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QUARTIC INTERACTION IN SUPERSTRING FIELD THEORY*

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ABSTRACT

We investigate the presence of contact interactions in Witten's superstring theory. A calculation of the four point amplitude is performed. Associativity of the string product is shown to fail due to short distance effects of the mid-point operator insertions. An explicit quartic interaction is induced in the superstring action. At the same time the gauge transformation is modified by an order g^2 term. The quartic term completes the theory; terms higher than quartic are not present.

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1. Introduction

The motivation for the construction of covariant, gauge invariant superstring field theory has been to establish a formalism in which questions about non-perturbative phenomena in strings could be addressed. The hope is that a field theory formulation can shed some light on the basic structure of string theory. This construction has proved to be a challenging task, and indeed in the case of closed strings it is still outstanding. Interacting theory of open bosonic strings, as well as superstrings, has been introduced by Witten[1]. These theories are characterized by purely cubic interactions.

Operator formulation of the theory has been established in a series of papers[2-4]. Here the interaction of strings follows from a 3-point vertex. Gauge invariance, associativity of the string product, and the purely cubic nature of the interactions are closely linked. In the bosonic string case this relation has been explicitly shown[2]. The superstring case parallels the bosonic, the new non-trivial ingredient being the insertion at the string mid-point[3,4] of the picture changing operator of Friedan, Martinec and Shenker.[8]

Recently a connection of the interaction vertex and the first quantized formalism has been given[9]. Using this, Wendt[10] calculated the scattering of four vectors in the Witten superstring. Surprisingly he found that the result differed from the dual model by an (infinite) contact term. Other authors argued against such terms in Witten's theory[11]. Since the original argument is based on a limiting procedure, questions of prescription dependence could be raised. Contact terms, however, were also known to appear in the light cone theory and it is possible that they remain even in the covariant approach[12].

This issue is sufficiently relevant that a more concrete investigation is needed if a complete superstring field theory is to be specified.

In this paper we will investigate these issues directly in operator field theory. This calculation is clear cut and does not involve any limiting procedures or prescriptions. We shall find that a 4-point contact term is indeed present. The source of the extra term is found to lie in the short distance behaviour of the operator insertions at the vertices. This results in an associativity anomaly.

We shall also consider the issue of higher order contact terms with the conclusion that no further terms appear in the action. Throughout, the stated connection of associativity and gauge invariance will be used and will play a central role.

This paper is organized as follows: We first give the mid-point insertion. We then proceed to calculate the scattering of four tachyons in the bosonic string. Next, the scattering of vectors in the superstring is calculated, and the contact term is isolated. In the second part of the paper the connection between associativity and gauge invariance is established. Failure of associativity induces a 4-string interaction that is needed in order to maintain gauge invariance. Also, a g^2 order term is added to the gauge transformation. We then show that 5-point and higher interaction terms are not present. Lastly, we give the complete action for Witten's superstring and show the agreement with our perturbative calculation.

Various Neumann coefficients for the vertices that were used in the calculations are listed in the appendix.

2. Superstring Insertions

All the problems we will look at in this paper reside in the Neveu-Schwarz sector of Witten's gauge invariant, covariant open superstring. The action can be written as

$$S_{Witten} = \frac{1}{2}\langle A|Q|A\rangle + \frac{1}{3}g\langle V_3|A\rangle|A\rangle|A\rangle \ , \tag{2.1}$$

where the string fields $|A\rangle$ have (first quantized) ghost number $-\frac{1}{2}$. The gauge

invariance is given by

$$\delta|A\rangle = Q|\Lambda\rangle + g|A * \Lambda - \Lambda * A\rangle \ , \tag{2.2}$$

and the string product $*$ is defined as

$$|A * B\rangle_1 = {}_2\langle A|_3\langle B|V_3\rangle_{123} \ .$$

Imposing the Siegel gauge $b_0|A\rangle = 0$ leads to the gauge-fixed action[14,15]

$$S = \frac{1}{2}\langle \Phi|Q|\Phi\rangle + \frac{1}{3}g\langle V_3|\Phi\rangle|\Phi\rangle|\Phi\rangle \ , \tag{2.3}$$

with $|\Phi\rangle$ representing the sum of string fields of all ghost numbers, and satisfying $b_0|\Phi\rangle = 0$. As can be seen, in this gauge the interaction is given by the same vertex $|V_3\rangle$ as in the gauge invariant action.

The explicit form of the interaction vertex has been constructed in a series of papers[2-4]. Certain results were also obtained independently in Ref. [5]. The vertex construction followed from the Witten string overlaps by the use of Neumann function techniques.

The overlaps are such that all strings meet at the mid-point, where a curvature singularity resides. It is thus possible to attach operators at this mid-point without changing the overlaps. The correct insertions are deduced from imposing BRST as well as K_n (see for example ref.[2]) invariances of the vertex with insertion.

In everything that has been said so far the NS superstring parallels the bosonic string. It is at this point that the two start to differ in an essential way. In the bosonic case it turns out that no insertion is needed for the three string vertex, while in the NS superstring one has to insert the so-called picture changing operator[8] $X(\frac{\pi}{2})$. As we shall see, the presence of the insertion in the cubic vertex is responsible for all the nontrivial effects that follow.

To be able to carry out explicit superstring calculations in the operator field theory we first introduce a mid-point insertion equivalent to the picture changing operator $X(\frac{\pi}{2})$, but given without reference to bosonized ghosts. This is achieved by working with the original ghost variables (γ_r, β_s) with the vacua being related by e^{ϕ}. We recall that in the dual model insertions of $G_{\frac{1}{2}}$ and $G_{-\frac{1}{2}}$ were needed. The Witten vertex is on-shell equivalent to the dual model vertex, therefore the operator connecting these two[14] is of the form $e^{A_n L_n}$. Having in mind the commutation relations $[G, L] = G$, we are led to the insertion $G(\frac{\pi}{2})$ (the mid-point being the only place where one may put an operator in Witten's theory without spoiling the string overlaps).

The four point scattering calculation of the next section can be thought of as proof of the correctness of $G(\frac{\pi}{2})$ as an insertion. Finally let us mention that $G(\frac{\pi}{2})$ is related to the insertion recently constructed by Suehiro[7] by *imposing* BRST invariance, so that it also in fact leads to a BRST invariant interaction. Taking into account the effect of different vacua (*i.e.* an $e^{-\phi}$ operation) our insertion $G(\frac{\pi}{2})$ is indeed equivalent to $X(\frac{\pi}{2})$ but easier for calculation.

The part of the vertex coming from the overlaps (*i.e.* without insertion) is

$$\begin{aligned}|V_3^0\rangle_{123} = \exp\Big(\frac{1}{2}\alpha^a_{-n} N^{ab}_{nm} \alpha^b_{-m} + \frac{1}{2}\psi^a_{-r} K^{ab}_{rs} \psi^b_{-s} + \\ + b^a_{-n} \tilde{N}^{ab}_{nm} m c^b_{-m} + \beta^a_{-r} \tilde{K}^{ab}_{rs} \gamma^b_{-s}\Big)|+\rangle_{123} \,.\end{aligned} \tag{2.4}$$

The Neumann coefficients N and $\tilde{N}$ are as in the bosonic case, and were constructed in ref.[2]. K's can be found in ref.[3]. The $\tilde{K}$ coefficients of the super ghosts are given in the review [4] where they were denoted by $\tilde{K}'$ (the $\tilde{K}$ coefficients of that reference correspond to the $\beta\gamma$-vertex constructed over a *different* vacuum state than the one used here).

The full cubic vertex is

$$|V_3\rangle_{123} = \overline{G}\, |V_3^0\rangle_{123} \,, \tag{2.5}$$

where the operator $\overline{G}$ is proportional to $G(\frac{\pi}{2})$, and *finite* when acting on V_3^0.

