theta n_{\alpha\beta} + 2\sigma^{\gamma}{}_{(\alpha}n_{\beta)\gamma} \tag{5}$$

and lastly, the contracted Bianchi identities yield an equation for the evolution of the matter density μ,

$$\dot{\mu} = -\gamma\mu\theta \tag{6}$$

where we have assumed the matter takes the form of a perfect fluid with an equation of state of the form $p = (\gamma - 1)\mu$ and $2/3 \leq \gamma \leq 2$. The system of equations is invariant under scaling transformations which allow the following dimensionless variables to be defined,

$$\begin{aligned}
\Sigma_{\alpha\beta} &= \frac{\sigma_{\alpha\beta}}{\theta} \\
N_{\alpha\beta} &= \frac{n_{\alpha\beta}}{\theta} \\
S_{\alpha\beta} &= \frac{S^{(3)}_{\alpha\beta}}{\theta^2} \\
K &= -\frac{3R^{(3)}}{2\theta^2} \\
\frac{d}{d\tau} &= -\frac{3}{\theta}\frac{d}{dt}
\end{aligned} \tag{7}$$

The new time-variable τ is defined such that $\tau \to \infty$ as $t \to 0$ which is suitable to study the approach to the singularity. We can rewrite the equations for $\sigma_{\alpha\beta}$ and $n_{\alpha\beta}$ by defining the deceleration parameter q by

$$q = -\frac{\ddot{l}\,l}{\dot{l}^2} \tag{8}$$

where $l(t)$ is an average length scale defined such that $\dot{l}/l = \theta/3$. Then, defining the quantities

$$\Sigma = \frac{3}{2}\Sigma_{\alpha\beta}\Sigma^{\alpha\beta} = 3\frac{\sigma^2}{\theta^2} \tag{9}$$

and

$$\Omega = 3\frac{\mu}{\theta^2} \tag{10}$$

we can write

$$q = \frac{1}{2}(3\gamma - 2)\Omega + 2\Sigma \tag{11}$$

and the constraint equation (4) becomes

$$\Sigma + K + \Omega = 1 \tag{12}$$

The evolution equations for $N_{\alpha\beta}$ and $\Sigma_{\alpha\beta}$ are thus

$$\begin{aligned} \Sigma'_{\alpha\beta} &= \ (2-q)\Sigma_{\alpha\beta} + 3S_{\alpha\beta} \\ N'_{\alpha\beta} &= -\, qN_{\alpha\beta} - 6\Sigma^{\lambda}_{\alpha}N_{\lambda\beta} \end{aligned}$$

where $'$ signifies differentiation with respect to τ. In the shear eigenframe the shear tensor $\Sigma_{\alpha\beta}$ has only two independent components. This allows us to define two new shear variables

$$\begin{aligned} \Sigma_+ &= \frac{3}{2}(\Sigma_{22} + \Sigma_{33}) \\ \Sigma_- &= \frac{\sqrt{3}}{2}(\Sigma_{22} - \Sigma_{33}) \end{aligned}$$

so that $\Sigma = \Sigma_+^2 + \Sigma_-^2$. Since in the shear eigenframe $N_{\alpha\beta}$ is diagonal [10], the dynamical system can be written in the form

$$\left.\begin{aligned} N'_1 &= -\,(q - 4\Sigma_+)N_1 \\ N'_2 &= -\,(q + 2\Sigma_+ + 2\sqrt{3}\Sigma_-)N_2 \\ N'_3 &= -\,(q + 2\Sigma_+ - 2\sqrt{3}\Sigma_-)N_3 \\ \Sigma'_+ &= \ (2-q)\Sigma_+ + 3S_+ \\ \Sigma'_- &= \ (2-q)\Sigma_- + 3S_- \end{aligned}\right\} \tag{13}$$

where

$$\left.\begin{aligned} S_+ &= \frac{1}{2}\left[(N_2 - N_3)^2 - N_1(2N_1 - N_2 - N_3)\right] \\ S_- &= \frac{\sqrt{3}}{2}(N_3 - N_2)(N_1 - N_2 - N_3) \\ q &= \frac{1}{2}(3\gamma - 2)\,\Omega + 2\Sigma \end{aligned}\right\} \tag{14}$$

