

## FRACTIONAL QUANTUM HALL STATES WITH NEGATIVE FLUX: EDGE MODES IN SOME ABELIAN AND NON-ABELIAN CASES

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We investigate the structure of gapless edge modes propagating at the boundary of some fractional quantum Hall states. We show how to deduce explicit trial wavefunctions from the knowledge of the effective theory governing the edge modes. In general, quantum Hall states have many edge states. Here, we discuss the case of fractions having only two such modes. The case of spin-polarized and spin-singlet states at filling fraction  $\nu = 2/5$  is considered. We give an explicit description of the decoupled charged and neutral modes. Then we discuss the situation involving negative flux acting on the composite fermions. This happens notably for the filling factor  $\nu = 2/3$  which supports two counterpropagating modes. Microscopic wavefunctions for spin-polarized and spin-singlet states at this filling factor are given. Finally, we present an analysis of the edge structure of a non-Abelian state involving also negative flux. Counterpropagating modes involve, in all cases, explicit derivative operators diminishing the angular momentum of the system.

*Keywords:* Fractional quantum Hall states; edge modes; composite fermions.

### 1. Introduction

Electrons confined in a plane and subjected to a quantizing magnetic field may display under appropriate circumstances, the fractional quantum Hall effect (FQHE). These FQHE states are gapped liquids without long-range order with unconventional properties like fractionally charged quasiparticle excitations. Our understanding of this phenomenon is largely based on explicit first-quantized many-body wavefunctions. Historically, the first of these wavefunctions was introduced by Laughlin<sup>1</sup> to describe electrons when the magnetic field is tuned so that the lowest Landau level (LLL) has a  $1/3$  filling. It was soon discovered that these liquids also form for other filling fractions, including the sequence of filling factors  $\nu = p/(2p + 1)$  which is experimentally prominent. Some of these states can be

described accurately in the framework of so-called composite fermions (CF).<sup>2</sup> In this scheme, one considers that an even number  $2p$  of fictitious flux tubes is attached to each electron, leading to a composite object called a CF. This implies that the CF now feel a reduced flux which is equal to  $B_{\text{eff}} = B - 2pn\phi_0$  where  $\phi_0 = hc/e$  is the flux quantum and  $n$  the electron density. Since the magnetic field is reduced, the degeneracy of the Landau levels also changes and there are magic fillings at which an integer number of Landau levels of CFs are filled. We then expect formation of a FQHE state. This heuristic scheme allows construction of highly precise microscopic wavefunctions for many of the quantum Hall states.<sup>3-5</sup> It also nicely explains why there is a compressible state at filling  $\nu = 1/2$  where  $B_{\text{eff}} = 0$  and the ground state is essentially a Fermi sea of CFs (albeit interacting).

The simple CF states, however, do not explain all FQHE states observed so far. The most studied exception is the fraction  $\nu = 5/2$  observed in the second orbital Landau level. Due to the energy scales of the problem it is reasonable to write this fraction as  $\nu = 2 + 1/2$  and to consider that there is an essentially inert background of electrons fully occupying the LLL with filling  $\nu = 2$  and, on top of it, a half-filled Landau level with interactions between electrons renormalized by the presence of the background. So while this is again a half-filled Landau level with zero effective magnetic field acting upon the CFs, the interaction has been changed with respect to the LLL case. This change of interaction is observed to destroy the CF Fermi sea which is an apt description of  $\nu = 1/2$  in the LLL and lead to formation of an incompressible state which is of a different kind of those already observed. This picture assumes complete spin polarization of electrons which seems to be the case at  $\nu = 5/2$ . The best wavefunction candidate to describe this new state so far is the so-called Moore–Read Pfaffian state.<sup>6,7</sup> This is a microscopic wavefunction that includes some kind of pairing and that has been constructed from correlation functions of operators that belong to a simple two-dimensional conformal field theory. Recent experiments have given evidence for many new fractions that do not fit easily in existing CF scheme.<sup>8,9</sup> These include filling factors  $\nu = 4/11, 5/13, 4/13, 6/17, 5/17$  and there is some weak evidence for some even-denominator states at  $\nu = 3/10$  and  $3/8$ . The state at filling  $3/8$  is also observed in the second orbital Landau level, i.e., at  $\nu = 2 + 3/8$ . When the filling factor is an odd-denominator fraction it is plausible to argue that a hierarchical mechanism *à la* Halperin–Haldane is at work. For example, the filling  $\nu = 4/11$  corresponds to an effective filling factor  $1 + 1/3$  for CFs and the pseudo-Landau level at filling  $1/3$  may also form a conventional Laughlin liquid. However, as in all hierarchical constructions for fermionic constituents, there is no room for even-denominator states.

There is an interesting family of wavefunctions generalizing the Pfaffian state that are called the Read–Rezayi states.<sup>10,11</sup> When written for elementary bosonic particles, they are given by an explicit polynomial that vanishes when  $k + 1$  particles are at the same point in space. For  $k = 1$ , one finds simply the Laughlin wavefunction for bosons at  $\nu = 1/2$  for  $k = 2$ , one has the (bosonic) Moore–Read

Pfaffian state with filling factor  $\nu = 1$ . Other members of this series have  $\nu = k/2$ . Multiplication by an antisymmetric Jastrow factor leads to fermionic candidate states at  $\nu = k/(k + 2)$ . If we allow an arbitrary odd power  $M$  of the Jastrow factor, this can be extended to a family of states at  $\nu = k/(Mk + 2)$ . It has been noted<sup>12,13</sup> that such states may be candidates even in the case where the effective CF flux  $B_{\text{eff}}$  is *negative*. This leads then to a generalized family of candidate states at  $\nu = k/(3k \pm 2)$  which, surprisingly, includes all the new fractions. There is even some limited evidence from numerical diagonalization of small systems of electrons that these states have to do with the true ground state of electrons in the LLL. While these new candidates are given by explicit formulas, there are some technicalities that prevent immediate analysis. First, the formulas involve an extensive number of derivatives due to a projection onto the LLL, rendering analytical manipulations difficult beyond  $N = 6$  particles. Then there is no Hamiltonian whose ground state reproduces these wavefunctions, contrary to the Read–Rezayi family. This precludes straightforward counting of quasiparticle states or edge modes. It is known that the effective theory describing the low-energy physics of a given quantum Hall state is encoded into the edge mode structure. For a generic hierarchical state this edge structure is intricate and involves several fields. However, we need to understand the edge structure in order to find this effective theory.