Before we determine $\overline{G}$ let us first establish some basic results and conventions that will be of use later in our calculations. The results of the next section will lead us to the conclusion that in order to get correct scattering amplitudes an explicit interaction of four strings needs to be incorporated in the action. Therefore the action will be

$$S_{Witten} = \frac{1}{2}\langle A|Q|A\rangle + \frac{1}{3}g\langle V_3|A\rangle|A\rangle|A\rangle + \frac{1}{4}g^2\langle V_4|A\rangle|A\rangle|A\rangle|A\rangle \,. \tag{2.6}$$

The V_4 piece is proportional to g^2 since it needs to be the same order in g as the terms coming from the contractions of two V_3's. At this point one may wonder whether higher contact terms ($V_5, V_6, \ldots$) will be needed. In the following sections we shall in fact show that the above is indeed the *full* (gauge invariant) superstring action.

Physical states acquire Z_2 grading due to the anticommuting modes, as well as the gradings of the two vacuums of the zero mode b, c sector. Following usual conventions we take $|+\rangle$ to be even and $|-\rangle$ to be odd. GSO projection ensures that all physical states have the same grading. Since the vector $\psi_{-\frac{1}{2}}|-\rangle$ is in the spectrum we see that $|A\rangle$ is even. From the free part of the action we see that $\langle A|$ has opposite grading to $|A\rangle$, *i.e.* the two point vertex $|V_2\rangle$ which moves bras into kets is odd (the same being true for $\langle V_2|$).

$$\langle A|_1 = \langle V_2|_{12}|A\rangle_2 \,. \tag{2.7}$$

From the non vanishing of the cubic interaction we see that for superstrings $|V_3\rangle$ is odd ($\langle V_3|$ is even). Similarly, the quartic term establishes that $|V_4\rangle$ is even ($\langle V_4|$ is also even).

The general N-point vertex can be written in terms of a pure overlap piece $|V_N^0\rangle$ and an insertion I_N. The cubic vertex $|V_3^0\rangle$ is constructed over the $|+++\rangle$

vacuum, so it is even. This implies that its insertion must satisfy

$$I_3 = \text{ odd } . \tag{2.8}$$

In section 4 the connection between the 4-string and two 3-string vertices will be shown to be

$$|V_4^0\rangle_{1234} = \langle V_2|_{56}|V_3^0\rangle_{126}|V_3^0\rangle_{345} . \tag{2.9}$$

From this we immediately see that

$$I_4 = \text{ odd } . \tag{2.10}$$

The definite grading of physical states enables us to commute the $|A\rangle$'s in (6). Using BRST invariances of the vertices and relabeling dummy indices we get the following cycling properties

$$\begin{aligned} \langle V_2|_{12} &= -\langle V_2|_{21} & |V_2\rangle_{12} &= |V_2\rangle_{21} \\ \langle V_3|_{123} &= \langle V_3|_{231} & |V_3\rangle_{123} &= |V_3\rangle_{231} \\ \langle V_4|_{1234} &= \langle V_4|_{2341} & |V_4\rangle_{1234} &= -|V_4\rangle_{2341} \end{aligned} \tag{2.11}$$

Due to the curvature singularity at the mid-point operator insertions sitting there are in general divergent when acting on the vertices. We regularize this by moving the insertions a bit away from the mid-point along one of the strings. Insertions I_N thus actually carry a label indicating along which string we displace the insertion. Cyclicity of $|V_3^0\rangle$ and equations (9) and (11) imply that

$$I_3^{(1)} = I_3^{(2)} = I_3^{(3)} , \tag{2.12}$$

when acting on the cubic vertex. Also, on the quartic vertex we have

$$I_4^{(1)} = -I_4^{(2)} = I_4^{(3)} = -I_4^{(4)} . \tag{2.13}$$

Ghost number balancing further implies that $g(I_3) = 0$, while $g(I_4) = -1$.

$G(\frac{\pi}{2})$ obviously has the correct ghost number and grading to be the insertion I_3. We shall now regularize this, extract the finite insertion $(\overline{G})$, and show that it satisfies (12). First, let us review the results of the Neumann function method for constructing the interaction vertex. We shall illustrate things on the ψ and $\beta\gamma$ parts of the vertex. The basic correlation function for $\psi(\omega)$ on the unit circle is

$$K(\omega,\omega') = \langle \psi(\omega)\psi(\omega')\rangle = \frac{1}{\omega-\omega'} . \tag{2.14}$$

Since ψ has conformal dimension $\frac{1}{2}$ it follows that under the conformal transformation $\omega \to \rho(\omega)$ which takes the unit circle into the Witten overlap geometry we have

$$K(\sigma,\sigma') = \left(\frac{\partial\omega}{\partial\rho}\right)^{1/2} \frac{1}{\omega-\omega'} \left(\frac{\partial\omega'}{\partial\rho'}\right)^{1/2} . \tag{2.15}$$

The vertex coefficients K_{rs}^{ab} are simply the Fourier coefficients of the scattering part of $K(\rho,\rho')$, *i.e.*

$$K^{ab}(\sigma,\sigma') = \sum_{r,s\geq 1/2}^{\infty} e^{ir\sigma} e^{is\sigma'} K_{rs}^{ab} + \delta_{ab} \sum_{r\geq 1/2}^{\infty} e^{-ir(\sigma-\sigma')} . \tag{2.16}$$

From the last two formulas it directly follows that acting on V_3 we have the identity

$$\psi^a(\sigma) = \int \frac{d\sigma}{2\pi} K^{ab}(\sigma,\sigma')\psi^b_{creation}(\sigma') . \tag{2.17}$$

Similarly for the $\beta\gamma$ part of the vertex one uses the Neumann function

$$\tilde{K}(\sigma,\sigma') = \left(\frac{\partial\omega}{\partial\rho}\right)^{-1/2} \frac{1}{\omega-\omega'} \left(\frac{\partial\omega'}{\partial\rho'}\right)^{3/2} \frac{\omega^3+1}{\omega'^3+1} . \tag{2.18}$$

The obvious difference with the result for ψ being that now the conformal dimensions of γ and β ($-\frac{1}{2}$ and $\frac{3}{2}$) are used. The $\frac{\omega^3+1}{\omega'^3+1}$ term has been inserted[3] in order that the vertex be written as an exponential of a quadratic form acting on the simple vacuum defined by $\beta_r|0\rangle = \gamma_r|0\rangle = 0$ for all positive r. The analogues of (17) are now

$$\begin{aligned} \beta^a(\sigma) &= \int \frac{d\sigma}{2\pi} \beta^b_{creation}(\sigma') \tilde{K}^{ba}(\sigma', \sigma) \\ \gamma^a(\sigma) &= \int \frac{d\sigma}{2\pi} \tilde{K}^{ab}(\sigma, \sigma') \gamma^b_{creation}(\sigma') \ . \end{aligned} \tag{2.19}$$

The conformal transformation $\rho = \ln \frac{\omega^{3/2}-i}{\omega^{3/2}+i} - i\frac{\pi}{2}$ gives us

$$\frac{\partial \omega}{\partial \rho} = \frac{i}{3} \frac{\omega^3 + 1}{z} \ , \tag{2.20}$$

where

$$\omega^a(\sigma) = \omega_a \left(\frac{1 + ie^{i\sigma}}{1 - ie^{i\sigma}} \right)^{2/3} \ . \tag{2.21}$$

The three phases ω_a may be written as $\omega_a = (1, e, \bar{e})\ e^{i\pi/3}$, where we introduce the notation $e = e^{\frac{2\pi i}{3}}$ and $\bar{e}$ is its hermitean conjugate. We also need $z(\sigma) = \sqrt{\omega(\sigma)}$ as well as $g(\sigma) = \sqrt{z(\sigma)}$, with the roots chosen so that we have

$$\begin{aligned} z^a(\sigma) &= z_a \left(\frac{1 + ie^{i\sigma}}{1 - ie^{i\sigma}} \right)^{1/3} \\ g^a(\sigma) &= g_a \left(\frac{1 + ie^{i\sigma}}{1 - ie^{i\sigma}} \right)^{1/6} \ , \end{aligned} \tag{2.22}$$

and also $z_a = (1, \bar{e}, e)\ e^{i\pi/6}$, as well as $g_a = (1, e, \bar{e})\ e^{i\pi/12}$. With these choices of the roots the Neumann functions satisfy the correct overlap equations. For example for the ψ correlator we have

$$K^{ab}(\sigma, \sigma') = iK^{a-1\ b}(\pi - \sigma, \sigma') \ ,$$

for $0 \le \sigma \le \frac{\pi}{2}$. Formulas (17), (19) as well as similar equations for P, b, c may be used to find the action of these fields when applied to the string mid-point.