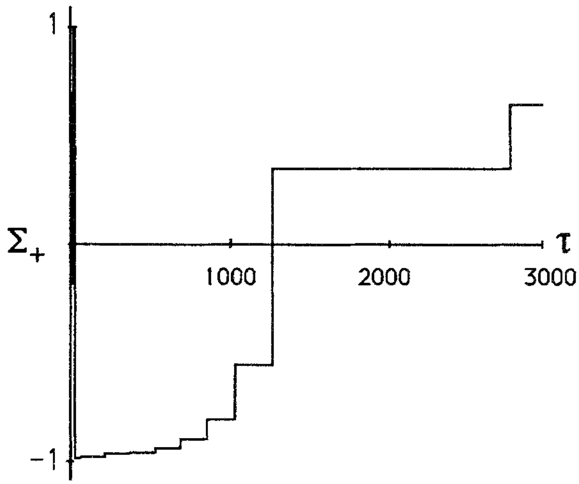

Figure 1
Plot of Σ_+ for a Bianchi IX model with initial conditions given by (16)

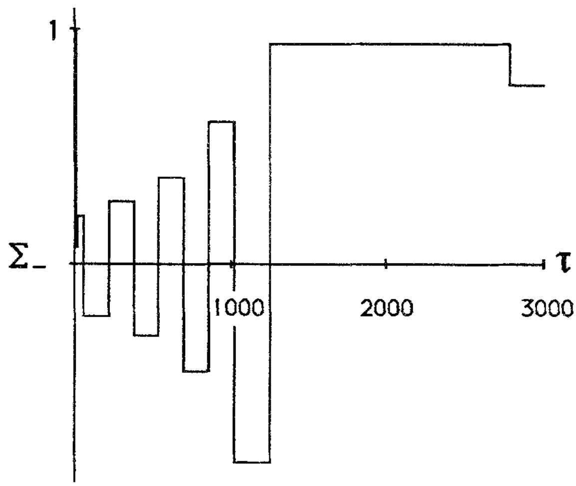

Figure 2
Plot of Σ_- for a Bianchi IX model with initial conditions given by (16)

We know that for a particular model the signs of the variables N_i cannot change. To ensure this is true in the numerical treatment, we follow Ma [6] and redefine the N_i using

$$Z_i = \ln(N_i{}^2). \tag{15}$$

For the Bianchi IX models, all N_i are of the same sign and non-zero whereas for the Bianchi VIII models they are nonzero and one has different sign; for the Bianchi I model, we have $N_i = 0$. Plots of these variables for the Bianchi IX model are shown in Figs. 1–5. The plots were made for a model with initial conditions given by

$$\left.\begin{aligned} N_1 &= 0.2 \\ N_2 &= 3.1 \\ N_3 &= 2.9 \\ \Sigma_+ &= 0.74 \\ \Sigma_- &= 1.005 \end{aligned}\right\} \tag{16}$$

Figs. 1 and 2 show the behaviour of the two shear variables Σ_+ and Σ_-. Both variables behave in qualitatively the same way and the rapid changes in the variables occur together. The sudden changes in Σ_+ and Σ_- correspond to the bounces from the potential wall in the Hamiltonian picture of the Mixmaster model [11]. The horizontal periods are where the model is close to a particular Kasner solution and the jumps occur when the model flips between one Kasner state and another.

The behaviour of the variables N_1, N_2 and N_3 is qualitatively different from that of the others.Spikes in the variables occur only when there is a corresponding change in the shear variables. At each jump, only one N_i changes, the others remaining close to zero. Away from the bounces, the N_i's remain close to zero so that the model is close to the Kasner model.

The important thing to notice is that there are long periods of a steady solution interspersed with suddden changes in some of the variables. The time for one of these sudden changes is much shorter than the time between them, which seems to be increasing. This behaviour is what one would of course expect from a billiards-type problem where the walls of the billiard table are expanding.

