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Citation: Am. J. Phys. 68, 540 (2000); doi: 10.1119/1.19481
View online: http://dx.doi.org/10.1119/1.19481
View Table of Contents: http://ajp.aapt.org/resource/1/AJPIAS/v68/i6
Published by the American Association of Physics Teachers

Extended charge in motion: Why is the Hamiltonian of a magnetic dipole $-\mathbf{m}\cdot\mathbf{B}$?

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(Received 2 November 1998; accepted 4 October 1999)

The interaction potential energy of a rigid moving mass and charge distribution (magnetic and electrical dipole in motion) in an electromagnetic field is expressed in terms of the center of mass coordinate, the rotation dyadic of a mass distribution, and magnetic and electric dipole moments.

I. INTRODUCTION

''Why is the Hamiltonian of a magnetic dipole $-\mathbf{m}\cdot\mathbf{B}$?'' is an interesting question raised by Griffiths.[1] At the same time Griffiths explained why he raised this question: ''If you integrate the energy density $\mathbf{B}_{\rm tot}^2/2\mu_0$, using $\mathbf{B}_{\rm tot}=\mathbf{B}+\mathbf{B}_d$, where $\mathbf{B}_d$ is the field of the dipole itself, you get[2] (for the interaction term) $W=\mathbf{m}\cdot\mathbf{B}$, with the 'wrong' sign.''

We read the Griffiths question just after completing the derivation of the interaction potential energy of a moving mass and charge distribution. In our derivation we used Rowe's[3] formulation of nonrelativistic equations of motion of a rigid body. Rowe applied[3] his formulation to derive the expression for the interaction potential energy of a moving mass and charge distribution in an external electrostatic field. In our derivation, which is explained in this paper, we consider the same problem but in the presence of electric and magnetic fields.

The answer to the Griffiths question follows from our derivation.

II. FORMULATION OF THE PROBLEM

We consider mass and charge distributions which are rigid and move together, i.e., we assume that relative positions of charge and mass elements do not change in time.

The configuration of the mass distribution is determined by the position vector $\mathbf{X}$ of its center of mass and a rotation dyadic R.[3,4] Here R describes the orientation of the frame $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ [attached with mass distribution (Fig. 1)], with respect to the laboratory reference frame $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$,

$$R(t)=\mathbf{e}_k(t)\mathbf{E}_k, \tag{1}$$

where the summation convention is used. The rotation dyadic satisfies

$$R\cdot\mathbf{E}_i=\mathbf{e}_i \tag{2}$$

and

$$R\cdot R^T=R^T\cdot R=1, \tag{3}$$

where

$$R^T=\mathbf{E}_k\mathbf{e}_k(t) \tag{4}$$

and

$$1=\mathbf{E}_k\mathbf{E}_k=\mathbf{e}_k(t)\mathbf{e}_k(t). \tag{5}$$

Charge distribution is described by a function $\varrho(x,t)$, which represents the density of charge when the center of mass is at the point $\mathbf{X}(t)$, and the orientation of the frame is determined by $R(t)$. To shorten the notation we shall write

$$\varrho(x,t)=\varrho_{X,R}(x)=\varrho(x),\quad x\in V(t)=V$$
$$\varrho_{X=0,R=I}(x)=\varrho_0(x),\quad x\in V_0, \tag{6}$$

where V_0 is the charge domain for $\mathbf{X}=\mathbf{0}$ and $R=I$ and V is the charge domain for arbitrary $\mathbf{X}$ and R.

The total charge we shall denote by

$$Q=\int_V\varrho(\mathbf{x})\,d\mathbf{x}=\int_{V_0}\varrho_0(\mathbf{x})\,d\mathbf{x}. \tag{7}$$

A point charge in an electromagnetic field is subject to the Lorentz force

$$\mathbf{F}=q(\mathbf{v}\times\mathbf{B}+\mathbf{E}). \tag{8}$$

With this force is associated[4] a generalized potential energy

$$U=-q\mathbf{v}\cdot\mathbf{A}+q\varphi, \tag{9}$$

so that

$$\mathbf{F}=-\frac{\partial U}{\partial\mathbf{x}}+\frac{d}{dt}\left(\frac{\partial U}{\partial\mathbf{v}}\right). \tag{10}$$

The generalization of the expression (9) to the case of a mass and charge distribution (an extended mass and charge) is straightforward. It is done by chopping an extended charge into infinitesimal pieces. Any piece dq is subject to the Lorentz force

$$d\mathbf{F}=dq(\mathbf{v}\times\mathbf{B}+\mathbf{E}). \tag{11}$$

The interaction potential energy of the charge distribution is obtained by integrating the elements of potential energy

$$dU=(-\varrho\mathbf{v}\cdot\mathbf{A}+\varrho\varphi)d\mathbf{x} \tag{12}$$

and by introducing a current density

$$\mathbf{j}=\varrho\mathbf{v}. \tag{13}$$

The result is

$$U=\int(-\mathbf{j}\cdot\mathbf{A}+\varrho\varphi)\,d\mathbf{x}. \tag{14}$$

However, when we try to generalize (10) to the case of a charge and mass distribution, we conclude that this is not so straightforward. We know that an extended charged body in an electromagnetic field is subject to a force and a torque. To find the precise expressions for these, it is necessary to write the interaction potential (14) as a function of external and internal coordinates $\mathbf{X}$ and R, respectively. This is the problem which we solve in the following two sections.

