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Non-Classical Behavior of Atoms in an Interferometer

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Using the time-dependent wave function we have studied the properties of the atomic transverse motion in an interferometer, and the cause of the non-classical behavior of atoms reported by Kurtsiefer, Pfau, and Mlynek [Nature **386**, *150 (1997)]. The transverse wave function is derived from the solution of the two-dimensional Schrödinger's equation, written in the form of the Fresnel–Kirchhoff diffraction integral. It is assumed that the longitudinal motion is classical. Comparing data of the space distribution and of the transverse momentum distribution in interferometers with one and two open slits, it follows that the atomic motion is influenced by the atomic matter wave and violates the laws of classical mechanics. However, the negative values of Wigner's function should not be taken as evidence that the atoms in an interferometer violate the classical statistical law of the addition of positive probabilities. This inference follows from the comparison of properties of Wigner's function and of the de Broglian probability density in phase space.*

KEY WORDS: atomic interference; compatible statistical interpretation; (non)violation of the classical probability laws; Wigner's function.

1. INTRODUCTION

The wave function $\psi(x, t)$ of the transverse motion of an atom in an interferometer is a linear superposition of states with maxima at two spatially separated locations. These states lead to negative values in Wigner's function $W(x, p_x, t)$, which is the quasi-probability distribution in phase space defined by position x and momentum p_x. Kurtsiefer *et al.*[(1)] and Pfau

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and Kurtsiefer[2] reconstructed $W(x, p_x, 0)$ from the measured distribution $|\psi(x, t)|^2$ of helium atoms in a double slit interferometer.[2] The authors conclude that the motion of atoms behave in a strongly non-classical manner.

Since there are at least two aspects of non-classicality, the characterization of motion (atomic behavior) as non-classical, in our opinion, requires further specification. First, non-classical motion may denote a motion, which does not obey the laws of classical mechanics. Second, it may denote a behavior, which does not obey the classical probability laws, in particular the classical law of the addition of positive probabilities, as suggested by Leibfried *et al.*[3] The aim of our study reported in this paper is to determine, which of these two aspects is revealed by the negative values of the Wigner function.

For the reason stated above, we have written in Sec. 2 the solution of Schrödinger's equation for an atom in an interferometer. This solution is written in the form of the Fresnel–Kirchhoff diffraction integral. In Sec. 3 we have derived the time dependent wave function $\psi(x, t)$ of the transverse motion. It describes the atomic matter wave that is non-classical property. Its modulus square $|\psi(x, t)|^2$ (graphically presented also in Sec. 3) is a probability density to find an atom at point x at time t.

The transverse momentum distribution $|c(p_x)|^2$ in the state $\psi(x, t)$ we have evaluated and presented graphically in Sec. 4. Also, we explained why $|c(p_x)|^2$ is an important characteristic of the state, and how it could be used for better description of the atomic non-classical behavior.

In Sec. 5 we study the problem of existence and form of a joint probability distribution of position and momentum of atoms in an interferometer. We have evaluated and presented graphically Wigner's phase space density $W(x, p_x, t)$ and the de Broglie probability distribution $P(x, p_x, t) = |\psi(x, t)|^2 \cdot |c(p_x)|^2$. We have compared (Secs. 5 and 6) their properties and found that the function $P(x, p_x, t)$ possess the general properties of a classical statistical distribution whereas the Wigner function does not possess such properties. From this comparison follows the conclusion (stated in Sec. 6) that there is no evidence that the observed nonclassical wavelike behavior of atoms violates the classical statistical laws.

2. THE APPLICATION OF THE FRESNEL–KIRCHHOFF DIFFRACTION FORMULA

We want to determine the wave function of an atom which travels with velocity $\vec{v} = v\vec{\imath}_y = (p/m)\,\vec{\imath}_y$ through region I (see Fig. 1), towards the

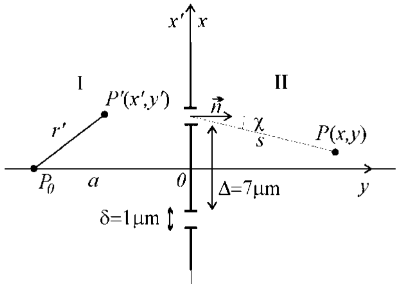

Fig. 1. Illustration of the diffraction formula presented with Eq. (7).

slits and is then sent through the slits to region II. Atom's behavior and motion is determined by its wave function, which is a solution of the time-dependent two dimensional Schrödinger equation

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Psi(x, y, t) = \mathrm{i}\hbar\frac{\partial}{\partial t}\Psi(x, y, t). \tag{1}$$

If the atomic source is far from the grating, the solution in front of the grating is a plane wave with the initial wave vector $\vec{k} = \vec{p}/\hbar$ along the longitudinal direction y. Behind the grating Tomonaga[4] invoked the approximation which is equivalent to the paraxial approximation in optics and wrote the solution in the form of a product of the longitudinal and transverse part, the former being plane wave. The transverse part was written in the form of a superposition of Gaussians (which spread in time) by Tomonaga,[4] Zurek,[5] Bonifacio, Olivares,[6] and others.