This system of equations has been used by Ma [6] to study the Class A Bianchi Models from a dynamical systems point of view. The results of the programs used in this paper agree with those that Ma uses. This is comforting, as will be discussed later.

3. THE LYAPUNOV EXPONENT

In this section we shall investigate the principal Lyapunov exponent for the Bianchi IX model using the equations given above. The numerical computations were carried out on an Orion 1/05 computer and the codes

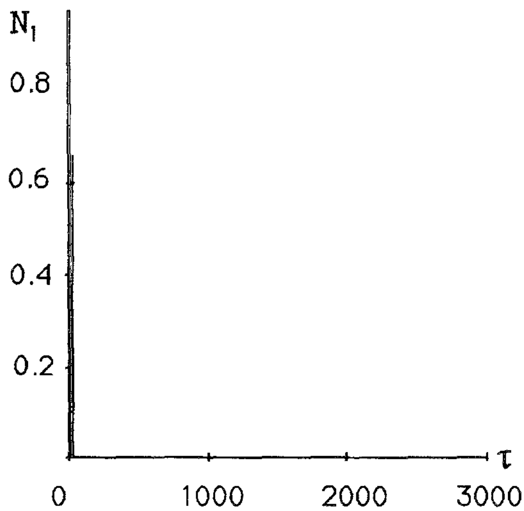

Figure 3
Plot of N_1 for a Bianchi IX model with initial conditions given by (16)

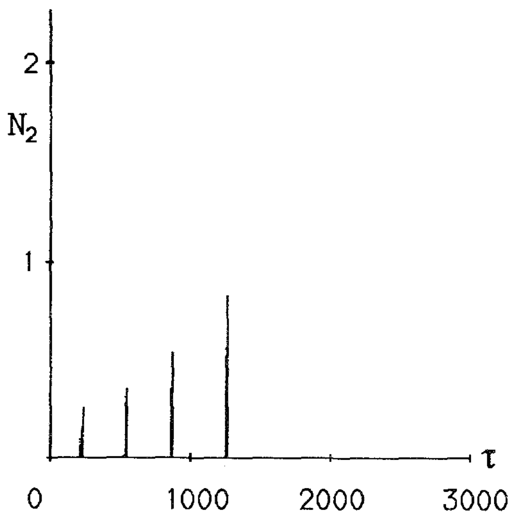

Figure 4
Plot of N_2 for a Bianchi IX model with initial conditions given by (16)

were written in the C programming language. The numerical integrations were performed using a Bulirsch-Stoer algorithm [12].

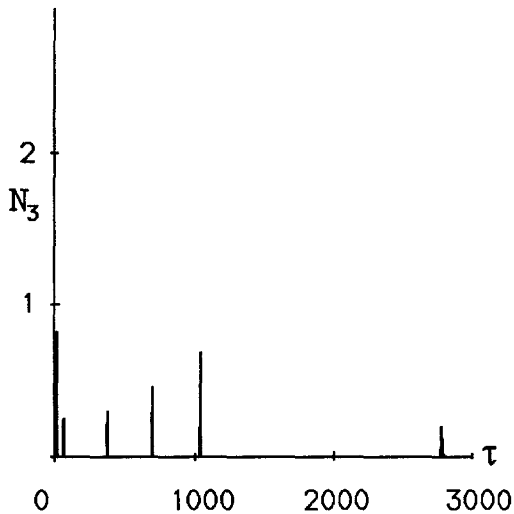

Figure 5
Plot of N_3 for a Bianchi IX model with initial conditions given by (16)

The Lyapunov exponents characterise the rate of divergence of neighbouring trajectories in the phase space of the system and are therefore a measure of the amount of information lost as the system becomes chaotic [13]. For an n-dimensional system, there will be n Lyapunov exponents giving information about the rate of divergence of trajectories in n orthogonal directions. The principal Lyapunov exponent is the λ_i with the greatest value. In this paper we shall concern ourselves only with the principal exponent. A positive Lyapunov exponent indicates that trajectories are diverging and that system has become chaotic; the principal Lyapunov exponent is the largest of the individual Lyapunov exponents. The method we use to calculate the Lyapunov exponents follows closely that described by Benettin and Galgani [14]. The code was checked using the Lorenz attractor as well as the Rossler 3-dimensional attractor; in both cases the code gave Lyapunov exponents which converged to previously obtained results [15][1] . For a system of ordinary differential equations which can be written in the form

$$\mathbf{x}_i = \mathbf{F}_i(\mathbf{x}) \tag{17}$$

[1] The results given by Wolf [16] do not seem to agree with those of Shimada and Nagashima. This is because Wolf et al. define the Lyapunov exponent using a logarithm of base two.