We see that for a field $A(\sigma)$ with conformal dimension J to *leading order* in ϵ we have

$$A\Big(\frac{\pi}{2} - \epsilon\Big)|V_3\rangle = \epsilon^{-\frac{1}{3}J} \overline{A}|V_3\rangle \ , \tag{2.23}$$

where $\overline{A}$ is a non-singular operator whose exact form can be calculated using formulas like (17) and (19).

Our insertion for V_3 is the finite part of $G(\frac{\pi}{2} - \epsilon)$. The operator $G(\sigma)$ is given by

$$G(\sigma) = \frac{1}{2}\Big(P(\sigma)\psi(\sigma) + b(\sigma)\gamma(\sigma) - 2c(\sigma)\partial_\sigma\beta(\sigma) - 3\partial_\sigma c(\sigma)\beta(\sigma) \Big) \ . \tag{2.24}$$

Using the identity $\partial_\sigma A(\frac{\pi}{2} - \epsilon) = -\partial_\epsilon A(\frac{\pi}{2} - \epsilon)$ we find

$$\begin{aligned} P(\frac{\pi}{2} - \sigma)\psi(\frac{\pi}{2} - \sigma)|V_3\rangle &= \epsilon^{-1/2}\overline{P}\ \overline{\psi}|V_3\rangle + \dots \\ b(\frac{\pi}{2} - \sigma)\gamma(\frac{\pi}{2} - \sigma)|V_3\rangle &= \epsilon^{-1/2}\overline{b}\ \overline{\gamma}|V_3\rangle + \dots \\ c(\frac{\pi}{2} - \sigma)\partial_\sigma\beta(\frac{\pi}{2} - \sigma)|V_3\rangle &= \frac{1}{2}\epsilon^{-7/6}\overline{c}\ \overline{\beta}|V_3\rangle + \dots \\ \partial_\sigma c(\frac{\pi}{2} - \sigma)\beta(\frac{\pi}{2} - \sigma)|V_3\rangle &= -\frac{1}{3}\epsilon^{-7/6}\overline{c}\ \overline{\beta}|V_3\rangle + \dots \end{aligned} \tag{2.25}$$

where dots indicate subleading terms in ϵ.

At first glance (25) seems to spell disaster for G as an insertion, since the $c\beta$ pieces diverge faster than the rest of G. The culprit is easily seen to be the derivation $\frac{\partial}{\partial\sigma}$. Therefore, although for primary fields we have leading behaviour as in equation (23) this does not hold for non-primary fields. For this reason one should in general expect problems when using non-primary fields as insertions on any vertex.

In the case of G, if we look closer, we see that the leading terms in $c\partial\beta$ and $\partial c\beta$ precisely cancel. Further, by expanding c and β to subleading terms we find that all of the pieces are of order $\epsilon^{-1/2}$; in other words

$$G\left(\frac{\pi}{2}-\epsilon\right)=\frac{1}{9}\left(\frac{2}{\epsilon}\right)^{1/2}\overline{G}\,. \tag{2.26}$$

The factor $\frac{\sqrt{2}}{9}$ has been put in just for convenience. The non-singular operator $\overline{G}$ is

$$\overline{G}=\alpha^a_{-n}\psi^b_{-r}F^{ab}_{nr}+p^a\psi^b_{-r}F^{ab}_{0r}+c^a_{-n}\beta^b_{-r}G^{ab}_{nr}+b^a_{-n}\gamma^b_{-r}H^{ab}_{nr}\,, \tag{2.27}$$

and it is to be used as an insertion for the cubic vertex. The coefficients F^{ab}_{nr}, G^{ab}_{nr}, H^{ab}_{nr} that have been used in this paper are listed in the appendix. Calculation of all of these coefficients is rather tedious. We shall present here calculations only for the β piece. As we have noted β is one of the fields for which both leading and subleading terms need to be evaluated.

Using formulas (18) and (19) we have

$$\beta^a\left(\frac{\pi}{2}-\epsilon\right)=\int\frac{d\sigma\prime}{2\pi}\beta^b_{creation}(\sigma\prime)f(\sigma\prime)\,, \tag{2.28}$$

where

$$f(\sigma\prime)=\left(\frac{\partial\omega'}{\partial\rho'}\right)^{-1/2}\frac{1}{\omega'-\omega}\left(\frac{\partial\omega}{\partial\rho}\right)^{-3/2}\frac{\omega'^3+1}{\omega^3+1}\,. \tag{2.29}$$

By expanding $\omega(\sigma)$, $z(\sigma)$ and $g(\sigma)$ in powers of ϵ we find the following useful results

$$\begin{aligned}\frac{1}{\omega-\omega'}&=-\frac{1}{\omega'}\left(1+\frac{\omega_a}{\omega'}\left(\frac{i\epsilon}{2}\right)^{2/3}+o(\epsilon^{4/3})\right)\\ \frac{1}{\omega^3+1}&=1+o(\epsilon^2)\\ \frac{\omega^3+1}{z}&=z_a^{-1}\left(\frac{i\epsilon}{2}\right)^{-1/3}\left(1+o(\epsilon^2)\right)\,.\end{aligned} \tag{2.30}$$

Therefore we have

$$\begin{aligned}f(\sigma')=&\frac{i}{3}z_a^{-3/2}\left(\frac{i\epsilon}{2}\right)^{-1/2}z'^{-3/2}(\omega'^3+1)^{1/2}+\\&+\frac{i}{3}z_a^{1/2}\left(\frac{i\epsilon}{2}\right)^{1/6}z'^{-7/2}(\omega'^3+1)^{1/2}+o(\epsilon^{5/6})\,.\end{aligned} \tag{2.31}$$

All we need do is expand $f(\sigma')$ in a Fourier series $f(\sigma')=\sum f_r e^{ir\sigma'}$ since this and equation (28) implies that

$$\beta^a\left(\frac{\pi}{2}-\epsilon\right)=\sum_{r>0}f_r\beta^b_{-r}\,. \tag{2.32}$$