The solution of Eq. (1) may be determined also by applying Fresnel–Kirchhoff diffraction formula.[7] This possibility, exploited by Zeilinger *et al.*,[8] Kurtsiefer *et al.*,[1] Božić *et al.*[9] and others, is due to the fact that the space dependent part $\Phi(x, y)$ of the stationary solution of Eq. (1)

$$\Psi(x, y, t) = e^{-\mathrm{i}\omega t}\Phi(x, y), \tag{2}$$

where $\hbar\omega = p^2/2m$ and $p = mv = \hbar k$, satisfies the Helmoholtz equation

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi(x, y) = \hbar\omega\Phi(x, y). \tag{3}$$

The solution of Eq. (3) in region I is[7] a spherical wave

$$\Phi(P') = \Phi(x', y') = \frac{Ae^{ikr'}}{r'}, \tag{4}$$

where A is a constant and r' is the distance (Fig. 1) from the source (P_0) to the point $P' = (x', y')$ in region I. The spherical wave at the slit points $(x', y' = 0)$ may be approximated by a plane wave, since the distance a of the double-slit screen from the source P_0 is very large compared to the width of the slits. Consequently, without a loss of generality, for $\Phi(x', y' = 0)$ at the border of region I we may choose the function

$$\phi_1(x', 0) = \begin{cases} 1/\sqrt{\delta}, & -\frac{\Delta}{2} \geqslant x' \geqslant -\frac{\Delta}{2} - \delta, \\ 0, & \text{all other values of } x', \end{cases} \tag{5}$$

for one open slit, and the function

$$\phi_2(x', 0) = \begin{cases} 1/\sqrt{2\delta}, & -\frac{\Delta}{2} \geqslant x' \geqslant -\frac{\Delta}{2} - \delta \\ 1/\sqrt{2\delta}, & \frac{\Delta}{2} + \delta \geqslant x' \geqslant \frac{\Delta}{2}, \\ 0, & \text{all other values of } x' \end{cases} \tag{6}$$

for two open slits. This means that in region II the solution of Eq. (3) is given by the formula of the Fresnel–Kirchhoff diffraction[7]

$$\Phi(x, y) = -\frac{iA}{2\lambda} \frac{e^{ika}}{a} \int_{\mathscr{A}} dx' \frac{e^{iks}}{s} [1 + \cos \chi], \tag{7}$$

where $s = \sqrt{y^2 + (x' - x)^2}$, $\cos \chi = y/s$, $\lambda = 2\pi/k$, while $\mathscr{A} = \{x'; -(\Delta/2) - \delta < x' < -(\Delta/2)\}$ when the lower slit is open and upper slit is closed, and $\mathscr{A} = \{x'; (\Delta/2) < x' < (\Delta/2) + \delta$ or $-(\Delta/2) - \delta < x' < -(\Delta/2)\}$ when the two slits are open. The constant A will be chosen from the normalization condition.

The spatial distribution of the transverse degree of freedom as a function of evolution time was investigated in a double slit experiment[1,2] with metastable helium atoms. A diagram of the apparatus is shown in Fig. 2.

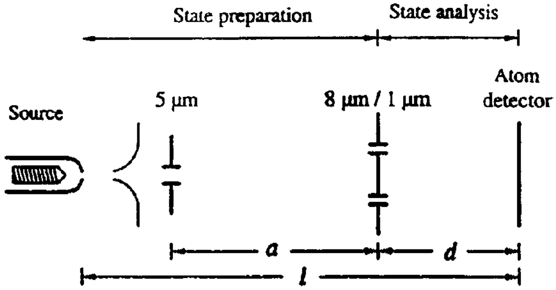

Fig. 2. Diagram of apparatus used in Ref. 1 to observe atomic interference patterns.