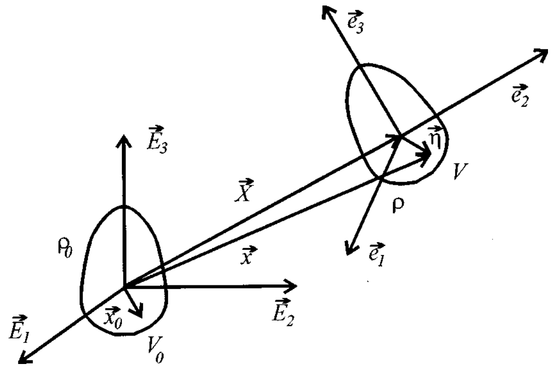

Fig. 1. The illustration of reference frames and notations.

III. INTERACTION POTENTIAL OF A MOVING CHARGE DISTRIBUTION IN AN ELECTROMAGNETIC FIELD

In order to write the integral in (14) in terms of coordinates $\mathbf{X}$ and R, we express $\mathbf{x}$ in terms of $\mathbf{X}$ and $\boldsymbol{\eta}$ (Fig. 1):

$$\mathbf{x}=\mathbf{X}+\boldsymbol{\eta}=\mathbf{X}+R\cdot\mathbf{x}_0. \tag{15}$$

From (15) it follows that

$$\mathbf{v}=\dot{\mathbf{x}}=\dot{\mathbf{X}}+\dot{R}\cdot\mathbf{x}_0=\dot{\mathbf{X}}+\dot{\boldsymbol{\eta}}. \tag{16}$$

By developing $\varphi(x)$ and $\mathbf{A}(x)$ in a Taylor series around the point $\mathbf{X}$ up to terms linear in $\boldsymbol{\eta}$, and using the relation (16), which contains the term $\dot{\boldsymbol{\eta}}$, we find

$$U=U_1+U_2+U_3+U_4+U_5+U_6, \tag{17}$$

where

$$\begin{aligned}
U_1&=-\int_V\varrho(\mathbf{x})\dot{\mathbf{X}}\cdot\mathbf{A}(\mathbf{X})\,d\mathbf{x},\\
U_2&=-\int_V\varrho(\mathbf{x})\dot{\mathbf{X}}\cdot\frac{\partial\mathbf{A}}{\partial X_i}(\mathbf{x}-\mathbf{X})_i\,d\mathbf{x},\\
U_3&=-\int_V\varrho(\mathbf{x})(\dot{\mathbf{x}}-\dot{\mathbf{X}})\cdot\mathbf{A}(\mathbf{X})\,d\mathbf{x},\\
U_4&=-\int_V\varrho(\mathbf{x})(\dot{\mathbf{x}}-\dot{\mathbf{X}})\cdot\frac{\partial\mathbf{A}}{\partial X_i}(\mathbf{x}-\mathbf{X})_i\,d\mathbf{x},\\
U_5&=\int_V\varrho(\mathbf{x})\varphi(\mathbf{X})\,d\mathbf{x},\\
U_6&=\int_V\varrho(\mathbf{x})\frac{\partial\varphi}{\partial\mathbf{X}}\cdot(\mathbf{x}-\mathbf{X})\,d\mathbf{x}.
\end{aligned} \tag{18}$$

Let us denote by $\mathbf{p}_0$ the electric dipole moment at $\mathbf{X}=\mathbf{0}$ for $R=I$:

$$\mathbf{p}_0=\int_{V_0}\varrho_0(\mathbf{x}_0)\mathbf{x}_0\,d\mathbf{x}_0. \tag{19}$$

It seems appropriate to generalize this well-known expression for the electric dipole moment of a stationary charge distribution to the case of a moving charge distribution as follows:

$$\begin{aligned}
\mathbf{p}&=\int_V\varrho(\mathbf{x})(\mathbf{x}-\mathbf{X})\,d\mathbf{x}\\
&=\int_{RV_0}\varrho_R(\boldsymbol{\eta})\boldsymbol{\eta}\,d\boldsymbol{\eta}\\
&=\int_{V_0}\varrho_0(\mathbf{x}_0)R\cdot\mathbf{x}_0\,d\mathbf{x}_0=R\cdot\mathbf{p}_0,
\end{aligned} \tag{20}$$

where

$$\varrho_R(\boldsymbol{\eta})\equiv\varrho(\mathbf{X}+\boldsymbol{\eta}). \tag{21}$$

The quantity $\mathbf{p}$ is the electrical dipole moment with respect to the center of mass at $\mathbf{X}$ for arbitrary R.