A general formula for the Fourier coefficients f_r is not too difficult to obtain. However, for our purposes here we shall be satisfied with the more modest task of determining only the coefficients that are needed in calculations that follow in this paper. In the $c\beta$ sector we in fact make use of only the coefficient $G^{11}_{1\frac{1}{2}}$ that multiplies $c^1_{-1}\beta^1_{-\frac{1}{2}}$. Therefore in the β calculation we only need to evaluate $f_{\frac{1}{2}}$. We easily determine that

$$\begin{aligned}\beta^1_{leading}\left(\frac{\pi}{2}-\epsilon\right)&=\frac{2}{3}\left(\frac{i\epsilon}{2}\right)^{-1/2}e^{-i\frac{\pi}{4}}\beta^1_{-\frac{1}{2}}+\dots\\ \beta^1_{subleading}\left(\frac{\pi}{2}-\epsilon\right)&=\frac{2}{3}\left(\frac{i\epsilon}{2}\right)^{1/6}e^{-i\frac{\pi}{4}}\beta^1_{-\frac{1}{2}}+\dots\end{aligned} \tag{2.33}$$

where dots indicate pieces proportional to other modes of β. Similarly working with $c(\frac{\pi}{2}-\epsilon)$ we obtain

$$\begin{aligned}c^1_{leading}\left(\frac{\pi}{2}-\epsilon\right)&=\frac{4}{3}i\left(\frac{i\epsilon}{2}\right)^{1/3}c^1_{-1}+\dots\\ c^1_{subleading}\left(\frac{\pi}{2}-\epsilon\right)&=\frac{4}{3}i\left(\frac{i\epsilon}{2}\right)c^1_{-1}+\dots\end{aligned} \tag{2.34}$$

It is now quite a straightforward matter to find that

$$-c\partial\beta-\frac{3}{2}\partial c\beta=\frac{1}{9}\left(\frac{2}{\epsilon}\right)^{1/2}\left(\frac{20}{3}ic^1_{-1}\beta^1_{-\frac{1}{2}}+\dots\right)\,, \tag{2.35}$$

in other words $G^{11}_{1\frac{1}{2}}=\frac{20}{3}i$. All the other coefficients of $\overline{G}$ can be determined in the same manner. In fact, the calculation there is much simpler since things need to be expanded only to leading order in ϵ.

A simple way to see that $\overline{G}$ satisfies the criterion of equation (12) will be presented in section 4. A more precise way is to show it by using the values of the coefficients F, G, H of $\overline{G}$. Since the $P\psi$ sector cannot mix with the ghost sector upon going from one string to another it is enough to look at the F coefficients. Using these (see appendix) it becomes very easy to show that (12) does indeed hold.

Let us again note that $\overline{G}$ is a well defined and finite insertion proportional to G at the mid-point. It is equivalent to inserting the picture changing operator at the mid-point, but does not make any reference to bosonized ghosts. We should stress that $G(\frac{\pi}{2}-\epsilon)$ is precisely tailored as an insertion for a cubic vertex. For any other vertex (23) does not hold, and so the leading singularity in G does not cancel.

The super–Virasoro algebra has, along with $G(\sigma)$, also the generators $L(\sigma)$ in it. We may wonder if $L(\frac{\pi}{2}-\epsilon)$ represents a well defined insertion on some V_N. From the conformal transformation appropriate to the four string vertex[2] it is easy to see that to leading order we have

$$A\Big(\frac{\pi}{2}-\epsilon\Big)\big|V_4\big\rangle = \epsilon^{-\frac{1}{3}J}\overline{A}\big|V_4\big\rangle \ . \tag{2.36}$$

Since $L(\sigma)$ is given by the expression

$$\begin{aligned} L(\sigma) = \frac{1}{2}\Big(P(\sigma)\cdot P(\sigma) &+ \partial_\sigma\psi(\sigma)\cdot\psi(\sigma) + \\ &+ 2c(\sigma)\partial_\sigma b(\sigma) + 4\partial_\sigma c(\sigma)b(\sigma) - \\ &- \gamma(\sigma)\partial_\sigma\beta(\sigma) - 3\partial_\sigma\gamma(\sigma)\beta(\sigma)\Big) \ , \end{aligned} \tag{2.37}$$

it follows that the leading singularity mismatch for L cancels only when applied to a four string vertex.

As we can see, there exists a very important connection between the super-Virasoro algebra and consistency conditions on insertions I_N. From the fact that

there is no counterpart to L and G that works on the five point or higher vertices one is tempted to assume that these interactions are absent from the theory. In the last section of this paper we shall show that by imposing gauge invariance one is led to a theory with only cubic and quartic vertices.

3. Four Point Scattering

The simplest four point scattering amplitude that one can evaluate in the superstring is that of four vectors. Calculating even this in the operator theory is quite a formidable task. The task in front of us will be made manageable by introducing an approximation. We shall calculate the scattering amplitude in powers of $x = \exp(-\tau)$, τ being the time of propagation of the internal particle. It will be seen that this is an excellent approximation to use in evaluating Feynman diagrams in Witten's string theory.

First, however, we put aside the complications of the superstring and proceed with a much simpler calculation of scattering in the bosonic string. Here the lowest state in the spectrum is the tachyon, so we choose tachyons as our external states.

By expanding $|\Phi\rangle$ and Q in zero modes, it is easy to see that the bosonic string propagator is simply

$$b_0\frac{1}{L_0-1} = b_0\int_0^1 dx\ x^{p^2/2-2}x^R \ , \tag{3.1}$$

where R is the mode piece of L_0. The three point vertex is

$$\begin{aligned} |V_3\rangle_{123} = \exp\Big(\frac{1}{2}\sum_{n,m=0}^{\infty}\sum_{a,b=1}^{3}\alpha^a_{-n}N^{ab}_{nm}\alpha^b_{-m} &+ \\ +\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{a,b=1}^{3} b^a_{-n}\tilde{N}^{ab}_{nm}mc^b_{-m}\Big)&|+\rangle_{123} \ . \end{aligned} \tag{3.2}$$

Coming now to the scattering of four tachyons we see that the s-channel amplitude for this process is simply

$$A_s = \int_0^1 dx\; x^{-s/2-2}\langle V_{34}(\overline{3}')|b_0 x^R|V_{12}(3')\rangle\ ,$$

where we have introduced the notation

$$|V_{12}(3')\rangle = {}_{12}\langle - |V_3\rangle_{123'}$$
$$\langle V_{34}(\overline{3}')| = {}_{3'4'}\langle V_2|V_{34}(4')\rangle\ .$$

Using the fact that $x^R\alpha_{-n}x^{-R} = x^n\alpha_{-n}$, as well as similar results for b's and c's, we find

$$A_s = \int_0^1 dx\; x^{-s/2-2}\langle V_{34}(\overline{3}')|b_0|V_{12}(\dot{3}')\rangle\ . \tag{3.3}$$

The dot on V indicates that α_{-n}, b_{-n}, c_{-n} have been replaced by $x^n\alpha_{-n}$, $x^n b_{-n}$, $x^n c_{-n}$. This matrix element is very difficult to do, however, we can expand $\dot{V}$ in powers of $x = e^{-\tau}$. Using the appropriate Neumann coefficients after a bit of straightforward algebra we find

$$A_s = \int_0^1 dx\; x^{-s/2-2}\Big(1 - \frac{11^2}{3^6}x^2 + \cdots\Big)e^{E(x)}\ .$$

The term in the exponent has the expansion

$$\begin{aligned} E(x) = -\frac{s}{2}\ln\Big(\frac{2^4}{3^3}\Big) &- \Big(\frac{s}{2}+2\Big)\Big(-\frac{2^3}{3^3}x + \frac{2^2\cdot 19}{3^6}x^2 + \cdots\Big) - \\ &- \Big(\frac{t}{2}+2\Big)\Big(-\frac{2^4}{3^3}x - \frac{2^6\cdot 7^2}{3^{10}}x^3 + \cdots\Big) - \\ &- \Big(\frac{2^4}{3^3}x + \frac{2^3}{3^6}x^2 + \cdots\Big) + \Big(\frac{26\cdot 5^2}{2\cdot 3^6}x^2 + \cdots\Big)\ . \end{aligned} \tag{3.4}$$

This result can easily be put in a much more manageable and useful form

$$A_s = \int_0^{z(1)} dz\; z^{-s/2-2}(1-z)^{-t/2-2}\ , \tag{3.5}$$

where

$$z(x) = \frac{2^4}{3^3}\,x\left(1 - \frac{2^3}{3^3}\,x + \frac{2^2}{3^3}\,x^2 + \cdots\right)\ . \tag{3.6}$$

To this order in x we have $z(1) \approx 0.50480$ which is very close to the exact

value $z(1) = \frac{1}{2}$ shown by Giddings.[13] Using this exact value the $s-t$ amplitude is simply

$$A(s,t) = A_s + A_t = \int_0^1 dz\; z^{-s/2-2}(1-z)^{-t/2-2}\ , \tag{3.7}$$

which is the familiar Veneziano formula.