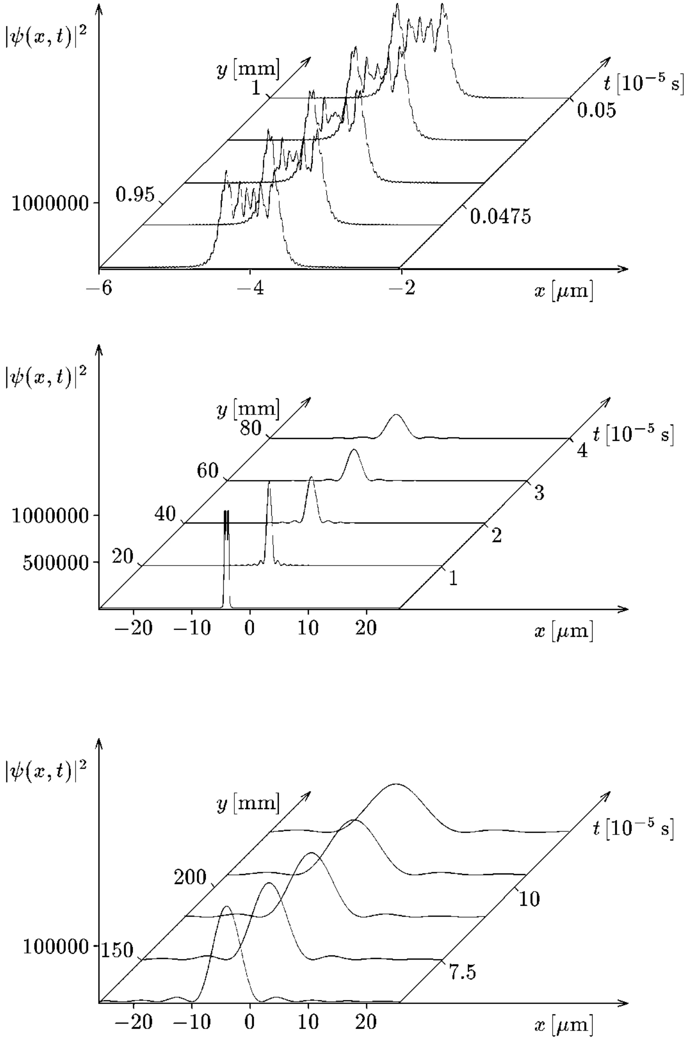

Fig. 3. The function $|\psi(x,t)|^2 \equiv |\Phi(x, y = vt)|^2/N^2$ for a single slit evaluated from Eq. (7) where $N^2 = \int |\Phi(x, y = vt)|^2 \, dx$ for a given y. Other parameters are: $k = 4\pi \cdot 10^{10}\ \mathrm{m}^{-1}$, $v = \hbar k/m = 1995.58$ m/s, $m = 6.64632 \cdot 10^{-27}$ kg is the mass of the Helium atom.

Atoms are emitted from a gas-discharge source operating in the pulse operation mode. The beam is collimated by a 5 μm-wide slit and then is sent through a double-slit structure with a slit separation $\Delta+\delta=8$ μm and an opening $\delta=1$ μm. The atoms then propagate for a distance d to a time- and space-resolving detector. Atom beam velocities lie between 1000 and 3000 ms^{-1}. We shall use the parameters of this experimental arrangement for the following calculations.

3. TIME-DEPENDENT WAVE FUNCTION OF THE TRANSVERSE MOTION

Assuming that the motion of an atom along the y-axis can be treated classically and that the transverse motion is quantized, one may use the relation $y=vt$ and determine the time dependent function of the transverse motion from the function $\Phi(x, y)$ given in (7), by the following definition

$$\Phi(x, vt)/N \equiv \psi(x, t), \tag{8}$$

where $N=\sqrt{\int |\Phi(x, vt)|^2\, dx}$. The graphs of the function $|\Phi(x, vt)|^2/N^2 \equiv |\psi(x, t)|^2$ for $k=4\pi\cdot 10^{10}\,\mathrm{m}^{-1}$ and for the chosen set of values of the coordinate y ($t=my/\hbar k$) are presented in Figs. 3 and 4.

Very close to the slit on the single-slit graphs (Fig. 3) we see the minima of the wave function for $x=x_c$, where $x_c=-4$ μm is the coordinate of the slit center. But, with increasing y, the maximum is present at $x=x_c$ for all y. This maximum becomes wider and wider with increasing y.

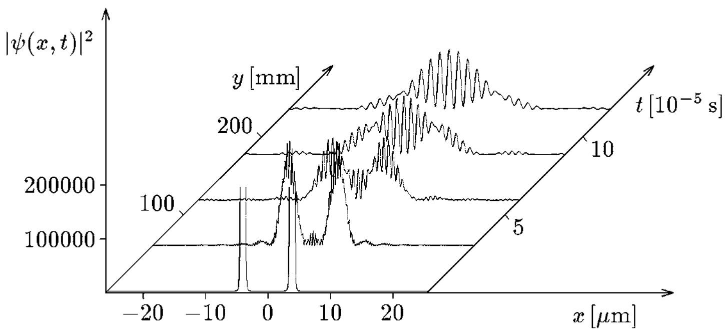

Fig. 4. The function $|\psi(x, t)|^2 \equiv |\Phi(x, y=vt)|^2/N^2$ for a double-slit evaluated from Eq. (7) where $N^2=\int |\Phi(x, y=vt)|^2\, dx$, for a given y. Other parameters are: $k=4\pi\cdot 10^{10}\,\mathrm{m}^{-1}$, $v=\hbar k/m=1995.58$ m/s, $m=6.64632\cdot 10^{-27}$ kg is the mass of the Helium atom.

On double-slit graphs (Fig. 4) we clearly see that near the slits the wave function consists of two widely separated Gaussian like maxima on which small oscillations are superimposed. With increasing distance from the slits the Gaussian-like maxima spread and start to overlap, so that the third maximum with superimposed oscillations start to develop. This region of y corresponds to Fresnel diffraction. With further increase of $y(t)$, distinct equally spaced oscillations develop, which correspond to the Fraunhofer diffraction limit.