In this paper, we construct explicit wavefunctions describing the edge structure of FQHE states involving negative flux in the CF sense. In the case of the Laughlin state, it is well-known that the one-quasihole wavefunction can be used as the generating function for the edge modes. We use a similar construction for the conventional fully spin-polarized CF state at filling  $\nu = 2/5$ . The two kinds of quasiholes leads then to two copropagating edge modes. There is also a similar picture in the case of the spin-singlet state which can be constructed also at the same filling by coupling spin-1/2 quasiholes edge modes. The simplest example of a state with negative flux is the fraction  $\nu = 2/3$ . While it can be considered as the particle–hole symmetric of the Laughlin state at  $\nu = 1/3$ , it can also be viewed as two filled pseudo Landau levels of CF in a negative effective field. Wavefunctions constructed along this line of thought are as successful as those with positive flux.<sup>14</sup> We give an explicit construction of the two counterpropagating modes from a quasihole construction. The edge modes that propagate in the “wrong” direction involve derivative operators instead of the symmetric polynomials that generate the global charge mode. We next apply this construction to the simplest non-Abelian state with negative flux which a Pfaffian state with bosonic filling  $\nu = 1$ , but has a relation between flux and number of particles different from the conventional Pfaffian state. The edge theory is now given by a bosonic mode — the charge mode — and a Majorana fermion that moves in the opposite direction.

In Sec. 2, we discuss the appearance of negative flux in the CF framework. Section 3 is devoted to the study of Abelian states with positive CF flux at filling factor  $\nu = 2/5$ , both spin-polarized and spin-singlet. In Sec. 4, we discuss the

simplest Abelian state with negative flux, the fraction  $\nu = 2/3$ . Then we apply the formalism developed in this section to the case of the Pfaffian with positive and negative flux state in Sec. 5. Our conclusions are given in Sec. 6.

## 2. Composite Fermions and Negative Flux

In this section, we discuss the appearance of negative flux states within the CF scheme. We consider states of two-dimensional electrons in the LLL. If we use the symmetric gauge, then the one-body orbitals are given by:

$$\phi_m(z) = \frac{1}{\sqrt{2\pi m!} 2^m} e^{-|z|^2/4}, \quad (1)$$

where  $z = x + iy$  is the complex coordinate in the plane and the positive integer  $m$  gives the angular momentum of the state:  $L_z = m\hbar$  (we have set the magnetic length to unity). A general  $N$ -body LLL quantum state can thus be written as:

$$\Psi(z_1, \dots, z_N) = f(z_1, \dots, z_N) e^{-\sum_i |z_i|^2/4}. \quad (2)$$

In the remainder of the paper, we will always omit the (universal) exponential factor. In an arbitrary Landau level, the one-body eigenstates now involve both  $z$  and  $z^*$ . A completely filled LLL state,  $\nu = 1$ , is the Slater determinant obtained by filling all orbitals, Eq. (1), up to some maximum  $m$  value. This (Vandermonde) determinant has the following closed form:

$$\Psi_{\nu=1}(\{z_i\}) = \prod_{i<j} (z_i - z_j). \quad (3)$$

This special antisymmetric product will be referred to as a Jastrow factor in the paper. In the CF construction,<sup>2,5</sup> since CFs feel a reduced flux they occupy higher Landau levels. Hence a generic CF wavefunction is:

$$\Psi_{\text{CF}} = \mathcal{P}_{\text{LLL}} \left\{ \prod_{i<j} (z_i - z_j)^{2s} \chi_{\nu^*} \right\}. \quad (4)$$

In this equation,  $\mathcal{P}_{\text{LLL}}$  is the LLL projector, and  $\prod_{i<j} (z_i - z_j)^{2s}$ ,  $2s$  an even integer, is the Jastrow factor that describes the flux attachment procedure. The filling factor if the CF state is then  $1/\nu = 2s + 1/\nu^*$ . When we have  $\nu^* = p$  an integer number of pseudo-Landau levels then  $\chi_p$  is just a Slater determinant of filled orbitals up to the  $p$ th Landau level. This leads to candidate wavefunctions for the prominent series of incompressible states at  $\nu = p/(2sp+1)$ . The effective magnetic field acting on the CF is then  $B_{\text{eff}} = B - 2spn\phi_0$ . If we fix integers  $s$  and  $p$  it is clear that one can have negative flux acting upon the CFs. For example, the simplest case is  $s = 1$  (we are thus dealing with <sup>2</sup>CFs in the notation of Jain<sup>5</sup>) and  $p = 2$ , i.e., at filling factor  $\nu = 2/3$ . In the CF formalism there is nothing that prevents the use of wavefunctions  $\chi_{\nu^*}$  with negative flux since they are simply given by the complex

conjugate of those of positive flux  $\chi_{\nu^*} = \chi_{\nu}^*$ . Note that in the case of  $\nu = 2/3$  there is no necessity of using the negative flux CF wavefunction since  $\nu = 2/3$  can also be viewed as the particle-hole conjugate of the positive flux state at  $\nu = 1/3$ . In fact, both approaches, negative flux or particle-hole symmetry, give equally good wavefunctions.<sup>14</sup>

Finally, we note that negative flux constructions also appear in multicomponent systems. Some convenient states with two components are those introduced<sup>15</sup> by Halperin:

$$\Psi_{mm'n} = \prod_{i,j \in A} (z_i - z_j)^m \prod_{k,l \in B} (z_k - z_l)^{m'} \prod_{a \in A, b \in B} (z_a - z_b)^n, \quad (5)$$

where the respective two-component indices belong to subsets  $A$  and  $B$ . This gives spin-polarized states. Concerning possible spin-singlet quantum Hall states, we note that there is evidence for an incompressible state at  $\nu = 2/3$  in the vanishing-Zeeman-splitting limit.<sup>16,17</sup> Numerical studies<sup>18</sup> are in agreement with a spin-singlet ground state for which the most prominent candidate is a state<sup>19</sup> with negative flux attachment:

$$\Psi_{2/3}^{(S=0)} = \mathcal{P}_{\text{LLL}} \left\{ \prod_{i < j} (z_{i\uparrow}^* - z_{j\uparrow}^*) \prod_{k < l} (z_{k\downarrow}^* - z_{l\downarrow}^*) \prod_{p < q} (z_p - z_q)^2 \right\}, \quad (6)$$

where  $p, q$  indices run over both spin values and we have omitted the spin part of the wavefunction.

### 3. Abelian States with Positive Flux

In this section, we explain how the consistent description of the edge of negative flux Abelian states requires the inclusion of edge states with derivative operators. Besides Abelian one-component states, we will consider spin-singlet, i.e., multicomponent states for which we will explicitly demonstrate that the existence of derivative operators in the neutral sector is still compatible with the charge — neutral sector separation that is expected and exists on the edge of a fractional quantum Hall system. This will facilitate our discussion and conclusions on the nature of non-Abelian negative flux states, which we will consider in the following section using their multicomponent formulation.