It should not be surprising that our approximation converges so rapidly to the exact value. This is a consequence of the specific conformal transformation, *i.e.* $z(x)$, that maps the Witten string overlap into the upper half plane of the dual model. Still, we should contrast this with the light-cone case where the series expansion for $z(x)$ does not converge at $x = 1$.[17]

We turn now to the scattering of vectors in the superstring. First we introduce the kinematic invariants for this scattering. General 4-point scattering has the usual invariants

$$\begin{aligned} s &\equiv -(p_1+p_2)^2 \\ t &\equiv -(p_1+p_4)^2 \\ u &\equiv -(p_1+p_3)^2 \end{aligned} \tag{3.8}$$

From momentum conservation and the fact that vectors are massless, it follows that

$$s + t + u = 0\ . \tag{3.9}$$

Under the re-labeling $P = \binom{1234}{4123}$ the s-channel diagram goes into the t-channel. The invariants s, t, u transform under P in a very simple way

$$\begin{aligned} s &\longrightarrow t \\ t &\longrightarrow s \\ u &\longrightarrow u\ . \end{aligned} \tag{3.10}$$

Using the polarization vectors one can construct six further invariants. We define

$$\begin{aligned} S &\equiv (\varsigma_1 \cdot \varsigma_2)(\varsigma_3 \cdot \varsigma_4) \\ T &\equiv (\varsigma_2 \cdot \varsigma_3)(\varsigma_1 \cdot \varsigma_4) \\ U &\equiv (\varsigma_1 \cdot \varsigma_3)(\varsigma_2 \cdot \varsigma_4) \quad , \end{aligned} \tag{3.11}$$

along with

$$\begin{aligned} \mathbf{S} =& \varsigma_1 \cdot p_4 \varsigma_3 \cdot p_2 \varsigma_2 \cdot \varsigma_4 + \varsigma_2 \cdot p_3 \varsigma_4 \cdot p_1 \varsigma_1 \cdot \varsigma_3 + \\ &+ \varsigma_1 \cdot p_3 \varsigma_4 \cdot p_2 \varsigma_2 \cdot \varsigma_3 + \varsigma_2 \cdot p_4 \varsigma_3 \cdot p_1 \varsigma_1 \cdot \varsigma_4 \\ \mathbf{T} =& \varsigma_2 \cdot p_1 \varsigma_4 \cdot p_3 \varsigma_3 \cdot \varsigma_1 + \varsigma_3 \cdot p_4 \varsigma_1 \cdot p_2 \varsigma_2 \cdot \varsigma_4 + \\ &+ \varsigma_2 \cdot p_4 \varsigma_1 \cdot p_3 \varsigma_3 \cdot \varsigma_4 + \varsigma_3 \cdot p_1 \varsigma_4 \cdot p_2 \varsigma_2 \cdot \varsigma_1 \\ \mathbf{U} =& \varsigma_1 \cdot p_2 \varsigma_4 \cdot p_3 \varsigma_3 \cdot \varsigma_2 + \varsigma_3 \cdot p_4 \varsigma_2 \cdot p_1 \varsigma_1 \cdot \varsigma_4 + \\ &+ \varsigma_1 \cdot p_4 \varsigma_2 \cdot p_3 \varsigma_3 \cdot \varsigma_4 + \varsigma_3 \cdot p_2 \varsigma_4 \cdot p_1 \varsigma_1 \cdot \varsigma_2 \quad . \end{aligned} \tag{3.12}$$

Under the action of P these transform into each other just like s, t, u.

As in the bosonic case, the s–channel amplitude here will be given in terms of integrals of the form $\int_0^{1/2} dz\ z^{-s/2-a}\ (1-z)^{-t/2-b}$. Three of these transform simply under P.

$$\begin{aligned} \beta_s &\equiv \int_0^{\frac{1}{2}} dz\ z^{-s/2}\ (1-z)^{-t/2-1} \\ \beta_t &\equiv \int_0^{\frac{1}{2}} dz\ z^{-s/2-1}\ (1-z)^{-t/2} \\ \beta_u &\equiv \int_0^{\frac{1}{2}} dz\ z^{-s/2}\ (1-z)^{-t/2} \ . \end{aligned} \tag{3.13}$$

It is easy to see that

$$\begin{aligned} \beta_s &\longrightarrow \overline{\beta}_t \\ \beta_t &\longrightarrow \overline{\beta}_s \\ \beta_u &\longrightarrow \overline{\beta}_u \ , \end{aligned} \tag{3.14}$$

where $\overline{\beta}$ denotes the same integrand as β but integrated over the complementary range, *i.e.* from $\frac{1}{2}$ to 1.

We proceed with the calculation. The propagator is now

$$b_0 \frac{1}{L_0 - \frac{1}{2}} \ , \tag{3.15}$$

while the vertex is $|V_3>=\overline{G}|V_3^0>$. Expressions for $\overline{G}$ and $|V_3^0>$ were given in the previous section.

The vertex with vectors on two of its legs is given by

$$\begin{aligned} \big|V_{12}(3)\big\rangle &= {}_1\big\langle - \big|(\psi^1_{\frac{1}{2}} \cdot \varsigma_1)\ {}_2\big\langle - \big|(\psi^2_{\frac{1}{2}} \cdot \varsigma_2)\big|V_3\big\rangle_{123} \\ &= \big(\mathcal{P}_{12}(3)\mathcal{Q}_{12}(3) + \mathcal{R}_{12}(3)\big)\exp(Z_{12}(3))\big| + \big\rangle_3 \ , \end{aligned} \tag{3.16}$$

where we have employed the following notation

$$\mathcal{P}_{12}(3) = c^3_{-n} G^{33}_{nr} \beta^3_{-r} + b^3_{-n} H^{33}_{nr} \gamma^3_{-r} \ , \tag{3.17}$$

$$\mathcal{Q}_{12}(3) = K^{12}_{\frac{1}{2}\frac{1}{2}} \varsigma_1 \cdot \varsigma_2 + K^{12}_{\frac{1}{2}s} K^{12}_{r\frac{1}{2}} \varsigma_1 \cdot \psi^3_{-r} \varsigma_2 \cdot \psi^3_{-s} \ , \tag{3.18}$$

$$\begin{aligned} \mathcal{R}_{12}(3) =& \Big(K^{12}_{\frac{1}{2}\frac{1}{2}}\ \varsigma_1 \cdot \varsigma_2\ p^a F^{a3}_{0s} + K^{23}_{\frac{1}{2}s} \varsigma_1 \cdot p^a F^{a1}_{0\frac{1}{2}} \varsigma_2 - K^{13}_{\frac{1}{2}s} \varsigma_2 \cdot p^a F^{a2}_{0\frac{1}{2}} \varsigma_1\Big) \cdot \psi^3_{-s} + \\ &+ \Big(K^{12}_{\frac{1}{2}\frac{1}{2}} \varsigma_1 \cdot \varsigma_2 \alpha_{-n} F^{33}_{ns} + K^{23}_{\frac{1}{2}s} \varsigma_1 \cdot \alpha_{-n} F^{31}_{n\frac{1}{2}} \varsigma_2 - K^{13}_{\frac{1}{2}s} \varsigma_2 \cdot \alpha_{-n} F^{32}_{n\frac{1}{2}} \varsigma_1\Big) \cdot \psi^3_{-s} + \\ &+ \Big(\alpha_{-n} \cdot \psi^3_{-r'} F^{33}_{nr'} + p^a \cdot \psi^3_{-r'} F^{a3}_{0r'}\Big) K^{12}_{\frac{1}{2}s} K^{12}_{r\frac{1}{2}} \varsigma_1 \cdot \psi^3_{-r} \varsigma_2 \cdot \psi^3_{-s} \ , \end{aligned} \tag{3.19}$$

as well as

$$Z_{12}(3) = \frac{1}{2}\alpha^3_{-n} N^{33}_{nm} \alpha^3_{-m} + \frac{1}{2}\psi^3_{-r} K^{33}_{rs} \psi^3_{-s} + b^3_{-n} \tilde{N}^{33}_{nm} m c^3_{-s} + \beta^3_{-r} \tilde{K}^{33}_{rs} \gamma^3_{-s} \ . \tag{3.20}$$