4. THE TRANSVERSE-MOMENTUM DISTRIBUTION

The time dependent function defined by Eq. (8) should be a solution of the one-dimensional time-dependent Schrödinger's equation. Therefore, we may assume[9] that it can be written in the form

$$\psi(x,t)=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} c(p_x)\, e^{ip_x x/\hbar} e^{-i\omega_x t}\, dp_x=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} c'(k_x)\, e^{ik_x x} e^{-i\omega_x t}\, dk_x, \tag{9}$$

where $\int_{-\infty}^{\infty} |c(p_x)|^2\, dp_x=\int_{-\infty}^{\infty} |c'(k_x)|^2\, dk_x=1$, $p_x=\hbar k_x$, $c'(k_x)=\sqrt{\hbar}\, c(p_x)$ and $\hbar\omega_x=p_x^2/2m$. From Eq. (9) we may determine the transverse-momentum distribution $|c(p_x)|^2=|c'(k_x)|^2/\hbar$ in the state $\psi(x,t)$. At first, one determines

$$C(k_x,t)\equiv\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \psi(x,t)\, e^{-ik_x x}\, dx \tag{10}$$

by performing the Fourier-transform of the function $\psi(x,t)$, defined by Eq. (8), taking t as a parameter. If Eq. (9) is valid, then it should be

$$C(k_x,t)=c'(k_x)\, e^{-i\omega_x t}. \tag{11}$$

Consequently,

$$|c'(k_x)|^2=|C(k_x,t)|^2. \tag{12}$$

The graph of $|c'(k_x)|^2=\hbar|c(p_x)|^2$ for one slit is given in Fig. 5a and for two slits in Fig. 5b.

Our numerical calculation for various values of t, show that $|c'(k_x)|^2$ is independent of t. This fact justifies the assumptions of Eqs. (8) and (9) as well as the statement of Kurtsiefer, Pfau, and Mlynek[1] that the longitudinal

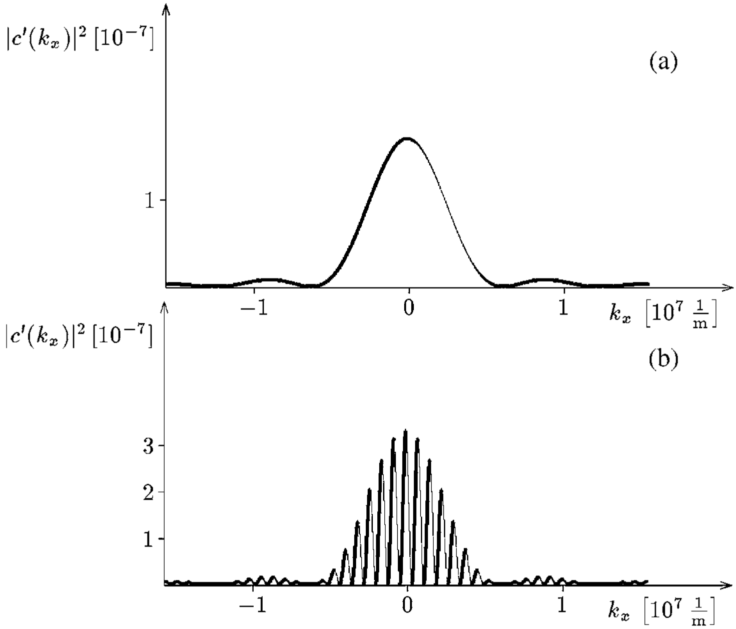

Fig. 5. Momentum distribution $|c'(k_x)|^2 = |c(p_x)|^2 \cdot \hbar$ in the state $\psi(x, t)$ with parameters given in the caption of Figs. 3 and 4. (a) One open slit; (b) two open slits.

motion of the atoms at velocities v of several thousand meters per second can be treated completely classically.

We compared also the transverse momentum distribution $|c'(k_x)|^2$ (evaluated as described above and presented in Fig. 4) with the absolute value square of the Fourier transform

$$F_i(k_x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi_i(x', 0)\, e^{-ik_x x'}\, dx' \tag{13}$$

of the function $\phi_i(x', 0)$, $i = 1, 2$. After the evaluation of the latter integral one finds

$$F_1(k_x) = \frac{i e^{ik_x \Delta/2}}{k_x \sqrt{2\pi\delta}} \{1 - e^{ik_x\delta}\},$$
$$|F_1(k_x)|^2 = \frac{2 \sin^2(k_x\delta/2)}{\pi\, \delta k_x^2} \tag{14}$$

and

$$F_2(k_x) = \frac{2}{k_x\sqrt{\pi\delta}} \sin\frac{k_x\delta}{2} \cos\frac{k_x(\Delta+\delta)}{2},$$
$$|F_2(k_x)|^2 = \frac{4}{k_x^2\pi\delta} \sin^2\frac{k_x\delta}{2} \cos^2\frac{k_x(\Delta+\delta)}{2}. \tag{15}$$

We found that $|c'(k_x)|^2$ for one slit is practically identical to $|F_1(k_x)|^2$ and that $|c'(k_x)|^2$ for two slits is practically identical to $|F_2(k_x)|^2$.

From these facts we conclude[9] that the function $|\psi(x, t)|^2$ defined by (9) and the function $|\Phi(x, y = vt)|^2/\int dx\,|\Phi(x, y = vt)|^2$ defined by (7) are equivalent. However, we have not found yet a general proof of this fact.