#### 3.1. Laughlin case

For  $N$  fully polarized fermions at filling  $1/m$  the physics of the FQHE ground state can be captured by the Laughlin wavefunction:

$$\Psi_L(\{z_i\}) = \prod_{i < j} (z_i - z_j)^m, \quad (7)$$

where  $m$  is an odd integer for antisymmetry and  $i, j = 1, \dots, N$ . Above this ground state one finds gapped excitations including charged quasiparticles. The quasihole excitation is given by the following formula:

$$\Psi_{\text{qh}}(\{z_i\}; w) = \prod_{i=1}^N (z_i - w) \Psi_L, \tag{8}$$

where  $w$  is the complex coordinate corresponding to the spatial location of the quasihole. One should think of Eq. (8) as a coherent state of a quasihole. This coherent state can be expanded as a superposition of definite angular momentum states:

$$\Psi_{\text{qh}}(\{z_i\}; w) = \sum_{n=0}^N (-w)^{N-n} s_n \Psi_L(\{z_i\}), \tag{9}$$

$$s_n = \sum_{i_1 < \dots < i_n} z_{i_1} \dots z_{i_n}, \tag{10}$$

where  $s_n$  are elementary symmetric polynomial of degree  $n$ . It is known<sup>20</sup> that the edge states are precisely given by the products  $s_n \Psi_L; n = 1, 2, \dots$ . This means that the quasihole wavefunction, Eq. (8), can be considered as a generating function for the edge states. Multiple quasihole constructions generate all combinations (products) of symmetric polynomials, which correspond to all possible edge states. They also emerge from the single boson effective description of the edge of the Laughlin state.<sup>20</sup> Since  $\Psi_L$  is the unique highest density zero-energy state of the hard-core interaction with interactions only for relative angular momentum unity between electrons (for  $m = 3$ ), these edge states are also zero-energy states. They will smoothly transform in a low-lying manifold of states in the presence of realistic interactions, provided the Laughlin-like physics is preserved.

### 3.2. The spin-singlet CF state at filling $\nu = 2/5$

Next in complexity, we consider the CF state which is spin-singlet for filling  $\nu = 2/5$ . We can fill the pseudo-LLL of the CFs by spin-singlet pairs only. Since we accommodate twice as many electrons as in the polarized construction, we obtain a global spin-singlet state at total filling  $\nu = 2/5$ . If we only write the orbital part of the wavefunction, it is given by:

$$\Psi_{2/5}^{S=0} = \prod_{i < j} (z_{i\uparrow} - z_{j\uparrow})^3 \prod_{k < l} (z_{k\downarrow} - z_{l\downarrow})^3 \prod_{p < q} (z_{p\uparrow} - z_{q\downarrow})^2. \tag{11}$$

This is a multicomponent Halperin (332) state in the notation introduced in Sec. 2. With the spin degree of freedom, it is clear that we can now have two simple generalizations of the quasihole:

$$\Psi_{\text{qh}}^\sigma = \prod_i^{N/2} (z_{i\sigma} - w) \Psi_{2/5}^{S=0}, \tag{12}$$

where  $\sigma = \uparrow$  or  $\sigma = \downarrow$ . Each wavefunction generates a set (ring) of symmetric polynomials that we note  $s_n^\sigma$  defined as in Eq. (10). The two sets describe the excitations of two chiral bosons on the edge of the system. For each pair of symmetric polynomials of degree  $n$ :  $\{s_n^\uparrow, s_n^\downarrow\}$  we can define charge and neutral superpositions:  $\{s_n^c = s_n^\uparrow + s_n^\downarrow, s_n^s = s_n^\uparrow - s_n^\downarrow\}$ , which are in one-to-one correspondence with two-boson states that describe charge and neutral excitations on the boundary of the spin-singlet system. This construction has been introduced first by Balatsky and Stone.<sup>21</sup>

Let us now consider the case of two quasiholes of opposite spin at locations  $w_1$  and  $w_2$ . The wavefunction involves a global factor given by:

$$\Psi_{2\text{qh}} = \prod_i (z_{i\uparrow} - w_1) \prod_k (z_{k\downarrow} - w_2) \times \Psi_{2/5}^{S=0}. \tag{13}$$

If we expand the two-quasihole factor we find:

$$\begin{aligned} & \prod_i (z_{i\uparrow} - w_1) \prod_k (z_{k\downarrow} - w_2) \\ &= \sum_m s_m^\uparrow w_1^{N/2-m} \sum_n s_n^\downarrow w_2^{N/2-m} \\ &= \frac{1}{4} \sum_{m,n} (s_m^\uparrow s_n^\downarrow + s_m^\downarrow s_n^\uparrow) (w_1^{N/2-m} w_2^{N/2-n} + w_2^{N/2-m} w_1^{N/2-n}) \\ &+ \frac{1}{4} \sum_{m,n} (s_m^\uparrow s_n^\downarrow - s_m^\downarrow s_n^\uparrow) (w_1^{N/2-m} w_2^{N/2-n} - w_2^{N/2-m} w_1^{N/2-n}). \end{aligned} \tag{14}$$

We thus have a sum of two kinds of superpositions of angular momentum eigenstates of  $w$ 's, each with a definite symmetry under the coordinate exchange  $w_1 \leftrightarrow w_2$ . Quasiholes in the case of the spin-singlet state at  $\nu = 2/5$  can be considered as spin-1/2 fermions.<sup>21,22</sup> Therefore, the first superposition is a spin-singlet ( $S = 0$ ), because it is symmetric under exchange and the second superposition is a triplet ( $S = 1$ ) state with  $S_z = 0$  as it is antisymmetric under exchange.