The amplitude in the s-channel may be written as

$$A_s = \int_0^1 dx\ x^{-s/2-3/2} \big\langle V_{34}(\overline{3}') \big| b_0 x^R \big| V_{12}(3') \big\rangle \ . \tag{3.21}$$

We denote the above matrix element as J. A bit of algebra gives us J in terms of $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{Z}$. We find that

$$
\begin{aligned}
J = &-\left\langle \mathcal{R}_{34}(\bar{3}')\mathcal{R}_{12}(\dot{3}')\right\rangle_{X\psi}\cdot\left\langle b_0\right\rangle_{bc\beta\gamma}+\\
&+\left\langle \mathcal{Q}_{34}(\bar{3}')\mathcal{Q}_{12}(\dot{3}')\right\rangle_{X\psi}\cdot\left\langle \mathcal{P}_{34}(\bar{3}')b_0\mathcal{P}_{12}(\dot{3}')\right\rangle_{bc\beta\gamma}\,.
\end{aligned}
\tag{3.22}
$$

By $\left\langle b_0\right\rangle_{bc\beta\gamma}$ for example, we mean

$$
{}_{3'}\!\left\langle + \mid \exp\left(\mathcal{Z}_{34}(\bar{3}')\right)\, b_0^{3'} \exp\left(\mathcal{Z}_{12}(\dot{3}')\right) \mid +\right\rangle_{3'}\,,
$$

and the subscript "$bc\beta\gamma$" indicates that we should take only that sector of the $\mathcal{Z}$'s in evaluating this expression.

Expanding J in powers of x to order $x^{\frac{3}{2}}$ we find

$$
J = x^{1/2}\Big(K_0\langle 1\rangle_{\beta\gamma}+xK_1\Big)\langle 1\rangle_X\,,
\tag{3.23}
$$

where the most tedious part of this calculation is determining the coefficients K_0 and K_1. A long but straightforward calculation gives us

$$
\begin{aligned}
K_0 = -(K^{12}_{\frac12\frac12})^2&\left(\varsigma_3\cdot\varsigma_4 p^b F^{b3}_{0\frac12}+\varsigma_3\cdot p^b F^{b1}_{0\frac12}\varsigma_4+\varsigma_4\cdot p^b F^{b2}_{0\frac12}\varsigma_3\right)\cdot\\
&\cdot\left(\varsigma_1\cdot\varsigma_2 p^a F^{a3}_{0\frac12}+\varsigma_1\cdot p^a F^{a1}_{0\frac12}\varsigma_2+\varsigma_2\cdot p^a F^{a2}_{0\frac12}\varsigma_1\right)
\end{aligned}
\tag{3.24}
$$

as well as

$$
\begin{aligned}
K_1 =&\left(K^{12}_{\frac12\frac12}\varsigma_3\cdot\varsigma_4 p^b F^{b3}_{0\frac32}+K^{23}_{\frac12\frac32}\varsigma_3\cdot p^b F^{b1}_{0\frac12}\varsigma_4-K^{23}_{\frac12\frac32}\varsigma_4\cdot p^b F^{b2}_{0\frac12}\varsigma_3\right)\cdot\\
&\left(K^{12}_{\frac12\frac12}\varsigma_1\cdot\varsigma_2 p^a F^{a3}_{0\frac32}+K^{23}_{\frac12\frac32}\varsigma_1\cdot p^a F^{a1}_{0\frac12}\varsigma_2-K^{23}_{\frac12\frac32}\varsigma_2\cdot p^a F^{a2}_{0\frac12}\varsigma_1\right)-\\
&-(K^{12}_{\frac12\frac12})^2\left(\varsigma_3\cdot\varsigma_4 p^b F^{b3}_{0\frac12}+\varsigma_3\cdot p^b F^{b1}_{0\frac12}\varsigma_4+\varsigma_4\cdot p^b F^{b2}_{0\frac12}\varsigma_3\right)\cdot\\
&\cdot\left(\varsigma_1\cdot\varsigma_2 F^{33}_{1\frac12}\,\mathbb{1}+\varsigma_1\varsigma_2 F^{32}_{1\frac12}+\varsigma_2\varsigma_1 F^{31}_{1\frac12}\right)\cdot p^{b'}N^{b'3}_{01}-\\
&-(K^{12}_{\frac12\frac12})^2\left(\varsigma_1\cdot\varsigma_2 p^a F^{a3}_{0\frac12}+\varsigma_1\cdot p^a F^{a1}_{0\frac12}\varsigma_2+\varsigma_2\cdot p^a F^{a2}_{0\frac12}\varsigma_1\right)\cdot\\
&\cdot\left(\varsigma_3\cdot\varsigma_4 F^{33}_{1\frac12}\,\mathbb{1}+\varsigma_3\varsigma_4 F^{32}_{1\frac12}+\varsigma_4\varsigma_3 F^{31}_{1\frac12}\right)\cdot p^{a'}N^{a'3}_{01}-\\
&-(K^{12}_{\frac12\frac12})^4 F^{a3}_{0\frac12}F^{b3}_{0\frac12}\Big(\varsigma_2\cdot\varsigma_4\varsigma_1\cdot p^b\varsigma_3\cdot p^a-\varsigma_2\cdot\varsigma_3\varsigma_1\cdot p^b\varsigma_4\cdot p^a\\
&\qquad-\varsigma_1\cdot\varsigma_4\varsigma_3\cdot p^a\varsigma_2\cdot p^b+\varsigma_1\cdot\varsigma_3\varsigma_4\cdot p^a\varsigma_2\cdot p^b\\
&\qquad+\varsigma_1\cdot\varsigma_4\varsigma_2\cdot\varsigma_3 p^a\cdot p^b-\varsigma_1\cdot\varsigma_3\varsigma_2\cdot\varsigma_4 p^a\cdot p^b\Big)-\\
&-(K^{12}_{\frac12\frac12}F^{33}_{1\frac12})^2\Big((D-2)\varsigma_1\cdot\varsigma_2\varsigma_3\cdot\varsigma_4-\varsigma_1\cdot\varsigma_3\varsigma_2\cdot\varsigma_4+2\varsigma_1\cdot\varsigma_4\varsigma_3\cdot\varsigma_2\Big)-\\
&-2G^{33}_{1\frac12}H^{33}_{1\frac12}(K^{12}_{\frac12\frac12})^2\varsigma_1\cdot\varsigma_2\varsigma_3\cdot\varsigma_4\,,
\end{aligned}
\tag{3.25}
$$

where a,a' run through $1,2,3'$; b,b' run through $3,4,\overline{3'}$. $\mathbb{1}$ is just a diagonal matrix with components $\delta_{\mu\nu}$.