Far from the slits (in the Fraunhofer region) the wave function in the coordinate representation is proportional[10] to the wave function of the transverse motion in the momentum representation

$$\psi(x, t = ym/\hbar k) = \frac{\sqrt{k}}{\sqrt{y}}\, e^{-i\pi/4} e^{ikx^2/2y} c'(kx/y), \tag{16}$$

where kx/y plays the role of k_x. This relation implies that the transverse momentum distribution in the state $\psi(x, t)$ may be determined by measuring the atoms position distribution $|\psi(x, t)|^2$ in the far field.

In order to verify experimentally the independence of the momentum distribution on y would require a direct measurement of the momentum distribution as a function of y. To the best of our knowledge such a measurement has not been performed.

By comparing the spatial distributions for one and two slits shown in Figs. 3 and 4, one must conclude that the presence of the second slit influences the motion of each atom, independent of the slit through which it has passed to region II (see Fig. 1). This influence is also very well seen by comparing the momentum distributions for one and two slits, presented in Fig. 5. Certain values of the particle's transverse momentum, which are allowed with one slit, are not allowed when both slits are open. This fact is also a signature of a non-classical atomic behavior that can be understood in a similar way to the quantization of the electronic orbits in atom based on de Broglie's wavelength. It appears that the atomic matter wave excludes certain values of transverse momentum and favors others, which is an evident quantum effect.

5. STATISTICAL LAWS OF THE ATOMIC NON-CLASSICAL BEHAVIOR

As pointed out in the previous section, due to the relation (16) the transverse momentum distribution may be measured by measuring the atomic position distribution very far from the slits. In photon optics this possibility follows from the far field limit of the Fresnel–Kirchhoff integral.[11] It is hard to explain why this possibility has been much less exploited in the study of statistical features of quantum interference than the assertion about the impossibility of joint measurement of position and momentum, deduced from Heisenberg's uncertainty relation.

The interest for the joint measurement of position and momentum is related to the long standing problems of the existence and form of the joint probability density in phase space,[12–19] and in general of the missing link between quantum mechanics and probability theory.[20]

The search for a phase space distribution was initiated by Wigner.[12] Wigner attempted to determine the phase space distribution function $W(x, p_x, t)$ in the pure state $\psi(x, t)$ by imposing the following requirements:

$$W(x, p_x, t) \geqslant 0, \tag{17}$$

$$\int W(x, p_x, t)\, dp_x = |\psi(x, t)|^2, \tag{18}$$

$$\int W(x, p_x, t)\, dx = |c(p_x)|^2, \tag{19}$$

$$\int \psi^*(x, t)\, \hat{F}(\hat{x}, \hat{p}_x)\, \psi(x, t) = \int F(x, p_x)\, dx\, dp_x, \tag{20}$$

where $\hat{F}(\hat{x}, \hat{p}_x)$ is an operator associated with a physical quantity $F(x, p_x)$.

Wigner showed[12, 15] that the function

$$W(x, p_x, t) = \frac{1}{\hbar\pi} \int \mathrm{d}\tilde{x}\, e^{2\mathrm{i}p_x\tilde{x}/\hbar} \psi^*(x+\tilde{x}, t)\, \psi(x-\tilde{x}, t) = W'(x, k_x, t)/\hbar, \tag{21}$$

satisfies (18)–(20) but does not satisfy (17). Later, Wigner proved[14] that positive joint distributions, which are bilinear in the wave function, do not exist.

Margenau and Hill[19] derived the analogous conclusion valid for two arbitrary random quantities. Their conclusion is expressed in terms of covariance and correlations of two quantities: "There is, however, a convincing argument which establishes the impossibility of introducing any sensible joint probability distribution that exhibits correlations."

We would like to point out that Margenau and Hill[19] defined covariance using for the average value the expression on the left hand side of the relation (20), which is a bilinear form in the wave function. This means that the conclusion of Margenau and Hill is a generalization of Wigner's proof given in Ref. 14.

Cohen and Zaparovanny[17] and Cohen[18] proposed to abandon the requirement that the joint probability is bilinear in the wave function. This implies to abandon the requirement (20) of Wigner, as well as the definition of covariance based on the quantum mechanical average value. By abandoning the requirement (20) Cohen and Zaparovanny found[17] the whole class of positive distributions in phase space. The special case in this class is the distribution

$$P(x, p_x, t) = |\psi(x, t)|^2\, |c(p_x)|^2 = P'(x, k_x, t)/\hbar = |\psi(x, t)|^2\, |c'(k_x)|^2/\hbar, \tag{22}$$

for uncorrelated x and p_x.

The function $P(x, p_x, t)$, called the de Broglie probability density by Božić and Marić,[21–23] is the probability density for the particle to have a momentum p_x and to be at position x at time t.[22] It is always positive and satisfies both marginal conditions (18) and (19) imposed by Wigner. For operators having the form $F(\hat{x}, \hat{p}_x) = F_1(\hat{x}) + F_2(\hat{p}_x)$, the probability density $P(x, p_x, t)$ satisfies also the requirement (20).