Similarly, the expression for the magnetic moment of a stationary charge distribution at $\mathbf{X}=\mathbf{0}$, for $R=I$,

$$\mathbf{m}_0=\frac{1}{2}\int_{V_0}\mathbf{x}_0\times\varrho_0(\mathbf{x}_0)\dot{\mathbf{x}}\,d\mathbf{x}_0, \tag{22}$$

is generalized to the case of a moving charge distribution as

$$\mathbf{m}=\frac{1}{2}\int_{RV_0}\boldsymbol{\eta}\times\varrho_R(\boldsymbol{\eta})\dot{\boldsymbol{\eta}}\,d\boldsymbol{\eta}. \tag{23}$$

Here $\mathbf{m}$ is the magnetic moment with respect to the center of mass at $\mathbf{X}$ for arbitrary orientation R.

Applying Green's theorem to the integral

$$\oint_S\mathbf{x}(d\mathbf{S}\cdot\mathbf{a})=\int_V(\mathbf{x}\,\mathrm{div}\,\mathbf{a}+\mathbf{a})\,dV, \tag{24}$$

which for $d\mathbf{S}\perp\mathbf{a}$ reduces to

$$\int_V\mathbf{a}\,d\mathbf{x}=-\int_V\mathbf{x}\,\mathrm{div}\,\mathbf{a}\,d\mathbf{x} \tag{25}$$

and the equation of continuity

$$\mathrm{div}\,\mathbf{j}+\frac{\partial\varrho}{\partial t}=\mathrm{div}(\varrho\mathbf{v})+\frac{\partial\varrho}{\partial t}=0, \tag{26}$$

the terms U_i are transformed (see the Appendix) into the following forms:

$$\begin{aligned}
U_1&=-Q\dot{\mathbf{X}}\cdot\mathbf{A}(\mathbf{X}),\\
U_2&=-\dot{\mathbf{X}}\cdot(\mathbf{p}\nabla)\mathbf{A},\\
U_3&=-\mathbf{A}(\mathbf{X})\cdot\dot{\mathbf{p}},\\
U_4&=-\mathbf{m}\cdot\mathbf{B}-\frac{1}{2}\frac{\partial\mathbf{A}}{\partial X_i}\cdot\int_{RV_0}\boldsymbol{\eta}\eta_i\frac{\partial}{\partial t}\varrho_R(\boldsymbol{\eta})\,d\boldsymbol{\eta},\\
U_5&=Q\varphi(\mathbf{X}),\quad U_6=\frac{\partial\varphi}{\partial\mathbf{X}}\cdot\mathbf{p}.
\end{aligned} \tag{27}$$

The total interaction potential energy U now reads

$$\begin{aligned}
U=&-Q\dot{\mathbf{X}}\cdot\mathbf{A}(\mathbf{X})-\dot{\mathbf{X}}\cdot(\mathbf{p}\nabla)\mathbf{A}-\mathbf{A}(\mathbf{X})\cdot\dot{\mathbf{p}}-\mathbf{m}\cdot\mathbf{B}\\
&+Q\varphi(\mathbf{X})+\frac{\partial\varphi}{\partial\mathbf{X}}\cdot\mathbf{p}-\frac{1}{2}\frac{\partial\mathbf{A}}{\partial X_i}\cdot\int_{RV_0}\boldsymbol{\eta}\eta_i\frac{\partial}{\partial t}\varrho_R(\boldsymbol{\eta})\,d\boldsymbol{\eta}.
\end{aligned} \tag{28}$$

Using a series of vector identities, the second, third, and sixth terms in the latter expression may be rearranged to

$$U_2+U_3+U_6=(\dot{\mathbf{X}}\times\mathbf{p})\cdot(\nabla\times\mathbf{A})-\frac{d}{dt}(\mathbf{A}\cdot\mathbf{p})-\mathbf{E}\cdot\mathbf{p}. \quad (29)$$

The term $d(\mathbf{A}\cdot\mathbf{p})/dt$ does not contribute to the equations of motion. Therefore, the interaction potential energy of a moving charge distribution in an electromagnetic field [in the linear approximation of potentials $\mathbf{A}(\mathbf{x})$ and $\varphi(\mathbf{x})$ by the potentials $\mathbf{A}(\mathbf{X})$ and $\varphi(\mathbf{X})$] takes the form

$$U=-Q\dot{\mathbf{X}}\cdot\mathbf{A}(\mathbf{X})+Q\varphi(\mathbf{X})-\mathbf{E}\cdot\mathbf{p}-\mathbf{m}\cdot\mathbf{B}-(\mathbf{p}\times\dot{\mathbf{X}})\cdot\mathbf{B}$$
$$-\frac{1}{2}\frac{\partial\mathbf{A}}{\partial X_i}\cdot\int_{RV_0}\boldsymbol{\eta}\eta_i\frac{\partial}{\partial t}\varrho_R(\boldsymbol{\eta})\,d\boldsymbol{\eta}. \quad (30)$$

The meanings of the first four terms in this expression are evident because their forms are known either from the theory of a point charge or from the theory of magnetic and electric dipoles at rest in an electromagnetic field. The first two terms represent the interaction energy of a point charge Q at $\mathbf{X}$, the third is the interaction energy of a dipole at rest, and the fourth is the interaction energy of an intrinsic magnetic moment (which is due to a rotation of a charge distribution centered at $\mathbf{X}$). The fifth term is associated with a motion of an electric dipole in a magnetic field (as distinct from the first one which is associated with a motion of a point charge in a magnetic field). It shows that a moving electrical dipole picks up[5] a magnetic dipole moment $\mathbf{m}_d=\mathbf{p}\times\dot{\mathbf{X}}$.