The important point is that the spin-singlet superposition:

$$\begin{aligned} & \prod_i (z_{i\uparrow} - w_1) \prod_k (z_{k\downarrow} - w_2) + (w_1 \leftrightarrow w_2) \\ &= \frac{1}{2} \sum_{m,n} (s_m^\uparrow s_n^\downarrow + s_m^\downarrow s_n^\uparrow) (w_1^{N/2-m} w_2^{N/2-n} + w_2^{N/2-m} w_1^{N/2-n}), \end{aligned} \tag{15}$$

generates only edge states of the charge sector exactly as a single Laughlin (spinless) quasihole. Indeed as we take  $w_1 = w_2$  in Eq. (14) the coefficients in terms of  $z$ 's do not change — they are the same as those that we get in the expansion of a single spinless quasihole that generates edge states in the charge sector. Therefore, the

family of quantities:

$$S(w_1, w_2) = \prod_i (z_{i\uparrow} - w_1) \prod_k (z_{k\downarrow} - w_2) + (w_1 \leftrightarrow w_2), \tag{16}$$

may be used as generators of the charge sector. In the case of four quasiholes we can use again:

$$S^{(4)}(w_1, w_2, w_3, w_4) = S(w_1, w_2)S(w_3, w_4) - S(w_1, w_4)S(w_3, w_2), \tag{17}$$

as generators of the edge charge sector. This is similar to the spin-singlet construction of the BCS state or in general in the case of a many-body spin-singlet state built out of spin-singlet pairs. Strictly speaking, in the case of the Abelian state at  $\nu = 2/5$ , we do not need antisymmetrization as in Eq. (17) to generate edge states, but to keep the discussion general and applicable to the cases that we will discuss later, we emphasize that the charge sector can be generated through spin-singlet ( $S = 0$ ) constructions of quasiholes that are made of collections of spin-singlet pairs.

The two kinds of states that appear in Eq. (14) are in fact orthogonal to each other. Therefore, the associated symmetric polynomials:  $(s_m^\uparrow s_n^\downarrow + s_m^\downarrow s_n^\uparrow)$  [resp.  $(s_m^\uparrow s_n^\downarrow - s_m^\downarrow s_n^\uparrow)$ ] can be expressed through  $s_m^c$  (resp.  $s_m^s$ ) only, because they belong to charge (neutral) sector. The superposition that is antisymmetric in the quasihole coordinate exchange:

$$T(w_1, w_2) = \prod_i (z_{i\uparrow} - w_1) \prod_i (z_{i\downarrow} - w_2) - (w_1 \leftrightarrow w_2), \tag{18}$$

[the last line in Eq. (14)], creates triplet ( $S = 1$ ) excitations that change the spin number of the ground state and thus generate edge states that belong to the *neutral* sector. Therefore, we infer that all possible collections of triplet pairs generate the neutral sector. For example, in the case of four quasiholes we can use the following combination:

$$\mathcal{T}^{(4)}(w_1, w_2, w_3, w_4) = T(w_1, w_2)T(w_3, w_4) - T(w_1, w_4)T(w_3, w_2), \tag{19}$$

that maximizes the spin of four quasihole construction to  $S_{\max} = 2$ .

The important conclusion is that the edge states of the neutral sector of  $\nu = 2/5$  state and, in fact, of any two-component spin-singlet state can be generated through maximum spin superpositions of coherent states of spin-1/2 quasiholes.

### 3.3. The spin-polarized CF state at $\nu = 2/5$

In the case of the Laughlin state the deg modes are exact zero-energy eigenstates of the special hard-core pseudopotential for which the Laughlin itself is the densest zero-energy state. But in the case of Jain's states at  $\nu = p/(2p + 1)$ ;  $p > 0$  the CF wavefunctions are not unique zero-energy ground states of special Hamiltonians. Nevertheless, we expect that the quasihole constructions will still lead to generators



of edge states. For the spin-polarized CF state at  $\nu = 2/5$ , Jain state can be written as:

$$\Psi_{2/5} = \mathcal{P}_{\text{LLL}} \left\{ \prod_{i < j} (z_i - z_j)^2 \cdot \chi_2 \right\}, \tag{20}$$

where  $\chi_2$  represents the Slater determinant of two filled pseudo-Landau levels:

$$\chi_2 = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_N \\ \vdots & \vdots & & \vdots \\ z_1^{N/2} & z_2^{N/2} & \cdots & z_N^{N/2} \\ z_1^* & z_2^* & \cdots & z_N^* \\ z_1^* z_1 & z_2^* z_2 & \cdots & z_N^* z_N \\ \vdots & \vdots & & \vdots \\ z_1^* z_1^{N/2} & z_2^* z_2^{N/2} & \cdots & z_N^* z_N^{N/2} \end{vmatrix}.$$

As in the Laughlin case, we can now construct two kinds of quasiholes,  $w_1$  and  $w_2$  by modifying the determinant in the following way:

$$\Psi_{2\text{qh}}(w_1, w_2) = \begin{vmatrix} (z_1 - w_1) & (z_2 - w_1) & \cdots & (z_N - w_1) \\ (z_1 - w_1)z_1 & (z_2 - w_1)z_2 & \cdots & (z_N - w_1)z_N \\ \vdots & \vdots & & \vdots \\ (z_1 - w_1)z_1^{N/2} & (z_2 - w_1)z_2^{N/2} & \cdots & (z_N - w_1)z_N^{N/2} \\ (z_1 - w_2)z_1^* & (z_2 - w_2)z_2^* & \cdots & (z_N - w_2)z_N^* \\ (z_1 - w_2)z_1^* z_1 & (z_2 - w_2)z_2^* z_2 & \cdots & (z_N - w_2)z_N^* z_N \\ \vdots & \vdots & & \vdots \\ (z_1 - w_2)z_1^* z_1^{N/2} & (z_2 - w_2)z_2^* z_2^{N/2} & \cdots & (z_N - w_2)z_N^* z_N^{N/2} \end{vmatrix}. \tag{21}$$

These two kinds of quasiholes correspond to the two possibilities to create a hole in an empty shell in the CF scheme: we can make a hole either in the pseudo-LLL or in the first excited orbital pseudo-LL. When  $w_1 = w_2 = w$  we recover the ordinary Laughlin quasihole construction: one can factor out  $\prod_i (z_i - w)$  in front of the ground state in Eq. (20).

The CF state at  $\nu = 2/5$  can be viewed as a state at integer filling factor,  $\nu = 2$ , of CFs. It is then natural to assign a pseudospin degree of freedom to CFs; those in the pseudo-LLL and those in the second pseudo-LL carry distinct values of  $S_z$ , the pseudospin number. The excitation of two quasiholes in Eq. (21) represents two holes of CFs in their respective LLs. Because the lowest-energy excitations of the

state at  $\nu = 2/5$  can be classified as excitations of CFs in two LLs, the pseudospin  $S_z$  quantum number can be used to classify excitations. Thus, exactly as in the case of the spin-singlet state at  $\nu = 2/5$ , we can consider symmetric and antisymmetric superpositions of two quasiholes and conclude that they carry pseudospin equal to  $S = 1$  and  $S = 0$ , respectively. Now the presence of spin is tied to a charge excitation and if a particular configuration of quasiholes has maximum spin value, it belongs solely to the *charge* sector. Therefore, the symmetrized state of two quasiholes:

$$\begin{aligned} S^J(w_1, w_2) &= (1 + e_{12})\Psi_{2\text{qh}}(w_1, w_2) \\ &= \Psi_{2\text{qh}}(w_1, w_2) + \Psi_{2\text{qh}}(w_2, w_1), \end{aligned} \tag{22}$$

where  $e_{ij}$  denotes the exchange operation,  $i \leftrightarrow j$ , is a generator of symmetric polynomials just as a single Laughlin quasihole. We can convince ourselves that this is true by examining the terms in the expansion of the determinants. The antisymmetric combination:

$$\begin{aligned} T^J(w_1, w_2) &= (1 - e_{12})\Psi_{2\text{qh}}(w_1, w_2) \\ &= \Psi_{2\text{qh}}(w_1, w_2) - \Psi_{2\text{qh}}(w_2, w_1), \end{aligned} \tag{23}$$

generates the edge states of the neutral sector ( $S = 0$ ). These states cannot be represented as symmetric polynomials multiplying the Slater determinant of the ground state. The following simple example of one electron in the LLL and one electron in second LL is an illustration of this (we write only the determinantal part of the wavefunction):

$$\begin{aligned} &\begin{vmatrix} (z_1 - w_1) & (z_2 - w_2) \\ (z_1 - w_2)z_1^* & (z_2 - w_2)z_2^* \end{vmatrix} \\ &= z_1 z_2 (z_2^* - z_1^*) + w_1 (z_1 z_1^* - z_2 z_2^*) - w_2 (z_1 z_2^* - z_2 z_1^*) + w_1 w_2 (z_2^* - z_1^*). \end{aligned} \tag{24}$$

In the symmetric combinations of two quasiholes, the second and the third term in the expansion of the determinant combine to give  $\sim (z_1 + z_2)(z_2^* - z_1^*)$ , which demonstrates the factorization, a symmetric polynomial  $\times$  Slater determinant in the charge sector, which is not possible in the neutral sector: we get  $(z_1 - z_2)(z_2^* + z_1^*)$ .

Therefore, all collections of spin-singlet pairs of quasiholes generate edge states of the neutral sector of Jain's  $2/5$  state. We can, for example, use the pseudospin-singlet combination of four quasiholes:

$$\begin{aligned} \mathcal{S}_{S=0}^{(4)}(w_1, w_2, w_3, w_4) \\ = [(1 - e_{12})(1 - e_{34}) + (1 - e_{14})(1 - e_{32})]\Psi_{4\text{qh}}(w_1, w_3; w_2, w_4), \end{aligned} \tag{25}$$

where we use the following definition:

$$\Psi_{4qh}(w_1, w_3; w_2, w_4) = \begin{vmatrix} (z_1 - w_1)(z_1 - w_3) & \cdots & (z_N - w_1)(z_N - w_3) \\ (z_1 - w_1)(z_1 - w_3)z_1 & \cdots & (z_N - w_1)(z_N - w_3)z_N \\ \vdots & & \vdots \\ (z_1 - w_1)(z_1 - w_3)z_1^{N/2} & \cdots & (z_N - w_1)(z_N - w_3)z_N^{N/2} \\ (z_1 - w_2)(z_1 - w_4)z_1^* & \cdots & (z_N - w_2)(z_N - w_4)z_N^* \\ (z_1 - w_2)(z_1 - w_4)z_1^*z_1 & \cdots & (z_N - w_2)(z_N - w_4)z_N^*z_N \\ \vdots & & \vdots \\ (z_1 - w_2)(z_1 - w_4)z_1^*z_1^{N/2} & \cdots & (z_N - w_2)(z_N - w_4)z_N^*z_N^{N/2} \end{vmatrix}. \tag{26}$$

Contrary to the previous case of neutral edge states for the  $\nu = 2/5$  spin-singlet state, there is no factorization property of the form: a symmetric polynomial odd under  $\uparrow$  and  $\downarrow$  exchange  $\times$  the ground state.

With respect to the spin-singlet state at  $2/5$ , we did not have a transparent spin structure from which we could deduce edge sectors; we used  $S$  number as effective charge number of the  $U(1) \times U(1)$  edge theory. In this case of Jain’s state at  $2/5$ , besides the insights from the spin-singlet state, we also used the knowledge of effective theories<sup>20,23</sup> to argue for the existence of two sets of generators; without going into a detailed description of the edge states that they generate in the neutral sector, we were able to identify them. The generators represent charge and neutral sector, also because, as we put all quasiholes at the same point in any generator ( $w_1 = w_2$  in Eq. (22), etc.) we get a charge hole in the charge sector, but in the neutral sector any expression (Eqs. (23), (25), etc.) vanishes as it cannot be connected to any charge excitation.

#### 4. Abelian States with Negative Flux

##### 4.1. Fully polarized CF state at $\nu = 2/3$

We know<sup>14</sup> that the fully polarized state at  $\nu = 2/3$  can be described by both particle-hole conjugation of  $\nu = 1/3$  state and negative flux CF wavefunction and thus, the edge theory should be the same in both descriptions. To perform the particle-hole transformation onto the parent Laughlin  $\nu = 1/3$  state, one needs a background droplet with filling  $\nu = 1$  of a size larger than that of the Laughlin droplet and one then makes the transformation in the interior region.<sup>20,24,25</sup> This leads to an inner region of filling  $\nu = 2/3$  separated from the vacuum by a ring with filling  $\nu = 1$ . As a consequence, the edge is now composite. One can have edge modes on the exterior boundary at  $\nu = 1$ : they will be generated by the symmetric polynomials we have described in Sec. 3.1 and the associated effective theory is a free boson.<sup>20</sup> There are also edge modes associated with the boundary

between the  $\nu = 2/3$  core and the  $\nu = 1$  annulus. These modes should propagate in the opposite direction from the outer modes. This microscopic picture has received detailed confirmation from numerical and experimental studies.<sup>26–32</sup> There are thus two counterpropagating modes that interact probably in a nonuniversal manner depending on the details of the confining potential.<sup>32</sup> The inner modes have angular momenta which are less than the total angular momentum of the droplet as a whole. Since its structure is exactly that of a free boson generated by symmetric polynomials, then explicit wavefunctions for the edge modes generators are obtained by replacing  $z$  factors by derivatives in the generating functional, Eq. (10). They are not exact eigenstates but are expected to be satisfactory trial wavefunctions.

