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Probability in Theories With Complex Dynamics and Hardy's Fifth Axiom

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Abstract L. Hardy has formulated an axiomatization program of quantum mechanics and generalized probability theories that has been quite influential. In this paper, properties of typical Hamiltonian dynamical systems are used to argue that there are applications of probability in physical theories of systems with dynamical complexity that require continuous spaces of pure states. Hardy's axiomatization program does not deal with such theories. In particular Hardy's fifth axiom does not differentiate between such applications of classical probability and quantum probability.

Keywords Probability · Complex dynamics · Axiomatization

1 Introduction

There have been many, more or less known, attempts to formulate an axiomatic basis of quantum physics (recent contribution with an extended list of relevant references is [1] and famous examples are [2–5]). Particularly influential in recent years is the system proposed by L. Hardy [6–8], because of its instrumental character and intuitive clarity of the formulation. The goal of Hardy's axiomatization program can be seen as: a) to derive quantum mechanics of systems with finite number of degrees of freedom from simple axioms and/or b) to pinpoint the clear difference between physically relevant applications of the classical probability theory (CT) on one side and the theory of quantum probability (QT) on the other side, by formulating a set of simple axioms that are all satisfied by QT and one of them is not satisfied by CT. Then the axiom not satisfied by CT would provide the understanding of the crucial difference between QT and CT.

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Hardy obviously and remarkably succeeds in the first goal, although the claim that the axioms and notions used in their formulation are intuitively clear and well formulated could be, and has been, criticized (see for example [1, 9, 10]).

However, Hardy fails in the second goal because he considers the model of classical probability that is to narrow to capture typical situations that occur in applications of classical probability theory in mechanics of macroscopic bodies. Hardy formulates his axioms as if the World consist of systems such that all possible, or at least all "fundamental", applications of any probability theory (classical or quantum) are formulated with discrete spaces of pure states. If this assumption of discrete pure state space is made than one of the axioms (the fifth in [6]) almost trivially and independently of other axioms rules out classical probability. However, countable pure state space assumption is to narrow or simply wrong in the classical probability as it is applied in mechanics (we sometimes abbreviate this as noted by CT) of systems of macroscopic bodies.

As we shall see in Sect. 3, where we formulate our main argument against the discrete pure state space restriction, this assumption is not valid for systems described by Hamiltonian dynamics. In fact there are observable phenomena modelled by Hamiltonian dynamics (and that, as it appears, can not be reproduced by QT) that require continuous space of pure states for their description. Approximation by discretization fails to explain and predict the phenomena.

Major paradigm of classical systems is for Hardy given by the classical digital computer, and his axioms indeed distinguish applications of probability theory to such systems from quantum probability. However, there are problems in classical dynamics so complex that they are hard to solve by the digital computer, but are constantly solved by analog computers provided in Nature. Furthermore, such problems are typical in the world of macroscopic bodies.

In the next section we recapitulate very briefly Hardy's axiomatization scheme. Our main argument is presented in Sect. 3, and is based on attempted discretization of a Hamiltonian dynamical system by techniques of symbolic dynamics. This will provide formal argument and intuitive understanding of the need to use continuous space of pure states. In Sect. 4 critical examination of Hardy's arguments for the assumed restriction on discrete spaces of pure states will be given. We present and discuss our conclusions in Sect. 5.

2 Hardy's Axiomatization Scheme

The basic notions in Hardy's approach are states, state transformations and probability measurements [6] (see also [7, 8]).

The state of a physical system is that mathematical object from which one can determine the probability of any conceivable measurement. In quantum theory states of a system with finite number of degrees of freedom[1] are density matrices on the corre-

[1]We use the standard notion of degrees of freedom, i.e. a particle in R^3 has three degrees, two such particles have six degrees and a scalar field has continuously infinite number of degrees of freedom. This is different from the notion of degrees of freedom used by Hardy.

sponding separable complex Hilbert space. In applications of classical probability on systems of finite number of classical particles the states are probability distributions on a finite dimensional smooth manifold M. Hardy considers only the applications of classical probability with finite (or discrete) space of elementary events, so M is a discrete lattice.

The probability measurement results in a number $0 \leq p \leq 1$ that is the probability of a certain measurement outcome.

Between the time t_0 (of preparation of the system in) the state $\rho(t_0)$ can be subjected to various transformations $Z(\rho)$, in particular its natural evolution, before the probability measurement is performed at some later time.