The interpretation of the last term is not clear because the integral in it is not expressible in terms of $\mathbf{X}$, R, $\mathbf{m}$, or $\mathbf{p}$. It contains second-order powers of η_i and we expect that it would contribute to quadrupole moments. Therefore, the interaction potential energy in the dipole approximation is determined by the first five terms:

$$U=-Q\dot{\mathbf{X}}\cdot\mathbf{A}(\mathbf{X})+Q\varphi(\mathbf{X})-\mathbf{E}\cdot\mathbf{p}-\mathbf{m}\cdot\mathbf{B}-(\mathbf{p}\times\dot{\mathbf{X}})\cdot\mathbf{B}. \quad (30')$$

As stated at the beginning, our aim is to write the interaction potential energy (14) in the form from which one could derive the generalized forces using the Lagrangian prescription. Neither with the form (30) nor with its dipole approximation (30′) have we reached this goal. This is because the fourth and sixth terms are not written as functions of generalized (collective) coordinates $\mathbf{X}$ and R and/or of the corresponding generalized velocities. In the next section we shall solve this problem for a particular charge distribution.

IV. MAGNETIC MOMENT OF A MOVING MASS AND CHARGE DISTRIBUTION

It follows from (23) that the magnetic moment of an extended charged particle, imagined as a charge and mass distribution, is associated with the current which is due to the particle's internal motion—joint instantaneous rotation of a mass and charge distribution centered at $\mathbf{X}$.

Therefore, it seems suitable to express the magnetic moment in terms of the angular velocity $\boldsymbol{\omega}$ of a body. It is defined[3,4] by the motion of the moving frame $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$:

$$\dot{\mathbf{e}}_k(t)=\boldsymbol{\omega}(t)\times\mathbf{e}_k(t), \quad k=1,2,3. \quad (31)$$

Now we may express $\dot{\boldsymbol{\eta}}$ in terms of $\boldsymbol{\omega}$, since[3]

$$\dot{R}=\dot{\mathbf{e}}_k\mathbf{E}_k=\boldsymbol{\omega}\times\mathbf{e}_k\mathbf{E}_k=\boldsymbol{\omega}\times R. \quad (32)$$

Therefore,

$$\dot{\boldsymbol{\eta}}=\dot{R}\cdot\mathbf{x}_0=\boldsymbol{\omega}\times(R\mathbf{x}_0)=\boldsymbol{\omega}\times\boldsymbol{\eta}. \quad (33)$$

By substituting the relation (33) into (23) and using the vector identity

$$\boldsymbol{\eta}\times(\boldsymbol{\omega}\times\boldsymbol{\eta})=(\boldsymbol{\eta}\cdot\boldsymbol{\eta})\boldsymbol{\omega}-(\boldsymbol{\eta}\cdot\boldsymbol{\omega})\boldsymbol{\eta}, \quad (34)$$

we find

$$\mathbf{m}=\frac{1}{2}\boldsymbol{\omega}\int_{RV_0}\varrho_R(\boldsymbol{\eta})\boldsymbol{\eta}^2\,d\boldsymbol{\eta}-\frac{1}{2}\int_{RV_0}\varrho_R(\boldsymbol{\eta})\omega\eta\cos\vartheta\,\boldsymbol{\eta}d\boldsymbol{\eta}, \quad (35)$$

where ϑ is the angle between $\boldsymbol{\omega}$ and $\boldsymbol{\eta}$. We see that the first term is proportional to $\boldsymbol{\omega}$ for any $\varrho_R(\boldsymbol{\eta})$. However, the second term is a more complicated function of $\boldsymbol{\omega}$. In the special case of a spherically symmetric charge distribution, the second term is also proportional to $\boldsymbol{\omega}$, as we now show.

For this purpose it is convenient to write the vector $\boldsymbol{\eta}$ in spherical coordinates, where the polar axis is taken to be along the angular velocity $\boldsymbol{\omega}$. In this way we find

$$\mathbf{m}=\frac{1}{2}\boldsymbol{\omega}\int_{RV_0}\varrho_R(\eta)\eta^4\,d\eta\int_0^{\pi}\sin\vartheta\,d\vartheta\int_0^{2\pi}d\varphi$$
$$-\frac{\omega}{2}\int\varrho_R(\eta)\eta^4\,d\eta\int_0^{\pi}d\vartheta\int_0^{2\pi}d\varphi\cos\vartheta$$
$$\times\sin\vartheta[\cos\vartheta\,\mathbf{i}_\omega+\sin\vartheta\cos\varphi\,\mathbf{i}_m+\sin\vartheta\sin\varphi\,\mathbf{i}_n],$$

where $\mathbf{i}_\omega$ is the unit vector along the $\boldsymbol{\omega}$ axis and $\mathbf{i}_m$ and $\mathbf{i}_n$ are two mutually orthogonal vectors in the plane normal to $\boldsymbol{\omega}$. After integration we get

$$\mathbf{m}=\boldsymbol{\omega}2\pi\int\varrho_R(\eta)\,d\eta\,\eta^4-\boldsymbol{\omega}\frac{2\pi}{3}\int\varrho_R(\eta)\,d\eta\,\eta^4,$$

i.e.,

$$\mathbf{m}=\boldsymbol{\omega}\frac{4\pi}{3}\int\varrho_R(\eta)\,d\eta\,\eta^4\equiv\kappa\boldsymbol{\omega}. \quad (36)$$

The constant κ is related to the gyromagnetic ratio γ by $\kappa=\gamma I$.