The function $P(x, p_x, t)$ is associated with the compatible statistical interpretation (CSI) of a wave function proposed by Božić and Marić.[21] Based on the $P(x, p_x, t)$ Božić and Marić[22] explained the coherence of the characteristic modulation of the momentum distribution at the exit of a neutron interferometer found by Kaiser *et al.*[24] Božić and Arsenović[25] compared this explanation with the explanation based on Wigner's function by Lerner, Rauch, and Suda[26] and Suda.[27]

According to the CSI of a wave function, in an ensemble of particles in a pure state presented by Eq. (9), different particles may have different momenta. Recall that the probability density of p_x is $|c(p_x)|^2$. It is independent of x. However, each particle, no matter the value of its momentum p_x, is surrounded by the same wave,[22, 23] because they all are in the same state $\psi(x, t)$. A particle and a wave are two different, but compatible entities. The whole picture implies the assumption that x and p_x are uncorrelated random variables. This assumption is not in contradiction with Heisenberg's uncertainty relations because, as pointed out by Cohen and Zaparovanny,[17] "dispersions of x and p_x depend *only* on the marginal distributions and hence any proper distribution which yields the proper quantum mechanical distribution of position and momentum will yield the uncertainty relations."

Since the simultaneous measurement of a coordinate and momentum is not possible, $P(x, p_x, t)$ cannot be measured in a single experiment. However, one could experimentally determine the probability density of a coordinate x and momentum p_x in the state $\psi(x, t)$, i.e., $P(x, p_x, t)$, by measuring separately the distributions $|\psi(x, t)|^2$ and $|c(p_x)|^2$. These distributions reflect the non-classical behavior, as pointed out in the previous section.

In Figs. 6 and 7 we present the graphs of the de Broglian probability density of a coordinate x and transverse momentum p_x, $P(x, p_x, t)$, for $y = 120$ mm ($t = y/v = 6.01 \times 10^{-5}$ s) and $y = 240$ mm ($t = y/v = 12.02 \times 10^{-5}$ s). For the same values of y we present in Figs. 8 and 9 the plots of the Wigner distribution function, evaluated from the expression (21).

It is clear from Figs. 6–9 that $W(x, p_x, t)$ and $P(x, p_x, t)$ are very different functions. Consequently, from their forms and properties are derived

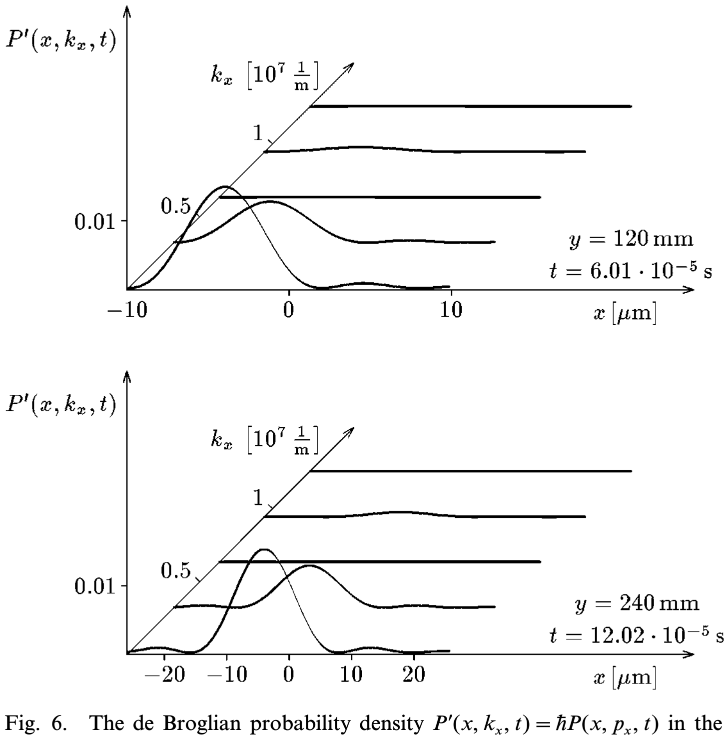

Fig. 6. The de Broglian probability density $P'(x, k_x, t) = \hbar P(x, p_x, t)$ in the single slit state $\psi(x, t)$ with parameters given in the caption of Fig. 3.