the Lyapunov exponent can be defined by

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \ln ||w(t)|| \tag{18}$$

where $w(t)$ is a solution of the variational equation

$$\dot{w} = \mathrm{A}[x(t)]w \tag{19}$$

where $x(t)$ is a solution of the original equations and **A** is a matrix Jacobian of **F**. When numerically integrating these expressions it is found that there is a computer overflow as $t \to \infty$ if $\lambda > 0$ along the trajectory. This is overcome by renormalising the vector w to unity. Details of this can be found in [15,17].

3.1 THE BIANCHI IX MODELS

The Bianchi IX models have been known to admit chaotic solutions for a long time. Using arbitrary intial conditions we can see the presence of chaos occurring in these models from the positive value of λ as shown in Fig. 6.

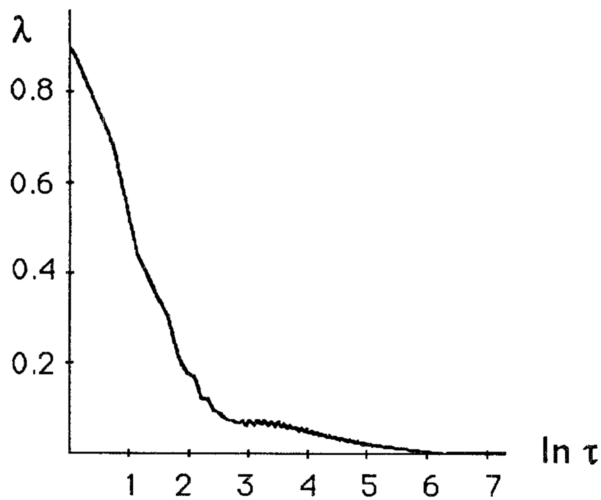

Figure 6.
The Lyapunov exponent against $\ln \tau$ for a Bianchi IX model with initial conditions given by (16)

The initial conditions for the Bianchi IX models were chosen such that

$$\Omega = \quad 1.183 \tag{20}$$

$$\Sigma = \quad 1.557 \tag{21}$$

$$K = -1.740 \tag{22}$$

and are given by (16). There is also an interesting feature in that the value of λ does not converge to a constant value but decreases very gradually to zero. This has also been noticed in a recent paper by Francisco and Matsas [8] and they argue that the reason for this is that the time between successive Kasner eras increases exponentially. This can also be thought of in terms of the Hamiltonian formalism used by Misner and others [18]. In this formalism, the evolution of the cosmological model is represented as a point bouncing in an expanding potential well. Here successive bounces occur after longer periods leading to the decrease in the Lyapunov exponent. These effects were not noted by Zardecki [5]. A further point to note is that the Lyapunov exponents calculated for the return map

$$x_{n+1} = x_n^{-1} - [x_n^{-1}] \tag{23}$$

where [...] denotes the integer part, is a constant. This seems to indicate that the above map does not contain all of the information concerning the chaotic behaviour of the system.

One should take care in studying chaotic systems using numerical methods that the noise created by round-off errors in the numerical calculations do not come to dominate the numerical solution. For certain simple systems it can be shown that the numerically calculated orbits do follow the actual orbits very closely, and so the numerical results can be trusted [19]. It is known however that the Einstein equations for a homogeneous cosmological model do not satisfy these requirements [4] and so one may be worried by the accuracy of these results. To attempt to see if the noise was dominating these calculations, we repeated all calculations using a range of tolerance values and found no discernable change in the observed results. Whilst this is by no means a convincing argument, it does suggest that the noise introduced by the round-off errors in the numerical computation is not dominating the calculation [19]. It should also be noted that the system (13) was integrated by Ma [6] using a Runge-Kutta-Fehlberg algorithm and his results for the evolution of the system agree with ours. We are thus confident that the numerical results can be relied upon. It appears that the slow decline in the value of λ is real and is probably due to the effect discussed in [8].