The sixth term in (30) is equal to zero in the case of a spherical charge distribution because $(\partial/\partial t)\varrho_R(\boldsymbol{\eta})=0$ in this case.

Therefore, it follows from (14) and (30′) that the interaction potential energy of the moving *spherical* charge distribution (whose $\mathbf{p}=\mathbf{0}$) in an electromagnetic field (in the dipole approximation) is the following function of the coordinates $\mathbf{X}$ and R and of the angular velocity $\boldsymbol{\omega}$:

$$U=-Q\dot{\mathbf{X}}\cdot\mathbf{A}(\mathbf{X})+Q\varphi(\mathbf{X})-\kappa\boldsymbol{\omega}\cdot\mathbf{B}. \quad (37)$$

The spherical rotator with a magnetic moment proportional to its angular momentum ($\mathbf{m}=\gamma I\boldsymbol{\omega}=\kappa\boldsymbol{\omega}$) was named the "magnetic top" by Barut *et al.*[6]

The interaction potential energy of the magnetic top with moving center of mass contains two parts:

$$U_{\text{ext}}=-Q\dot{\mathbf{X}}\cdot\mathbf{A}(\mathbf{X})+Q\varphi(\mathbf{X}), \quad (38a)$$

$$U_{\text{int}}=-\kappa\boldsymbol{\omega}\cdot\mathbf{B}. \quad (38b)$$

Here U_{ext} depends on the CM position and velocity and U_{int} is determined by the internal motion.

Consequently, the Lagrangian of the moving magnetic top is given by

$$L = L_{\text{ext}} + L_{\text{int}}, \tag{39}$$

where

$$L_{\text{ext}} = T_{\text{tr}} - U_{\text{ext}} = \frac{m\dot{\mathbf{X}}^2}{2} + Q\dot{\mathbf{X}}\cdot\mathbf{A}(\mathbf{X}) - Q\varphi(\mathbf{X}), \tag{40a}$$

$$L_{\text{int}} = T_{\text{rot}} - U_{\text{int}} = \frac{I\boldsymbol{\omega}^2}{2} + \kappa\boldsymbol{\omega}\cdot\mathbf{B}. \tag{40b}$$

Evidently, in a homogeneous field the external and the internal motion of the magnetic top are mutually independent and may be studied separately. In an inhomogeneous field the internal and external motion are coupled (mutually dependent).

The classical internal motion of the magnetic top in a homogeneous magnetic field was extensively studied by Barut *et al.*[6] and Arsenović *et al.*,[7] using Euler's angles as the internal coordinates, and by Marić *et al.*,[8] using spinors as the internal coordinates.

The external motion in a homogeneous electromagnetic field is the standard motion under the Lorentz force.

The equations of the internal motion may be written using any one of the following methods: (i) Lagrange's equations for the independent orientation coordinates,[4] (ii) Lagrange's equations for systems with constraints[4] (method of Lagrange multipliers), (iii) the variational method developed by Rowe[3] for the system described by the rotation dyadic—the set of coordinates which satisfy certain conditions (constraints).

By choosing Euler's angles φ, χ, ϑ for three independent orientation coordinates Barut *et al.*[6] applied method (i) and wrote Lagrange's equations of motion for the magnetic top:

$$\frac{d}{dt} I[\dot{\varphi} + \dot{\chi}\cos\vartheta] = F_{\varphi} = 0,$$

$$\frac{d}{dt} I[\dot{\chi} + \dot{\varphi}\cos\vartheta] = F_{\chi} = \gamma BI \sin\vartheta\cdot\dot{\vartheta}, \tag{41}$$

$$\frac{d}{dt} I\dot{\vartheta} + I\dot{\chi}\dot{\varphi}\sin\vartheta = F_{\vartheta} = -\gamma BI\sin\vartheta\cdot\dot{\chi}.$$

Here F_{φ}, F_{χ}, and F_{ϑ} are generalized forces associated with Euler's angles φ, χ, and ϑ:

$$F_{\alpha} = -\frac{\partial U_{\text{int}}}{\partial q_{\alpha}} + \frac{d}{dt}\left(\frac{\partial U_{\text{int}}}{\partial \dot{q}_{\alpha}}\right), \quad \alpha = \varphi, \chi, \vartheta. \tag{42}$$

By choosing the complex components of Cartan's spinor for the set of four orientation coordinates which satisfy one constraint, Marić *et al.*[8] applied method (ii). Both approaches[6–8] lead to the same equation

$$\frac{d\vec{s}}{dt} = \gamma\mathbf{s}\times\mathbf{B} \tag{43}$$

for the canonical spin angular momentum, defined[6] by

$$\mathbf{s} = I\boldsymbol{\omega} + \gamma I\mathbf{B}. \tag{44}$$

The equation (43) of spin motion is the so-called torque equation,[4] because on its right-hand side is the torque

$$\mathbf{T} = \gamma\mathbf{s}\times\mathbf{B}. \tag{45}$$

It follows from these studies[5–7] that the magnetic top is an appropriate classical model of a particle with spin.