The matrix element $\langle 1\rangle_X$ has been evaluated in the bosonic calculation. Carefully transcribing that result (bearing in mind that tachyons satisfied $s+t+u=-8$ while for vectors $s+t+u=0$) we find

$$
\begin{aligned}
A_s = \int_0^1 dx\,&\left(\frac{2^4}{3^3}x\big(1-\frac{2^3}{3^3}x+\cdots\big)\right)^{-s/2-1}\left(1-\frac{2^4}{3^3}x\big(1-\frac{2^3}{3^3}x+\cdots\big)\right)^{-t/2}\cdot\\
&\cdot\left(1-\frac{2^3}{3^3}x+\cdots\right)\Big(K_0\langle 1\rangle_{\beta\gamma}+xK_1\Big)
\end{aligned}
\tag{3.26}
$$

Using equation (6) for $z(x)$, as well as the fact that $z(1)=\frac{1}{2}$, leads us to

$$\int_0^{1/2} dz\, z^{-s/2-1}(1-z)^{-t/2}(1+\frac{1}{2}z+\cdots)(K_0\langle 1\rangle_{\beta_7}+\frac{27}{16}zK_1+\cdots) =$$
$$= \int_0^{1/2} dz\, z^{-s/2-1}(1-z)^{-t/2}\Big(K_0+z(\frac{5}{4}K_0+\frac{27}{16}K_1)+\cdots\Big)\,. \tag{3.27}$$

The amplitude A_s consists of two types of terms. The first are terms of the form $(\varsigma\cdot\varsigma)(\varsigma\cdot p)(\varsigma\cdot p)$. In terms of the invariants (8,11,12), as well as the integrals (13), we find this piece to be just

$$A'_s = (\mathbf{T}-\mathbf{U})\beta_t + (\mathbf{S}-\mathbf{U})\beta_s\,. \tag{3.28}$$

The second set of terms is of the form $(\varsigma\cdot\varsigma)(\varsigma\cdot\varsigma)$. It may be written in the following way

$$A''_s = \frac{t}{2}T\beta_t + \frac{s}{2}S\beta_s + (1+\frac{u}{2})(U-S-T)\beta_u + A_s^{extra}\,, \tag{3.29}$$

where we have

$$A_s^{extra} = (s-2)\beta_u(U-\frac{3}{2}T-2S) - \frac{s}{2}\beta_t S + \frac{t}{2}\beta_u S + \dots\,, \tag{3.30}$$

and dots indicate terms that come from higher powers of x. Collecting the two pieces, and using the transformation properties of the invariants and integrals under P we have

$$A(s,t) = A_s + A_t = (\mathbf{T}-\mathbf{U}+\frac{t}{2}T)(\beta_t+\overline{\beta}_t) + (\mathbf{S}-\mathbf{U}+\frac{s}{2}S)(\beta_s+\overline{\beta}_s)+$$
$$+(1+\frac{u}{2})(U-S-T)(\beta_u+\overline{\beta}_u) + A(s,t)^{extra}\,. \tag{3.31}$$

The integrals $\beta+\overline{\beta}$ are now given in terms of Euler beta functions, so that

$$A(s,t) = \Big(\frac{1}{4}st(U-S-T) - \frac{1}{2}t(\mathbf{T}-\mathbf{U}+\frac{t}{2}T) - \frac{1}{2}s(\mathbf{S}-\mathbf{U}+\frac{s}{2}T)\Big)\cdot$$
$$\cdot\frac{\Gamma(-s/2)\Gamma(-t/2)}{\Gamma(1-s/2-t/2)} + A(s,t)^{extra}\,. \tag{3.32}$$

Using (9) this can be cast in a more symmetric looking form

$$A(s,t) = -\Big(\frac{1}{2}s\mathbf{S}+\frac{1}{2}t\mathbf{T}+\frac{1}{2}u\mathbf{U}-\frac{1}{4}stU-\frac{1}{4}suT-\frac{1}{4}tuS\Big)\cdot$$
$$\cdot\frac{\Gamma(-s/2)\Gamma(-t/2)}{\Gamma(1-s/2-t/2)} + A(s,t)^{extra}\,. \tag{3.33}$$

Except for the extra piece this is the correct dual model result.

We see that an extra four point term is indeed present. In order to be able to cancel it with an added contact term it must not have any poles. To the order that we have calculated we see that the first poles *do* cancel. Extending this we can get the exact extra four point term by always adding pieces in such a way that order by order we cancel all poles. Obviously this extension is not unique. In the next section we shall find the *exact* expression for the contact term. The main reason for the previous calculation is that it exhibits the existence of an extra term in a straightforward way and does not involve any limiting procedures.

4. Associativity

One of the central axioms of Witten's string field theory is associativity of his string product $*$, *i.e.* the assumption that

$$A*(B*C) = (A*B)*C\,.$$

Associativity is crucial in showing gauge invariance of the string action. The operator representation of Witten's theory gives a concrete representation for $*$. We must now check if the assumed associativity indeed holds. We have

$$|A*(B*C)\rangle_1 = {}_2\langle A|_5\langle B*C||V_3\rangle_{125}\,.$$

Writing out $B*C$, and using cyclicity of V_3, we find

$$|A*(B*C)\rangle_1 = {}_2\langle A|_3\langle B|_4\langle C|\langle V_3|_{56}|V_3\rangle_{634}|V_3\rangle_{512}\,. \tag{4.1}$$

Using equations (2.5) as well as (2.26) this becomes equal to

$$\left(\frac{3^2}{2^{3/2}}\,\epsilon^{1/2}\right)^2 {}_2\langle A|_3\langle B|_4\langle C|\langle V_2|_{56}\, G^6\!\left(\frac{\pi}{2}-\epsilon\right)G^5\!\left(\frac{\pi}{2}-\epsilon\right)|V_3^0\rangle_{634}|V_3^0\rangle_{512}\,.$$

Since $G = \frac{1}{2}\, P\,\psi\; + \;\ldots$ the V_2 overlap for G is simply

$$\langle V_2|_{12}\big(G^1(\sigma) - i\, G^2(\pi-\sigma)\big) = 0 \qquad \text{for all } \sigma\,.$$

We may now move both G's to the same leg, so that

$$|A*(B*C)\rangle_1 = \frac{3^4}{2^3}\,\epsilon\, i\;{}_2\langle A|_3\langle B|_4\langle C|\langle V_2|_{56}\; G_+^5 G_-^5\,|V_3^0\rangle_{634}|V_3^0\rangle_{512}\,. \tag{4.2}$$

where $\pm$ simply denotes the points $\frac{\pi}{2} \pm \epsilon$.