In order to uniquely fix the state of the system at time t_0 i.e. to fix probabilities of all possible measurements at time $t > t_0$ one needs K real parameters. In quantum mechanics K is the number of reals that specify the density operator ρ. In applications of probability in classical mechanics K is continuously infinite, i.e. $\{\rho(x), x \in M\}$. In the case of probability on a discrete space of elementary events, considered by Hardy, K equals the number of elementary events. Mixed and pure states are defined as usual in probability theory. Number of real parameters needed to fix a pure state in classical mechanical applications is the dimension of M, i.e. values of the coordinates of the point $x \in M$.

A very important number indicates the crucial difference between the quantum and the classical mechanics. This is the number of perfectly distinguishable pure states denoted N by Hardy. In QT N equals the dimension of the system's Hilbert space. In CT all pure states are perfectly distinguishable, thus N is uncountably infinite. Next section deals with attempts to discretize M.

We now recollect the five axioms postulated by Hardy in [6].

Axiom 1 *Probabilities*. Relative frequencies (measured by taking the proportion of times a particular outcome is observed) tend to the same value (which we call the probability) for any case where a given measurement is performed on a ensemble of n systems prepared by some given preparation in the limit as n becomes infinite.

Axiom 2 *Simplicity*. K is determined by a function of N (i.e. $K = K(N)$) where $N = 1, 2, \ldots$ where, for each given N, K takes the minimum value consistent with the axioms.

Axiom 3 *Subspaces*. A system whose state is constrained to belong to an M dimensional subspace (i.e. have support on only M of a set of N possible distinguishable states) behaves like a system of dimension M.

Axiom 4 *Composite systems*. A composite system consisting of subsystems A and B satisfies $N = N_A N_B$ and $K = K_A K_B$.

Axiom 5 *Continuity*. There exists a continuous reversible transformation of a system between any two pure states of that system.

3 Probability and Classical Mechanics

Hamiltonian formulation of mechanics of conservative systems of classical particles provides a typical example of a classical theory [11–13]. A Hamiltonian dynamical system with finite n degrees of freedom (in the usual sense of the word) (M^{2n}, S, U) is given by a symplectic manifold M^{2n} of dimension $2n$, which is called the systems phase space, the space of states $\rho \in S$, which is the set of probability distributions on M^{2n}, and a symplectic mapping U on M^{2n} which defines the evolution of states by the adjoint mapping on the state space

$$(U^{\dagger}\rho)(x) = \rho(U^{-1}x). \tag{1}$$

Often the evolution is described by the Hamiltonian function $H \in C^{\infty}(M^{2n})$ which generates the evolution of states via the Liouville evolution equation:

$$\dot{\rho} = \{H, \rho\}. \tag{2}$$

The continuous time evolution, given by the Hamiltonian, is in a neighborhood of a periodic orbit uniquely described, via the construction of the Poincare map, by the symplectic map on manifold of $2n - 2$ dimensions [12]. So without loss of generality and for our purpose we shell consider the evolution of the Hamiltonian system as given by the symplectic mapping (1).

Pure states of a Hamiltonian system can be identified with the points of the phase space. Of course, there is continuously infinite number of perfectly distinguishable pure states, and in this sense Hardy's axiomatization trivially does not apply, i.e. it does not say anything about this example of a classical theory. In order for Hardy's axiomatization to apply also in this case we must investigate if the space of pure states can be discretized without observable effects. It is a much studied question [14, 15], motivated by standard applications and problems of the phase space discretization, weather a Hamiltonian dynamical system can be equivalently described by a countable set of pure states. It turns out, as will be briefly recapitulated in the next paragraph, that the answer to this question depends on the dynamics of the Hamiltonian system. There is a class of simple evolutions which can be uniquely described by a countable phase space, but the dynamics of a typical Hamiltonian system is so complex that its description necessarily involves continuous space of pure states [15]. In other words, a preparation that gives probabilities of a discrete set of events does not fix the probabilities of results of all possible future measurements. Distributions over the continuous sets of pure states must be used. Furthermore, such mathematically typical systems are also often encountered as explanations of real physical systems and this property is typical, i.e. stable under perturbations [15].