A magnet in the form of a spinning top is called a magnetic top by Simon *et al.*[9] and a levitron by Hones[10] and Berry.[11] The magnetic top of Barut *et al.*[6] and the magnetic top of Simon *et al.*[9] have in common that the magnetic moment and angular momentum point in the same direction. However, the former has a magnetic moment which is due to the rotation of its charge distribution, whereas the latter possesses at rest an intrinsic magnetic moment, it is uncharged, and it is made of a ferromagnetic material. Its spinning in the direction of its magnetic moment is induced by an external torque.

V. SUMMARY

The integral expression (14) for the interaction potential energy of a moving charge distribution in an electromagnetic field is transformed into the sum of six terms [Eq. (30)], using the linear approximation of potentials $\mathbf{A}(\mathbf{x})$ and $\varphi(\mathbf{x})$ by the potentials $\mathbf{A}(\mathbf{X})$ and $\varphi(\mathbf{X})$. (It is assumed that the relative positions of masses and charges do not change during a motion and that the structure is rigid.)

The first five terms have a clear physical meaning and their sum (30′) represents the interaction potential energy in the dipole approximation. The first two terms represent the interaction energy of a point charge Q at $\mathbf{X}$, the third is the interaction energy of a dipole at rest, and the fourth is the interaction energy of an intrinsic magnetic moment (which is due to the rotation of a charge distribution at point $\mathbf{X}$). The fifth one is associated with the motion of an electric dipole in a magnetic field (as distinct from the first one which is associated with a motion of a point charge in a magnetic field).

The sixth term contains second-order powers of η_i. Therefore, it would contribute to the quadrupole moments together with other terms which would appear in the second-order approximation of potentials $\mathbf{A}(\mathbf{x})$ and $\varphi(\mathbf{x})$.

It is important to note that, in this approach, the magnetic moment of a particle is associated with an internal current which is due to a joint internal motion (rotation) of a mass and a charge. The magnetic moment $\mathbf{m}$ is due to a time-varying current associated with a rotation of the charge. In our opinion, this classical model is more appropriate for a (charged) particle with spin than the model based on a (macroscopic) current loop. The relative motion of a charge and mass is an important characteristic of the loop.

VI. THE ANSWER TO QUESTION #66

From the results in this paper, an answer follows to question #66 raised by Griffiths:[1] "Why is the Hamiltonian of a magnetic dipole $-\mathbf{m}\cdot\mathbf{B}$?" The question was raised because the interaction term of a magnetic dipole derived in the field theory (by the reasoning[1,2] quoted at the beginning of this article) has a positive sign. Our answer contains two parts.

The first part of our answer is as follows: From the Lorentz force law and from the dipole approximation (30′) of the interaction potential energy of an extended charge in motion (14), it follows that the term $-\mathbf{m}\cdot\mathbf{B}$, which represents the interaction energy of the intrinsic magnetic moment in an electromagnetic field, has a negative sign.

In the field theory, the plus sign arises because the interaction energy is defined by the integral[2]

$$W_{\rm int}=\frac{1}{\mu_0}\int \mathbf{B}_d\cdot\mathbf{B}\,d\mathbf{x}, \tag{46}$$

where $\mathbf{B}_d$ is the field of the dipole itself. Using the relations

$$\mathbf{B}=\nabla\times\mathbf{A},\quad \nabla\times\mathbf{B}_d=\mu_0\mathbf{j}, \tag{47}$$

the integral (46) is then transformed to

$$W_{\rm int}=\frac{1}{\mu_0}\int(\nabla\times\mathbf{B}_d)\cdot\mathbf{A}\,d\mathbf{x}=\int\mathbf{j}\cdot\mathbf{A}\,d\mathbf{x}. \tag{48}$$

If we compare the latter integral with the first integral in (14) we conclude that they differ only in sign. Therefore, this shows that the $\mathbf{m}\cdot\mathbf{B}$ term, derived in the field theory from (46), has $+$ sign and why the $-\mathbf{m}\cdot\mathbf{B}$ term, derived in mechanics from (14), has $-$ sign.

The second part of our answer to question #66 deals with the construction of the Hamiltonian from the Lagrangian (39), of an extended charge in motion. The interaction potential energy (30) is dependent both on external and internal velocities, which means that the Hamiltonian of an extended charge in motion is not equal to the sum of kinetic and potential energy. Its form has to be determined from the explicit dependence of the interaction energy on velocities and coordinates.