We now discuss the edge modes as seen from the negative flux CF picture. Here we have two filled pseudo-Landau levels of  ${}^2\text{CFs}$ . A naive reasoning based on the results for the fraction  $\nu = 2/5$  would lead to two copropagating modes. However, writing trial wavefunctions is in general not enough to guarantee that their energies are as expected. For example, in a general Jain state with  $p$  filled levels, it is immediately clear what are the lowest-lying quasiholes or quasielectrons since there are many possibilities for excited states. Here, we know from the reasoning of the previous paragraph that the counterpropagating mode is generated by derivative operators; this means that it has to be found amongst the modes generated by the quasielectron operator:

$$\mathcal{O}_{\text{qe}}(w) = \prod_i \left( 2 \frac{\partial}{\partial z_i} - w^* \right). \quad (27)$$

Expansion in powers of  $w^*$  leads to the same modes as proposed in the previous particle-hole approach (we will not use Jain's quasielectron construction<sup>2,33</sup> because it is harder to implement in the case of many excitations and, as we consider only effective, lowest-energy physics of edge states, the Laughlin construction is adequate). Of course, this operator does not always generate low-lying modes. For example, if applied onto the Laughlin state all modes derived from it are gapped. For example, action of  $\sum_i (\partial/\partial z_i)$  will lead to state with an excitation energy of the order of the quasielectron itself. It is only when we act on some special states that one generates low-energy states. Hence, the operator itself does not contain all the information on the edge theory; this is also encoded in the ground state wavefunction.

With our identification of edge modes derived from the quasielectron, we conclude that we can use antisymmetric combinations of pairs of *quasiparticles* with pseudospin  $S = 0$  to generate the neutral sector that moves in the opposite direction of the charge sector.

#### 4.2. Spin-singlet state at $\nu = 2/3$

We now discuss the spin-singlet state  $\nu = 2/3$ . We know from effective theories<sup>20</sup> that there is a chiral boson in the edge theory that describes a neutral sector and

moves in the opposite direction with respect to the chiral boson of the charge sector. With the insight gained considering the edge states for two-component spin-singlet 2/5 and one-component Jain’s 2/5 and 2/3 ground states, we conclude that the edge states in the neutral sector of this negative flux spin-singlet state at 2/3 can be constructed by the action of derivative operators — symmetric polynomials of derivatives:

$$\tilde{s}_n^s = \tilde{s}_n^\uparrow - \tilde{s}_n^\downarrow, \tag{28}$$

where we define:

$$\tilde{s}_n^\sigma = \sum_{i=1}^{N/2} \frac{\partial^n}{\partial z_{i\sigma}^n}, \quad n = 1, 2, 3, \dots \tag{29}$$

that act on the ground state. They create excitations of the neutral sector that move in the opposite direction from the charge excitations. The generators of this neutral sector are in fact the maximum spin coherent states of spin-1/2 quasiparticles.

As emphasized in Ref. 21, at the edge of the  $\nu = 2/5$  spin-singlet state, we have spin-charge separation and the separation is expected in any gapped spin-singlet state. In the case of the  $\nu = 2/3$  spin-singlet state the existence of the spin-charge separation implies that  $s_n^c$  and  $\tilde{s}_m^s$  commute. Indeed, they do commute provided we confine our description to the lowest-energy sector of the system, i.e., the edge. This conclusion comes from the following simple algebra:

$$[s_n^c, \tilde{s}_m^s] = [s_n^\uparrow, \tilde{s}_m^\uparrow] - [s_n^\downarrow, \tilde{s}_m^\downarrow] = [s_n^s, \tilde{s}_m^s]. \tag{30}$$

Here by  $s_n^s$  and  $\tilde{s}_m^c$  we mean  $s_n^s = s_n^\uparrow - s_n^\downarrow$  and  $\tilde{s}_m^c = \tilde{s}_m^\uparrow + \tilde{s}_m^\downarrow$ . Because the last expression does not belong to the lowest-energy sector, when the projection to the edge sector is done, we find that  $s_n^c$  and  $\tilde{s}_m^s$  commute.

## 5. Edge Modes of Pfaffian and Negative-Flux Pfaffian States

### 5.1. The Pfaffian state in multicomponent formulation and its edge

We now study the edge mode structure of the simplest non-Abelian quantum Hall state, the so-called Pfaffian state introduced by Moore and Read.<sup>6,7</sup> We consider the bosonic case with no loss of generality since it can be multiplied by one or more odd powers of the Jastrow factor to give an antisymmetric trial state. For bosons, its filling factor is  $\nu = 1$  and the wavefunction is explicitly given by:

$$\Psi_{\text{MR}} = \text{Pf} \left( \frac{1}{z_i - z_j} \right) \cdot \prod_{i < j} (z_i - z_j), \tag{31}$$

where the Pfaffian symbol stands for the following sum over permutations of  $N$  indices:

$$\text{Pf} \left( \frac{1}{z_i - z_j} \right) = \sum_{\sigma \in S_N} \text{sign } \sigma \frac{1}{z_{\sigma(1)} - z_{\sigma(2)}} \dots \frac{1}{z_{\sigma(N-1)} - z_{\sigma(N)}}. \tag{32}$$

There is an alternate way<sup>34</sup> to write this wavefunction:

$$\Psi_{\text{MR}} = \mathcal{S} \left\{ \prod_{i_1 < j_1} (z_{i_1} - z_{j_1})^2 \prod_{i_2 < j_2} (z_{i_2} - z_{j_2})^2 \right\}, \tag{33}$$

where the sum is over all possible partitions of  $N$  particles into two groups denoted by 1 and 2. It is this expression that admits a generalization with grouping particles in  $k$  subsets.<sup>10,11</sup>

The equivalence between these two formulas can be proved by the following manipulations:

$$\begin{aligned} \Psi_{\text{MR}} &= \mathcal{S} \left\{ \prod_{i_1 < j_1} (z_{i_1} - z_{j_1})^2 \prod_{i_2 < j_2} (z_{i_2} - z_{j_2})^2 \right\} \\ &= \mathcal{S} \left\{ \frac{\prod_{i_1 < j_1} (z_{i_1} - z_{j_1}) \prod_{i_2 < j_2} (z_{i_2} - z_{j_2})}{\prod_{i_1 < i_2} (z_{i_1} - z_{i_2})} \right\} \cdot \prod_{i < j} (z_i - z_j) \\ &= \mathcal{S} \left\{ \text{Det} \left( \frac{1}{(z_{i_1} - z_{i_2})} \right) \right\} \cdot \prod_{i < j} (z_i - z_j) \\ &\propto \text{Pf} \left( \frac{1}{(z_i - z_j)} \right) \cdot \prod_{i < j} (z_i - z_j), \end{aligned} \tag{34}$$

where we used the Cauchy determinant identity and the fact that a sum of determinants gives a Pfaffian.