4. THE SHORT TIME AVERAGE

In the study of turbulence in fluid flows, one often works with averaged features of the attractors. However, one finds that real turbulent flows are characterised by transient, non-stationary features such as fluid intermittency [20]. This appears as bursts in the time series of the fluid after it has been passed through a high frequency filter. One suggested way to study this behaviour is to use short-time-averages (STA) of the Lyapunov exponent [21]. This involves averaging over times which are intermediate between the numerical integration time and the width of the burst.

In the case of the cosmological model it is instructive to use a similar technique. Since all the variables only ever change at a bounce against the potential wall, we use an STA timescale which is of the order of the bounce-time. What we expect to find is periods when the Lyapunov exponent is zero with regions of non-zero λ appearing when the bounces occur. A similar result would hold in an analysis of the standard billiard table problem except there the wall of the billiard table is hard whereas for the cosmological case, the wall has an exponential profile.

The results of the numerical calculations are shown in Fig. 7.

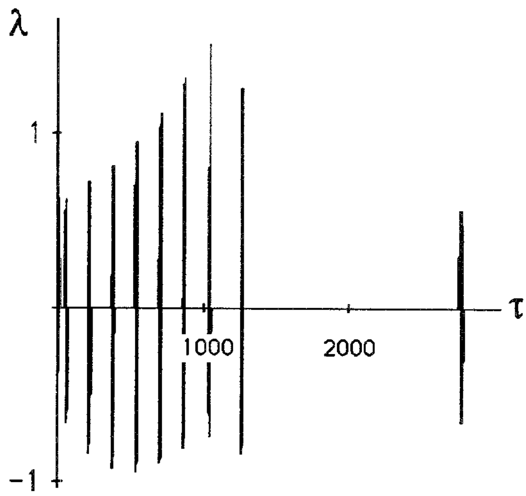

Figure 7.
The Lyapunov exponent against τ for a Bianchi IX model with initial conditions given by (16) using the short-time average.

As can be seen, λ is non-zero during the periods of the bounces and zero in between the bounces as expected. The plots also show that there is considerable structure in the bounces. This is particularly noticeable as $\tau \rightarrow \infty$. Although λ goes negative during the bounce, the contribution to λ averaged over the whole bounce is positive. Francisco and Mastas argue that as one approaches the singularity, the structure of the bounce cannot be neglected when considering the continuous evolution of the model. This is borne out by the structure mentioned above. The behaviour of the Lyapunov exponent calculated using the STA reveals why we were obtaining the results presented in Section 3.1. Usually when calculating the Lyapunov exponent numerically, one averages over the whole integration time. However, since there are only small periods when $\lambda \neq 0$, this average will decrease.

When studying the onset of chaotic behaviour, it is usual to consider families of equations. For example, when examining the onset of turbulent fluid flow, the Reynolds number is the parameter which is varied. In the case we consider here, the eqution of state paprameter γ may play a similar role. To see if this was the case, we performed some calculations with different values of γ and found that for $\gamma < 2$, the motion was chaotic but for $\gamma > 2$, the motion was regular; Fig. 8 shows the behaviour of the principal Lyapunov exponent for this case.

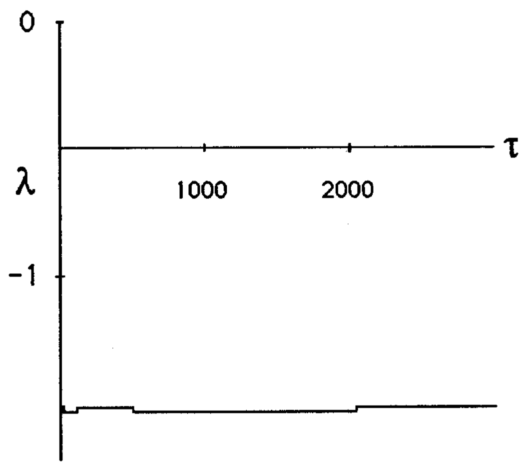

Figure 8.
The Lyapunov exponent against τ for a Bianchi IX model with $\gamma = 3.0$.