In the case of a magnetic top, this problem was solved by Barut *et al.*[6] The Hamiltonian associated with the Lagrangian (40) of a magnetic top reads

$$H=\frac{m\dot{\mathbf{X}}^2}{2}+\frac{I\boldsymbol{\omega}^2}{2}=H_{\rm ext}+H_{\rm int}=\frac{(\mathbf{p}-Q\mathbf{A})^2}{2m}+\frac{(\mathbf{s}-\kappa\mathbf{B})^2}{2I}, \tag{49}$$

where $\mathbf{p}$ is the canonical momentum $\mathbf{p}=m\dot{\mathbf{X}}+Q\mathbf{A}$ and $\mathbf{s}$ is the canonical angular momentum-spin, given by (44).

The internal Hamiltonian may be written in the form

$$H_{\rm int}=\frac{(\mathbf{s}-\kappa\mathbf{B})^2}{2I}=\frac{\mathbf{s}^2}{2I}-\boldsymbol{\mu}\cdot\mathbf{B}+\frac{\kappa^2B^2}{2I}, \tag{50}$$

where

$$\boldsymbol{\mu}\equiv\gamma\mathbf{s}=\mathbf{m}+\gamma\kappa\mathbf{B} \tag{51}$$

is the canonical magnetic moment. Here $H_{\rm int}$ contains three terms: the first is the constant of motion,[6–8] and the third is constant in the case of a homogeneous time-independent field. The second term has the same form as the Hamiltonian of a magnetic dipole in quantum theory (Pauli term). However, we point out that $\vec{\mu}$ in (50) is a canonical magnetic moment related to the magnetic moment $\mathbf{m}$ by the relation (51).

The results obtained in this paper, together with the results of Barut *et al.*,[6] show that the explanation of the linear relation between magnetic moment and spin, postulated in quantum theory, could be based on the classical electrodynamics of an extended charge in motion.

ACKNOWLEDGMENTS

One of us (M.B.) is grateful to Professor Martin Rivas for a discussion about spin models which we had in Wien during the Fifth Wigner Symposium, and for later correspondence. This discussion initiated the present work. Both authors are grateful to the referee of this article, whose questions and suggestions have been extensively used in writing the final version of this paper. This work is supported by Ministry of Science and Technology of Republic Serbia, under Contract No. 01M01.

APPENDIX: COMPACT FORM OF THE POTENTIAL TERMS

The transformations of the terms U_i defined in (18) into the forms (27):

$$U_1=-\int_V\varrho(\mathbf{x})\dot{\mathbf{X}}\cdot\mathbf{A}(\mathbf{X})\,d\mathbf{x}=-\dot{\mathbf{X}}\cdot\mathbf{A}\int_V\varrho(\mathbf{x})\,dV=-Q\dot{\mathbf{X}}\cdot\mathbf{A}(\mathbf{X}),$$

$$U_2=-\int_V\varrho(\mathbf{x})\dot{\mathbf{X}}\cdot\frac{\partial\mathbf{A}}{\partial X_i}(\mathbf{x}-\mathbf{X})_i\,d\mathbf{x}=-\int_V\varrho(\mathbf{x})\dot{\mathbf{X}}\cdot\frac{\partial\mathbf{A}}{\partial X_i}(R\cdot\mathbf{x}_0)_i\,d\mathbf{x}=-\dot{\mathbf{X}}\cdot\frac{\partial\mathbf{A}}{\partial X_i}\int_V\varrho(\mathbf{x})(R\cdot\mathbf{x}_0)_i\,d\mathbf{x}$$
$$=-\dot{\mathbf{X}}\cdot\frac{\partial\mathbf{A}}{\partial X_i}\left[R\cdot\int_{V_0}\varrho_0(\mathbf{x}_0)\mathbf{x}_0\,d\mathbf{x}_0\right]_i=-\dot{\mathbf{X}}\cdot\frac{\partial\mathbf{A}}{\partial X_i}p_i=-\dot{\mathbf{X}}\cdot(\mathbf{p}\nabla)\mathbf{A},$$

$$U_3=-\int_V\varrho(\mathbf{x})(\dot{\mathbf{x}}-\dot{\mathbf{X}})\cdot\mathbf{A}(\mathbf{X})\,d\mathbf{x}=-\int_V\varrho(\mathbf{x})(\dot{R}\cdot\mathbf{x}_0)\cdot\mathbf{A}(\mathbf{X})\,d\mathbf{x}=-\mathbf{A}(\mathbf{X})\cdot\int_V\varrho(\mathbf{x})(\dot{R}\cdot\mathbf{x}_0)\,d\mathbf{x}$$
$$=-\mathbf{A}(\mathbf{X})\cdot\dot{R}\cdot\mathbf{p}_0=-\mathbf{A}(\mathbf{X})\cdot\dot{\mathbf{p}},$$

$$U_4=-\int_V \varrho(\mathbf{x})(\dot{\mathbf{x}}-\dot{\mathbf{X}})\cdot\frac{\partial \mathbf{A}}{\partial X_i}(\mathbf{x}-\mathbf{X})_i\,d\mathbf{x}=-\int_V \varrho(\mathbf{x})(\dot{R}\cdot\mathbf{x}_0)\cdot\frac{\partial \mathbf{A}}{\partial X_i}(R\cdot\mathbf{x}_0)_i\,d\mathbf{x}=\frac{\partial \mathbf{A}}{\partial X_i}\cdot\int_{RV_0}\boldsymbol{\eta}\,\mathrm{div}\,(\varrho_R(\boldsymbol{\eta})\dot{\boldsymbol{\eta}}\eta_i)\,d\boldsymbol{\eta}$$