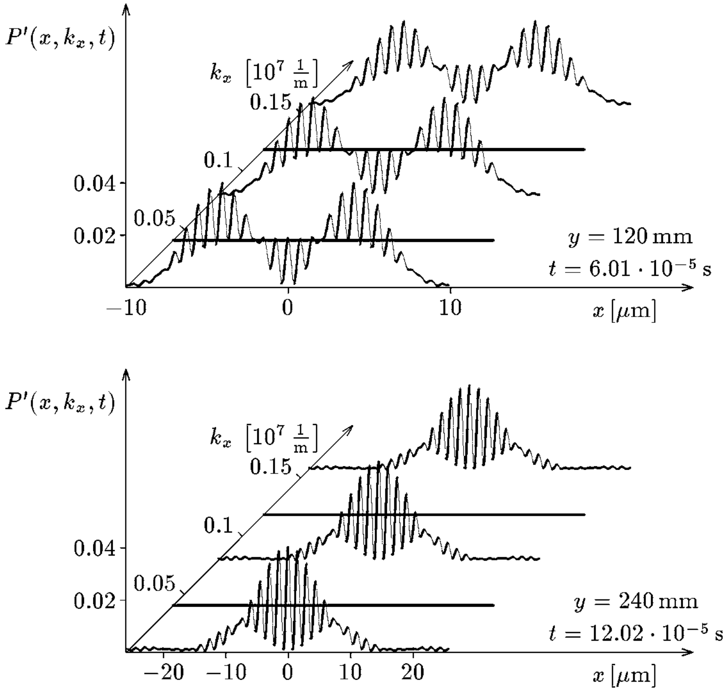

Fig. 7. The de Broglian probability density $P'(x, k_x, t) = \hbar P(x, p_x, t)$ in the double-slit state $\psi(x, t)$ with parameters given in the caption of Fig. 4.

different interpretations of the behavior of quantum particles. It was shown by Janicke and Wilkens[28], Kurtsiefer, Pfau, and Mlynek[1] that Wigner's function $W(x, p_x, 0)$ may be reconstructed from evaluated and measured values of $|\psi(x, t)|^2$ for various values of t. The negative values of $W(x, p_x, 0)$ were interpreted as a signature of violation of classical statistical laws of addition of positive probabilities.[3] These negative values are also associated with the requirement of Heisenberg's uncertainty relationship that a quantum mechanical particle has to be described by an area of uncertainty in phase space no smaller than $\Delta x \, \Delta p_x = \hbar/2$.[3] The authors also pointed out that the negative values reflect the impossibility of joint measurement of position and momentum.

However, we interpret the de Broglie probability density, presented in Figs. 6 and 7, as an objective probability density of particle coordinate and momentum. The eventual impossibility of simultaneous measurements of a particle's x and p_x does not forbid us from assuming that their joint distribution objectively exists. The important fact is that this assumption does not lead to any contradiction with the facts derived from measurable

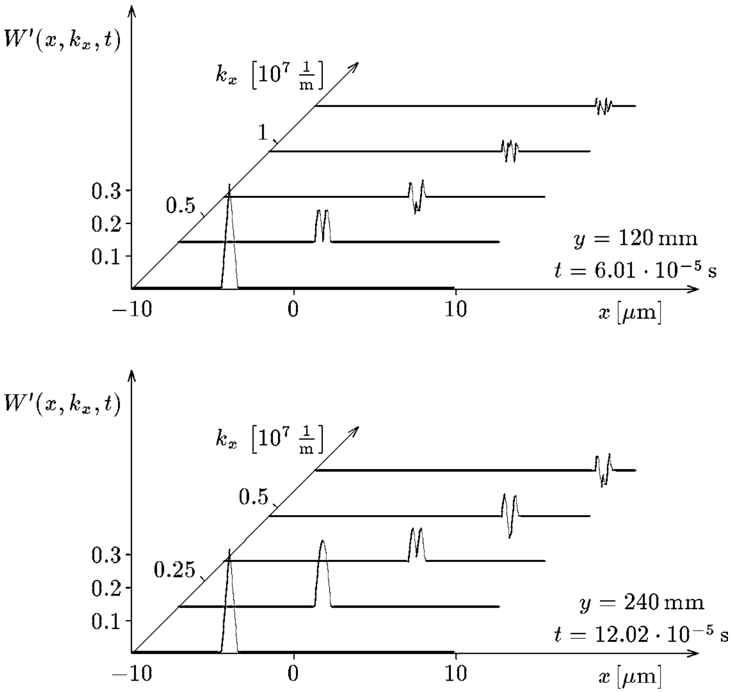

Fig. 8. Wigner's function $W'(x, k_x, t)$ associated with the single slit state $\psi(x, t)$ and evaluated from (21). Parameters are given in the caption of Fig. 3.

distributions. One can see that this joint probability density is consistent with the measurable probability density of position and the measurable probability density of momentum. For example, for values of $\tilde{p}_x$ for which $|c(\tilde{p}_x)|^2 = 0$, the joint distribution $P(x, \tilde{p}_x, t)$ is also equal to zero. Thus, if there is no particle with a certain value of momentum $\tilde{p}_x$, this value can not be found *anywhere* during the measurement of momentum. Similar reasoning is valid for space points $\tilde{x}$ in which $|\psi(\tilde{x}, t)|^2 = 0$, since $P(\tilde{x}, p_x, t)$ is also equal to zero in these space points for any value of momentum. Therefore, at a point $\tilde{x}$ *no* particle will be detected in the experiment.