The case with $\lambda > 2$ corresponds to a fluid which is unphysical in the sense that the sound speed in the fluid is greater than the speed of light.

The Lyapunov exponent is sensitive to the initial conditions of the trajectory and to investigate the phase space a little further we examined trajectories which satisfy current constraints on the values of Ω and Σ. Limits on the amount shear that is admissible today can be derived from limits on the microwave background anisotropy [22]

$$\frac{\sigma}{\theta} < 2.6 \times 10^{-5} \tag{24}$$

for closed cosmological models with $\Omega_0 \simeq 1$. For initial conditions satisfying this constraint and taking $\tau = 0$ to be the present day we find that λ is positive from a very early time (Fig. 9).

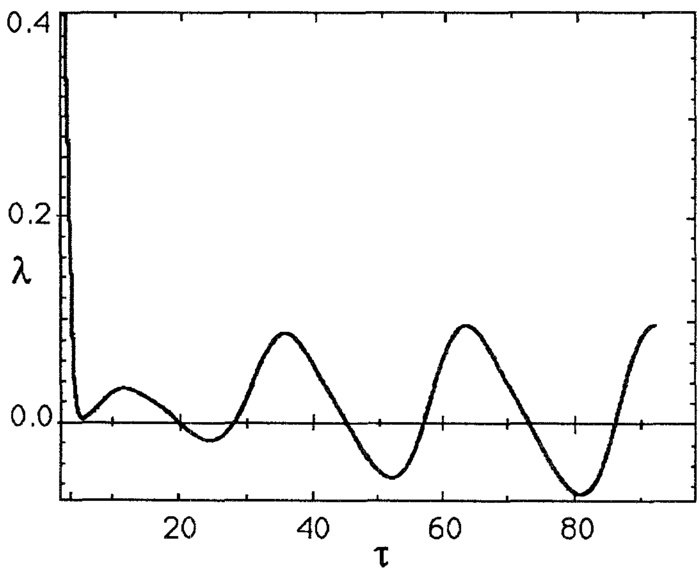

Figure 9.
The Lyapunov exponent against τ for a Bianchi IX model with $\gamma = 1.0$ and initial conditions constrained by observed limits on the shear and density.

This leads one to suspect that chaos could have been a quite recent feature of the universe and may have had effects upon baryosynthesis and nucleosynthesis.

5. CONCLUSIONS

In this paper we have investigated the behaviour of the Lyapunov exponents for one of the class A Bianchi models with a perfect fluid. The results of these investigations show that there is a lot of structure in the chaotic behaviour of these solutions. These results warrant further investigation and in particular we are investigating how the approximate mappings used by Belinskii and Khalatnikov [2] and Barrow [4] represent the chaotic behaviour and whether other mappings can be constructed. Our results show that there is considerable structure in the bounce which may be due to the complex shape of the potential surface, which is evolving with time. The contributions of the bounce as $\tau \rightarrow \infty$ appears to become more prominent and we are investigating the nature of these bounces futher.

The chaotic behaviour comes in bursts, similar to the intermittent behaviour observed in turbulent fluid flow. This would presumably have important consequences for an observer living in one of these models. If he happened to exist during one of the calm periods, our friend would not observe anything unusual. If, however, he lived in a time just prior to a burst of chaotic behaviour, then he would indeed be living in an interesting epoch. These sudden changes indicate a sudden change in the underlying attractor of the dynamical system.

There still remains the question of the reliability of the numerical calculations and this is a question that remains unanswered even amongst those working in the field of chaotic dynamics. A further check that can be made is to do more calculations with greater tolerances. It may be possible to write an integrator for this purpose using LISP which allows for infinite precision integer arithmetic [23] and the feasibility of such a system is currently being investigated. Until a definite answer to this question can be given, numerical results must still be taken with due care.

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