$$=\frac{\partial \mathbf{A}}{\partial X_i}\cdot\int_{RV_0}\boldsymbol{\eta}[\varrho_R(\boldsymbol{\eta})\dot{\boldsymbol{\eta}}\cdot\mathrm{grad}\,\eta_i+\eta_i\,\mathrm{div}\,(\varrho_R(\boldsymbol{\eta})\dot{\boldsymbol{\eta}})]\,d\boldsymbol{\eta}$$

$$=-\frac{1}{2}\frac{\partial \mathbf{A}}{\partial X_i}\cdot\int_{RV_0}\{\varrho_R(\boldsymbol{\eta})\dot{\boldsymbol{\eta}}(\boldsymbol{\eta}\cdot\mathbf{E}_i)-\boldsymbol{\eta}(\varrho_R(\boldsymbol{\eta})\dot{\boldsymbol{\eta}}\cdot\mathbf{E}_i)-\boldsymbol{\eta}\eta_i\,\mathrm{div}\,(\varrho_R(\boldsymbol{\eta})\dot{\boldsymbol{\eta}})\}\,d\boldsymbol{\eta}$$

$$=-\frac{\partial \mathbf{A}}{\partial X_i}\cdot\int_{RV_0}\frac{1}{2}(\boldsymbol{\eta}\times\varrho_R(\boldsymbol{\eta})\dot{\boldsymbol{\eta}})\mathbf{E}_i\,d\boldsymbol{\eta}+\frac{1}{2}\frac{\partial \mathbf{A}}{\partial X_i}\cdot\int_{RV_0}\boldsymbol{\eta}\eta_i\,\mathrm{div}\,(\varrho_R(\boldsymbol{\eta})\dot{\boldsymbol{\eta}})\,d\boldsymbol{\eta}$$

$$=-\frac{\partial \mathbf{A}}{\partial X_i}\cdot(\mathbf{m}\times\mathbf{E}_i)+\frac{1}{2}\frac{\partial \mathbf{A}}{\partial X_i}\cdot\int_{RV_0}\boldsymbol{\eta}\eta_i\,\mathrm{div}\,(\varrho_R(\boldsymbol{\eta})\dot{\eta})\,d\boldsymbol{\eta}$$

$$=-\mathbf{m}\cdot\mathrm{rot}\,\mathbf{A}-\frac{1}{2}\frac{\partial \mathbf{A}}{\partial X_i}\cdot\int_{RV_0}\boldsymbol{\eta}\eta_i\frac{\partial}{\partial t}\varrho_R(\boldsymbol{\eta})\,d\boldsymbol{\eta}$$

$$=-\mathbf{m}\cdot\mathbf{B}-\frac{1}{2}\frac{\partial \mathbf{A}}{\partial X_i}\cdot\int_{RV_0}\boldsymbol{\eta}\eta_i\frac{\partial}{\partial t}\varrho_R(\boldsymbol{\eta})\,d\boldsymbol{\eta},$$

$$U_5=\int_V \varrho(\mathbf{x})\varphi(\mathbf{X})\,d\mathbf{x}=\varphi(\mathbf{X})\int_V \varrho(\mathbf{x})\,d\mathbf{x}=Q\varphi(\mathbf{X}),$$

$$U_6=\int_V \varrho(\mathbf{x})\frac{\partial \varphi}{\partial \mathbf{X}}(\mathbf{x}-\mathbf{X})\,d\mathbf{x}=\int_V \varrho(\mathbf{x})\frac{\partial \varphi}{\partial \mathbf{X}}(R\cdot\mathbf{x}_0)\,d\mathbf{x}=\frac{\partial \varphi}{\partial \mathbf{X}}\cdot\int \varrho(\mathbf{x})(R\cdot\mathbf{x}_0)\,d\mathbf{x}=\frac{\partial \varphi}{\partial \mathbf{X}}\cdot\mathbf{p}.$$

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[5]We are grateful to the referee for this remark as well as for his justification of the remark based on the example of two charges $\pm q$ moving at speed v in the x direction for a time t. With $+q$ at $y=d$ and $-q$ at $y=0$, the effective current around the ''loop'' is $i=q/t$ and the area is $a=d(vt)$, so the magnetic dipole moment is $m_d=Ia=(qd)v=pv$, where $p=qd$ is the electric dipole moment. Putting in the direction $\mathbf{m}_d=\mathbf{p}\times\mathbf{v}$ we find precisely the form that emerges in the fifth term of Eq. (30).

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DEMONSTRABLE LIES

Since history always involved too many versions and misunderstandings, it had never had much attraction for T. He preferred the demonstrable truths of mathematics and the demonstrable lies of quantum physics that sometimes hid stupendous truths.

Gore Vidal, *The Smithsonian Institution—A Novel* (Random House, New York, 1998), p. 86.