One can see in Figs. 8 and 9 that Wigner's function $W(x, p_x, t)$ may take values different from zero at the points $\tilde{x}$ and $\tilde{p}$ in which either $\psi(\tilde{x}, t) = 0$ or $c(\tilde{p}_x) = 0$. Despite this property, inconsistent with a notion of a joint probability, the Wigner function satisfies the marginal conditions stated by Eqs. (18) and (19). It is well known that Wigner's function may assume negative values, even though it is a joint probability distribution by definition. Because of this, it is possible to satisfy Eqs. (18) and (19). Thus,

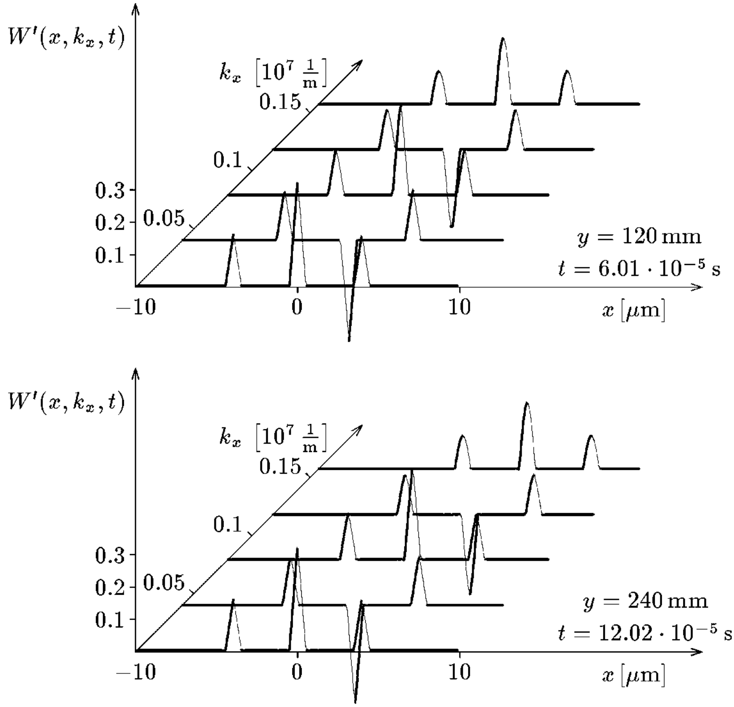

Fig. 9. Wigner's function $W'(x, k_x, t)$ associated with the double-slit state $\psi(x, t)$ and evaluated from (21). Parameters are given in the caption of Fig. 4.

two different properties of Wigner's function, inconsistent with a notion of a joint probability, cancel each other and make it possible to satisfy two marginal conditions. This is clearly seen by comparing results presented in Figs. 6 and 7 and 8 and 9. We note in Fig. 8 and 9 the negative peaks in the x-dependence of the Wigner function for those values $\tilde{p}_x$ of momentum for which $|c(\tilde{p}_x)|^2 = 0$.

6. CONCLUSION

The properties of atoms in the atomic interferometer and cause of non-classical behavior are studied using the solution of the two-dimensional Schrödinger's equation, written in the form of the Fresnel–Kirchhoff diffraction integral. The time dependent wave function of the transverse motion was derived from it.

By comparing the spatial distributions and the transverse momentum distribution for one and two open slits, we conclude that independent of the slit through which it has passed, the presence of the second slit influences the motion of each atom. This is a signature of a non-classical atom behavior reflecting nonvalidity of the laws of classical mechanics.

By taking into account the de Broglie,[29] the Bohm and Vigier[30] and the Selleri[31] understandings of wave-particle duality, we conclude that non-classical atomic behavior is due to a real atomic wave that is associated with each atom and that influences its motion. The obstacle in front of the incoming atoms determine the actual form of this influence. Therefore, the application of methods for determination of the amplitude and phase structure of the atomic wave field, similar to the method of Raymer, Beck, and McAlister,[32] would be of great importance.

By comparing the de Broglian probability density and Wigner's function in the state $\psi(x, t)$, we conclude that the negative values of Wigner's function should not be taken as a signature of a violation of classical statistical laws. This follows from the fact that the de Broglian probability density has the general properties of a statistical distribution, whereas Wigner's function does not possess such properties. Their fundamental properties are:

(B1) the De Broglian probability density is always positive.

(B2) the De Broglian distribution is consistent with the measurable probability density of position and measurable probability density of momentum.

(B3) In the phase space points (x', p'_x) in which either $|\psi(x, t)|^2$ is equal to zero or the $|c(p_x)|^2$ is equal to zero, the de Broglian probability density $P(x, p_x, t)$ is also equal to zero.

(W1) In certain phase space points Wigner's function takes negative values. So, despite the fact that it satisfies marginal conditions, it does not satisfy the classical law of addition of positive probabilities.

(W2) There exist phase space points (x', p'_x) in which $|\psi(x', t)|^2 = 0$ but $W(x', p'_x, t)$ is different from zero. There exist phase space points (x'', p''_x) in which $|c(p''_x)|^2 = 0$ but $W(x'', p''_x, t)$ is different from zero. In classical statistics this property is not allowed, because this property is contradictory to the very definition of a joint probability as a positive quantity. This property contradicts also the law of addition of positive probabilities.

Therefore, it does not follow that the motion of atoms in an interferometer violates the classical statistical laws neither from the evaluated and measured space distribution,[1,2] nor from the transverse momentum distribution evaluated in this paper, nor from the negative values of Wigner's function in the state $\psi(x, t)$. However, it follows that this motion violates the laws of classical mechanics. This is because no wave is associated with a classical particle while an atom, whose motion is governed by the Schrödinger equation, is accompanied by the matter wave.

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