Mathematical formalism that attempts the program of discretization of a smooth dynamical system is known by the name of symbolic dynamics [14]. Briefly, a finite partition Π of the phase space into N cells is iterated by the dynamical map U (and U^{-1}). Intersections of the elements of the partition after $i-1$-th and after i-th iteration generate the partition at i-th step. The initial partition Π is called the generating partition if the cells obtained in the doubly infinite limit $i \to \pm\infty$ (where negative i corresponds to the iteration of U^{-1}) converge on the points of the phase space. In this case the continuous space of pure states x can be uniquely encoded by the countable space of doubly infinite sequences N^Z of letters from the alphabet of the generating partition Π. The dynamics is represented by shifts on the sequences. However, and this is crucial for our argument, only atypical Hamiltonian systems such as integrable or strongly chaotic, posses the generating partition. The typical systems, i.e. those with the mixed phase space possessing the regular (periodic and quasi-periodic) and irregular (chaotic) orbits do not posses the generating partition. For a typical Hamiltonian system it is necessary to use continuous space of pure states in order to uniquely fix the probabilities of future measurements. Intuitively, neighbouring points of a typical system will generate qualitatively different orbits that could move over quite different parts of the phase space so that the measured values of the physical variables will be quite different. The question of possibility of effective approximate simulation on a discrete computer of such an evolution over bounded times, and construction of the corresponding algorithms, is a problem that does not concern the fundamental theory, but is important in applications [15, 16].

4 Hardy's Arguments Against Continuous State Spaces

Hardy restricts his axiomatization program right from the start only on the theories that do not involve continuous state spaces. Hardy suggests several arguments of different character in favor of the assumption that the continuous dimensional state spaces should not be seriously considered in any fundamental theory of Nature. We shall now analyze each of these arguments in the order as they appear in [6].

The first argument employs Axiom 3 to claim that subsystems of a system must have finite dimensional state spaces. This is true only if the pure state space of the total system is already assumed finite, or if this is not the case if it is assumed that if a system is a subsystem then it must have finite dimensional space of pure states. Obviously, in either case what is to be demonstrated is already assumed, so this can not be considered as a valid argument against possibility of continuous pure state spaces. Of course, one can always chose a finite set of states and consider the description with these states as corresponding to a "subsystem". However, if one is considering, not only the kinematics but also the role of states in the description of the dynamics then the qualitative properties of the dynamics dictate if there is a subset of pure states which is invariant under the evolution. There are dynamical systems with complex dynamics such that no approximation by finite set of states is good enough. So, again it must be assumed that the finite set of pure states is enough to describe the evolution of the "subsystem" (or the approximation of the original system), and there are evolutions which do not allow this.

The second argument involves description of the space–time and physical systems at the Plank scale. Even if it is supposed that the theories of quantum gravity clearly imply that there is only a countable pure state space of the quantum space–time, it is not clear how this is relevant to the axiomatization of a theory like say Hamiltonian dynamics which successfully describes the world of objects in its domain.

We see that both arguments discussed so far exclude probability theory of classical hamiltonian mechanics without any reference to the Axioms 1–5. It is then hard to see what is the relevance of the conclusion that the classical probability theory is incompatible with Axiom 5. The two stated arguments against continuous pure state spaces, like in the classical mechanics, actually express a belief that classical Hamiltonian dynamics and the corresponding probability theory should not be considered as examples of fundamental theory. Discussion of such a belief poses a question of what should be considered as a fundamental theory. Such a discussion should be of no relevance for the attempt to rule out the classical probability theory on the bases of reasonable but sufficiently general axioms. Nevertheless, we could argue that as yet there is no completely successful theory of quantum-to-classical transition that derives all the formal aspects of the Hamiltonian mechanics and all observable effects in the world of macroscopic objects from the quantum mechanics as we know it [17–19]. Until that is achieved the Hamiltonian classical mechanics with the corresponding probability theory should be considered as a fundamental theory of a possible world, namely of the world of conservative systems of macroscopic bodies. This conclusion may be appreciated on a formal level, and it does not involve reference to any philosophical or methodological position.

The third argument that the main motivation for the continuous pure state space of classical mechanics is the intuition that there are no jumps in the Nature is difficult to discuss, and might be true from the historical point of view. However we have seen that the reason for the continuous pure state space of typical physical systems described by Hamiltonian classical mechanics is the observed complexity of their evolution.

The final argument that the motion of a classical point mass along a continuous line from A through B to C establishes inequivalent relations between AB and AC fails to say anything about pure states of the classical one degree of freedom system. The pure states of such a system are points in the phase plane spanned by the coordinate and the momentum. Points A, B, C are values of the coordinate only and do not specify the pure states. Equivalent relations between all pairs of pure states are established if the pure states are described as they are, i.e. by pairs $(p, q) \in U \subset R^2$.