In the “pairing” formulation with the explicit Pfaffian equation (31), edge states are neutral fermion excitations created by breaking some of the pairs:

$$\Psi_{\text{MR}}^{(n_p)} = \mathcal{A} \left\{ z_1^{m_1} \cdots z_{2n_p}^{m_{2n_p}} \frac{1}{z_{2n_p+1} - z_{2n_p+2}} \cdots \frac{1}{z_{N-1} - z_N} \right\} \cdot \prod (z_i - z_j), \tag{35}$$

where  $0 \leq m_1 < m_2 < \cdots < m_{2n_p}$  are integers,  $n_p$  denotes the number of broken pairs and  $\mathcal{A}$  stands for antisymmetrization as in the Pfaffian definition. These states can be obtained<sup>35</sup> from the non-Abelian quasihole constructions<sup>6,7</sup> for two quasiholes at positions  $w_1$  and  $w_2$ :

$$\Psi_{\text{MR}}^{2\text{qh}} = \prod_{i < j} (z_i - z_j) \cdot \text{Pf} \left( \frac{(z_i - w_1)(z_j - w_2) + (w_1 \leftrightarrow w_2)}{z_i - z_j} \right). \tag{36}$$

In addition to these neutral fermion modes, there is the usual charge sector generated by symmetric polynomials as we have seen in all previous examples.

The multicomponent formulation, Eq. (33), suggests that the quasihole factors can be introduced in each of the two groups of particles before symmetrizing the whole expression. This leads to the following alternate generating functional:

$$\mathcal{G}(\tilde{w}_1, \tilde{w}_2) = \mathcal{S} \left\{ \prod_{i_1} (z_{i_1} - \tilde{w}_1) \prod_{i_2} (z_{i_2} - \tilde{w}_2) \prod_{i_1 < j_1} (z_{i_1} - z_{j_1})^2 \prod_{i_2 < j_2} (z_{i_2} - z_{j_2})^2 \right\}. \tag{37}$$

One can also construct the neutral sector from the multicomponent formulation:

$$\Psi_{\text{MR}}^{\{\{n_i\}\}} = \mathcal{S} \left\{ s_{n_1}^s \cdots s_{n_k}^s \prod_{i_1 < j_1} (z_{i_1} - z_{j_1})^2 \prod_{i_2 < j_2} (z_{i_2} - z_{j_2})^2 \right\}, \quad (38)$$

where we have inserted  $s_n^s = s_n^1 - s_n^2$ . The generators — quasihole coherent constructions of the states of the form given by Eq. (38) can be easily specified in analogy with how it was done in our discussion of the two component  $\nu = 2/5$  case.

In fact, these two sets of edge states are exactly the same. The set, Eq. (38), seems to be overcomplete if we count the number of edge states at fixed  $\Delta M$  — the increase of the angular momentum with respect to the ground state. But there are linear dependencies among set members, Eq. (38), that will reduce their number to the number derived from Eq. (35) at fixed  $\Delta M$ . To show that this is true, we come back to the string of equivalences in Eq. (34). Any quasihole construction inside the Pfaffian (at the end of the string) [like Eq. (36)] is also a valid quasihole construction for each determinant in the sum over partitions. Now we have to use the relationship between this quasihole construction and the usual Laughlin quasihole constructions, for each determinant, explained in Ref. 35, from which it follows that the spaces of edge states generated in these two ways are the same. This follows from the Abelian nature of the Cauchy determinant pairing. This assertion for the Cauchy determinants automatically translates in the same assertion for the Pfaffian: the two kinds of quasihole constructions in the case of the Pfaffian are different but they are superpositions of zero-energy states — edge states that belong to the same subspace of states. Both can be used to generate this subspace and its neutral sector. Besides Eq. (35), the neutral sector can be also represented by Eq. (38), although one should remember that this set is overcomplete.

## 5.2. Negative flux Pfaffian

We now discuss the simplest non-Abelian state with negative flux. The basic idea is quite simple: we start from the CF definition, Eq. (4) and we note that the wavefunction  $\chi_{\nu^*}$  need not necessarily be a Slater determinant. It may be itself a state with non-Abelian correlations as advocated in Refs. 12 and 13. The simplest example is to use a Pfaffian state for bosons. If the flux of this state is taken as positive then we simply pile up powers of the overall jastrow factor but a more interesting possibility is to use a negative flux state which is non-Abelian. For example, the simplest example is given by  $\chi_{\nu^*=-1} = \Psi_{\text{MR}}^*$ , leading to:

$$\Psi_{\text{MR}}^{\text{neg.flux}} = \mathcal{P}_{\text{LLL}} \left[ \mathcal{S} \left\{ \prod_{i_1 < j_1} (z_{i_1}^* - z_{j_1}^*)^2 \prod_{i_2 < j_2} (z_{i_2}^* - z_{j_2}^*)^2 \right\} \prod_{i < j} (z_i - z_j)^2 \right]. \quad (39)$$

This state has filling factor  $\nu = 1$  but is different from the usual Pfaffian, for example, if written on the sphere, it has a flux–particle relationship given by  $N_\phi = N - 1$  while the Pfaffian requires a different tuning:  $N_\phi = N - 2$ . Not much is known

about these states; some of them have interesting overlap properties as measured in exact diagonalization of small systems.<sup>12,13</sup> In this section, we construct the edge modes of the negative flux Pfaffian. We define a set of edge states which is complete and support charge–neutral sector separation. Our proof follows from the formalism developed in Sec. 3. We first note that the edge states in the neutral sector can be obtained by inserting derivative operators in the multicomponent formulation:

$$\mathcal{P}_{\text{LLL}} \left[ \mathcal{S} \left\{ \tilde{s}_n^s \cdots \tilde{s}_m^s \prod_{i_1 < j_1} (z_{i_1}^* - z_{j_1}^*)^2 \prod_{i_2 < j_2} (z_{i_2}^* - z_{j_2}^*)^2 \right\} \prod_{i < j} (z_i - z_j)^2 \right]. \quad (40)$$

This set is obtained by a quasiparticle coherent state construction: this is similar to the case of the neutral sector of the two component  $\nu = 2/5$  state. The projection onto the LLL,  $\mathcal{P}_{\text{LLL}}$ , does not induce any extra linear dependencies among the states because the symmetrization process is analogous to Eq. (38) where every coordinate is replaced by a derivative. The number of edge states at fixed  $-(\Delta M)$  is then the same as the number of modes at  $+\Delta M$  for Eq. (38). Therefore Eq. (40), though overcomplete, describes edge states corresponding to a massless Majorana fermion CFT, associated to a neutral mode moving in the opposite direction with respect to the charge mode. We argue next that the set of modes, Eq. (40), allows charge–neutral sector separation on the edge. Indeed, as in the multicomponent Abelian case, we have the following commutation:

$$[s_n^c, \tilde{s}_m^s] = 0. \quad (41)$$

We also expect that the quantities  $s_n^c$ , after commuting with  $\tilde{s}_m^s$ 's, act on the charge part of the ground state which is separate from the spin part on the edge.<sup>21</sup> From the multicomponent formulation of the non-Abelian negative flux state, we can thus conclude that the separation between modes occurs because we can express the Pfaffian as a sum of Cauchy determinants and each of them represents an Abelian multicomponent construction that has complete separation between the modes. So the conclusion is that we expect one usual charge mode and one counterpropagating Majorana mode.

### 6. Conclusion

In this paper, we have given explicit expression for some edge state wavefunctions for fractional quantum Hall states involving more than one mode. We discussed spin-polarized as well as spin-singlet states when there are only two edge modes. Explicit expressions are based on the knowledge of the effective field theory of the edge.<sup>20</sup> Indeed, if one guesses a candidate wavefunction it is not clear that it has to do with the edge properties. For example, if we start from the simple Laughlin state at  $\nu = 1/3$  and act upon the wavefunction with derivative operators proposed for the counterpropagating mode at  $\nu = 2/3$ , one simply generates states that have a gap of the order of the quasielectron gap. It is only the quasihole that generates edge excitations. So the knowledge of the ground state wavefunction is not enough,



one should also have some knowledge of the effective theory. We have also studied the case of negative flux states: some of them belong to experimentally prominent quantum Hall fractions like  $\nu = 2/3$ . While they were already studied by effective theory approaches<sup>26–28,30</sup> and exact diagonalizations,<sup>24,25,32</sup> no microscopic expression for the wavefunction was proposed before our work. Finally, we have studied the case of a negative flux state build upon the Moore–Read Pfaffian whose edge modes can be constructed in a straightforward way from the formalism we developed. The edge states are given by a bosonic charge mode and a counterpropagating Majorana fermion.

Of course, the true electronic system is very complex due to interactions between modes. In the case of  $\nu = 2/3$ , it has been shown<sup>32</sup> that there is a regime with clear separation between the two counterpropagating modes. This depends notably on the confining potential that may reconstruct the edge. However, our explicit wavefunctions give precise guidance to detailed numerical studies of the edge phenomena.

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## References

1. R. B. Laughlin, *Phys. Rev. Lett.* **50**, 1395 (1983).
2. J. K. Jain, *Phys. Rev. Lett.* **63**, 199 (1989).
3. S. Das Sarma and A. Pinczuk (eds), *Perspectives in Quantum Hall Effect* (Wiley, New York, 1996).
4. O. Heinonen (ed.), *Composite Fermions: A Unified View of the Quantum Hall Regime* (World Scientific, Singapore, 1998).
5. J. K. Jain, *Composite Fermions* (Cambridge University Press, Cambridge, 2007).
6. G. Moore and N. Read, *Nucl. Phys. B* **360**, 362 (1991).
7. N. Read and G. Moore, *Prog. Theor. Phys. (Kyoto) Suppl.* **107**, 157 (1992).
8. W. Pan, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer, K. W. Baldwin and K. W. West, *Phys. Rev. Lett.* **90**, 016801 (2003).
9. J. S. Xia, W. Pan, C. L. Vincente, E. D. Adams, N. S. Sullivan, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer, K. W. Baldwin and K. W. West, *Phys. Rev. Lett.* **93**, 176809 (2004).
10. N. Read and E. H. Rezayi, *Phys. Rev. B* **54**, 16864 (1996).
11. N. Read and E. H. Rezayi, *Phys. Rev. B* **59**, 8084 (1999).
12. Th. Jolicoeur, *Phys. Rev. Lett.* **99**, 036805 (2007).
13. Th. Jolicoeur, Talk given at les Houches summer school “Exact Methods in Low-dimensional Statistical Physics and Quantum Computing”, e-print arXiv:0907.1538
14. G. Möller and S. Simon, *Phys. Rev. B* **72**, 045344 (2005).
15. B. I. Halperin, *Helv. Phys. Acta* **56**, 75 (1983).
16. J. P. Eisenstein, H. L. Stormer, L. N. Pfeiffer and K. W. West, *Phys. Rev. B* **41**, 7910 (1990).

17. L. W. Engel, S. W. Hwang, T. Sajoto, D. C. Tsui and M. Shayegan, *Phys. Rev. B* **45**, 3418 (1992).
18. X. C. Xie, Y. Guo and F. C. Zhang, *Phys. Rev. B* **40**, 3487 (1989).
19. X. G. Wu, G. Dev and J. K. Jain, *Phys. Rev. Lett.* **71**, 153 (1993).
20. X.-G. Wen, *Int. J. Mod. Phys. B* **6**, 1711 (1992).
21. A. Balatsky and M. Stone, *Phys. Rev. B* **43**, 8038 (1991).
22. F. D. M. Haldane, in *The Quantum Hall Effect*, eds. R. E. Prange and S. M. Girvin, 2nd edn. (Springer-Verlag, New York, 1990).
23. N. Read, *Phys. Rev. Lett.* **65**, 1502 (1990).
24. A. H. MacDonald, *Phys. Rev. Lett.* **64**, 220 (1990).
25. M. D. Johnson and A. H. MacDonald, *Phys. Rev. Lett.* **67**, 2060 (1991).
26. F. D. M. Haldane, *Phys. Rev. Lett.* **74**, 2090 (1995).
27. C. L. Kane, M. P. A. Fisher and J. Polchinski, *Phys. Rev. Lett.* **72**, 4129 (1994).
28. C. L. Kane and M. P. A. Fisher, *Phys. Rev. B* **52**, 17393 (1995).
29. A. M. Chang, L. N. Pfeiffer and K. W. West, *Phys. Rev. Lett.* **77**, 2538 (1996).
30. U. Zülicke, A. H. MacDonald and M. D. Johnson, *Phys. Rev. B* **58**, 13778 (1998).
31. E. V. Deviatov, V. T. Dolgoplov, A. Lorke, W. Wegscheider and A. D. Wieck, *J. Exp. Theor. Phys. Lett.* **82**, 539 (2005).
32. Z.-X. Hu, H. Chen, K. Yang, E. H. Rezayi and X. Wan, *Phys. Rev. B* **78**, 235315 (2008).
33. G. S. Jeon and J. K. Jain, *Phys. Rev. B* **68**, 165346 (2003).
34. A. Cappeli, L. S. Georgiev and I. T. Todorov, *Nucl. Phys. B* **599**, 499 (2001).
35. M. Milovanović and N. Read, *Phys. Rev. B* **53**, 13559 (1996).