We see that none of the arguments presented can be used to rule out the possibility of theories with continuous spaces of pure states.

5 Discussion

The difference, stressed by Hardy, between QT and CT is that in the first $K = N^2$ and in the second $K = N$. The last equality is valid in classical applications of probability irrespective of the discrete or continuous structure of the phase space. Hardy attempts to axiomatize the difference between $K = N$ and $K = N^2$ cases by his fifth axiom. This is successful if N is discrete but fails if the space of pure states is a continuous manifold. In the case of classical mechanics of systems with finite number of degrees of freedom the space of pure states is a smooth manifold and there is a canonical continuous and invertible transformation between any two pure states. This application of classical probability theory cannot be ruled out by the fifth axiom. In Hardy's approach this case is in fact excluded not by axioms but by restricting the types of considered physical situations, i.e. by the type of classical probability theory.

If the probability theories with continuous spaces of pure states are included in the consideration than the fifth axiom does appear useless.

Hardy's reasons for excluding the continuous spaces of pure states have been discussed in Sect. 4. Hardy's main arguments are centered around the positions of what should be the properties of a fundamental theory of Nature. Theories that are not fundamental must be entirely derivable from what is currently considered as the fundamental theory. Furthermore, axiomatic foundations of all possible generalized probability theories are certainly restricted and incomplete if the existence and properties of a one final fundamental theory are presupposed. We have stressed that there are consistent theories of dynamical systems that (a) successfully model observed phenomena, and (b) the phenomena are so complex that their description and prediction within the theory require a continuous space of pure states. It is not inconceivable that there could be other processes on different levels in Nature that are so **dynamically complex** that their description requires continuous spaces of pure states. Probability theories with continuous spaces of pure states that might need to be applied in such situations should not be excluded from consideration in an axiomatization program.

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References

1. Dakić, B., Brukner, Č.: Quantum theory and beyond: is entanglement special? arXiv:0911.0695v1 [quant-ph]
2. Birkhoff, G., von Neumann, J.: The logic of quantum mechanics. Ann. Math. **37**, 823–843 (1936)
3. Mackey, G.W.: The Mathematical Foundations of Quantum Mechanics. Benjamin, New York (1963)
4. Piron, C.: Axiomatique quantique. Helv. Phys. Acta **37**, 439–468 (1964)
5. Ludwig, G.: Foundations of Quantum Mechanics. Springer, New York (1983)
6. Hardy, L.: Quantum mechanics from five reasonable axioms. arXiv:quant-ph/0101012
7. Hardy, L.: Why quantum theory? In: Butterfield, J., Placek, T. (eds.): Proceedings of the NATO Advanced Research Workshop on Modality, Probability and Bell' s Theorem, pp. 61–73. IOS, Amsterdam (2002)
8. Hardy, L.: Probability theories in general and quantum theory in particular. Sud. Hist. Phylos. Mod. Phys. **34**, 381–391 (2001)
9. Schack, R.: Quantum theory from four of Hardy's axioms. arXiv:quant-ph/0210017v1
10. Duck, I.: Discovering quantum mechanics one again. arXiv:quant-ph/0307121v1
11. Arnold, V.I.: Mathematical Methods of Classical Mechanics. Springer, New York (1978)
12. Abraham, R., Marsden, J.E.: Foundations of Mechanics. Benjamin/Cummings, New York (1980)
13. Arnol, V.I., Avez, A.: Ergodic Problems of Classical Mechanics. Benjamin, New York (1968)
14. Robinson, C.: Dynamical Systems: Stability, Symbolic Dynamics and Chaos. CRC, New York (1999)
15. MacKay, R.S., Meiss, J.D. (eds.): Hamiltonian Dynamical Systems. Hilger, Bristol (1987)
16. Burić, N., Percival, I.C.: Modular smoothing and KAM tori. Physica D **71**, 39–54 (1994)
17. Landsman, N.P.: Mathematical Topics Between Classical and Quantum Mechanics. Springer, New York (1998)
18. Ford, J., Mantica, G.: Does quantum mechanics obeys correspondence principle? Is it complete? Am. J. Phys. **60**, 1086–1098 (1992)
19. Buric, N.: Hamiltonian quantum dynamics with separability constraints. Ann. Phys. (NY) **233**, 17–33 (2008)