Now that both insertions are on the same string we may use the familiar operator product expansion (given in the z-plane of fig.3)

$$G(z)G(z') = \frac{\frac{1}{2}}{z-z'}L(z') + \cdots\,,$$

where dots indicate non-singular pieces in ϵ. From the z-plane overlap it follows that

$$G_+^a G_-^a = E^a\,\frac{e^{i\frac{\pi}{3}}}{2\sqrt{3}}\,\frac{1}{\epsilon}\,L_-^a + \cdots\,, \tag{4.3}$$

with $E^a = (1, e, \bar{e})$, and $e \equiv e^{\frac{2\pi}{3}i}$. From now on we establish a convention to always label the string on which the OPE's are calculated "1", so that $E^1 = 1$ will be used. As a result of this we are left with a single insertion $\propto L$ residing on the internal string. Therefore $A*(B*C)$ is

$$\frac{3^4}{2^3}\, i\, \frac{e^{i\frac{\pi}{3}}}{2\sqrt{3}}\;{}_2\langle A|_3\langle B|_4\langle C|\langle V_2|_{56} L_-^5 |V_3^0\rangle_{634}|V_3^0\rangle_{512}\,.$$

Using expressions for operators at $\frac{\pi}{2} - \epsilon$ acting on the cubic vertex (as in

section 2) we have[5]

$$P_-^1 = e P_-^2 = \bar{e} P_-^3$$
$$\psi_-^1 = \bar{e}\psi_-^2 = e\psi_-^3\,.$$

From this, for example, we directly see that $G_-^1 = G_-^2 = G_-^3$, which as we saw in (2.12) is a necessary property for $\overline{G}$ to be a good insertion for V_3. It also follows that $L(\sigma) = \frac{1}{2}P(\sigma)^2 + \cdots$ obeys

$$L_-^1 = \bar{e} L_-^2 = e L_-^3\,. \tag{4.4}$$

With the help of this relation we may move the L insertion to the leg labeled "1." Finally having moved the insertion to an external leg we have the result

$$|A*(B*C)\rangle_1 = i\,\frac{3^3\sqrt{3}}{2^4}\,e^{i\frac{\pi}{3}}\,\bar{e}\cdot{}_2\langle A|_3\langle B|_4\langle C|L^1\!\left(\frac{\pi}{2}-\epsilon\right)|V_4^0\rangle_{1234}\,, \tag{4.5}$$

where we have defined (as promised in section 2) the "bare" four point vertex by

$$|V_4^0\rangle_{1234} = \langle V_2|_{56}|V_3^0\rangle_{512}|V_3^0\rangle_{634}\,. \tag{4.6}$$

We will have more to say about V_4^0 in the next section. Let us note here that our calculations in this section so far depend crucially on the OPE equation (6) as well as the phases in equation (8). The consistency of the two may readily be tested by performing the cyclings of insertions and OPE's in different orders. It is easy to convince oneself that the final result is independent of this order.

The $(A*B)*C$ calculation proceeds in the same way as the previous one. We find that

$$|(A*B)*C\rangle_1 = {}_2\langle A|_3\langle B|_4\langle C|\langle V_2|_{56}|V_3\rangle_{623}|V_3\rangle_{541}\,. \tag{4.7}$$

Moving both insertions, as before, to the fifth leg and performing the OPE we find

$$\frac{3^4}{2^3}\, i\, \frac{e^{i\frac{\pi}{3}}}{2\sqrt{3}}\; {}_2\langle A|_3\langle B|_4\langle C|\langle V_2|_{56} L_-^5 |V_3^0\rangle_{623}|V_3^0\rangle_{541} \; .$$

We next take the L insertion to the external leg with the help of equation (8). Using the fact that

$$\langle V_2|_{56}|V_3^0\rangle_{634}|V_3^0\rangle_{512} = \langle V_2|_{56}|V_3^0\rangle_{623}|V_3^0\rangle_{541} \; , \tag{4.8}$$

(which has been proven in the bosonic case,[3] and just represents associativity for that string) we arrive at the final expression for $(A*B)*C$, namely

$$|(A*B)*C\rangle_1 = i\,\frac{3^3\sqrt{3}}{2^4}\, e^{i\frac{\pi}{3}}\, e\cdot {}_2\langle A|_3\langle B|_4\langle C|L^1\Big(\frac{\pi}{2}-\epsilon\Big)|V_4^0\rangle_{1234} \; . \tag{4.9}$$

The reason for the difference in the phase (e *vs.* $\bar{e}$) of these two results is that in moving L from the internal line where it is created *via* the OPE to our fixed external line we once proceed clockwise and the other time counter-clockwise along the legs of a vertex.

We thus find that associativity does not hold, but rather that due to the effect of the insertions, it suffers what at first seems to be only a slight generalization

$$|A*(B*C)\rangle = e|(A*B)*C\rangle \; . \tag{4.10}$$

To recapitulate, associativity fails because L (arising through the short distance expansion of two G's), unlike the G's themselves, acquires a phase while cycling on the legs of the cubic vertex. As we have seen, G is tailor-made for the cubic vertex, but L does not have this property. In fact, the only vertex on which L makes sense as an insertion is a quartic one. This in fact represents a non-trivial check of the consistency of these calculations.

5. Gauge Invariance

The failure of associativity has great repercussions on the theory. The central property of the action in (2.1) was invariance under the gauge transformation (2.2). Proof of gauge invariance relies on associativity. Thus the action, as well as the gauge transformation, may have to be changed a bit. From the result of four point scattering we are already resigned to adding an explicit four string interaction

$$S_{Witten} = \frac{1}{2}\langle A|Q|A\rangle + \frac{1}{3}g\langle V_3|A\rangle|A\rangle|A\rangle + \frac{1}{4}g^2\langle V_4|A\rangle|A\rangle|A\rangle|A\rangle \; . \tag{5.1}$$

We shall now show that *imposing* gauge invariance completely determines both the action and the gauge transformation. Both acquire explicit order g^2 terms.

One easily sees that invariance of the above action under the transformation

$$|\delta A\rangle = |\delta_0 A\rangle + g|\delta_1 A\rangle + g^2|\delta_2 A\rangle \tag{5.2}$$

implies the following equations (in orders of g)

$$\langle A|Q|\delta_0 A\rangle = 0 \tag{5.3}$$

$$\langle V_3|_{123}|\delta_0 A\rangle_1|A\rangle_2|A\rangle_3 + \langle A|Q|\delta_1 A\rangle = 0 \tag{5.4}$$

$$\langle V_4|_{1234}|\delta_0 A\rangle_1|A\rangle_2|A\rangle_3|A\rangle_4 + \langle V_3|_{123}|\delta_1 A\rangle_1|A\rangle_2|A\rangle_3 + \langle A|Q|\delta_2 A\rangle = 0 \tag{5.5}$$

$$\langle V_4|_{1234}|\delta_1 A\rangle_1|A\rangle_2|A\rangle_3|A\rangle_4 + \langle V_3|_{123}|\delta_2 A\rangle_1|A\rangle_2|A\rangle_3 = 0 \tag{5.6}$$

$$\langle V_4|_{1234}|\delta_2 A\rangle_1|A\rangle_2|A\rangle_3|A\rangle_4 = 0 \; . \tag{5.7}$$

Equations (3) and (4) are as before, *i.e.* without g^2 modifications. They determine $\delta_0 A$ and $\delta_1 A$. (5) is for us the central equation and it determines the

g^2 terms V_4 and $\delta_2 A$. Since we are not admitting higher order terms in either the action or the gauge transformation law then equations (6) and (7) play the role of consistency conditions. We should note that in the bosonic case we have no V_4 or $\delta_2 A$, so that (5) is the consistency condition. It is satisfied in the bosonic string because of associativity. (3) trivially implies $|\delta_0 A\rangle = Q|\Lambda\rangle$. Using this the V_3 term in (4) becomes

$$ {}_3\langle A|_2\langle A|_1\langle \Lambda|Q_1|V_3\rangle_{123} \, . $$

Using BRST invariance of $|V_3\rangle$, as well as the odd grading of $\langle A|$, this is equal to

$$ -{}_2\langle A|Q_2\, {}_3\langle A|_1\langle \Lambda||V_3\rangle_{123} + {}_3\langle A|Q_{3}\,{}_2\langle A|_1\langle \Lambda||V_3\rangle_{123} \, . $$

Re-labeling the first piece $\binom{123}{312}$ and the second $\binom{123}{231}$, and using cyclicity of $|V_3\rangle$ gives

$$ |\delta_1 A\rangle = |A * \Lambda - \Lambda * A\rangle \, , \tag{5.8} $$

which is just the well known result. Let us look now to equation (5). We would like to satisfy this equation without changing the usual form of gauge invariance. It will turn out that this cannot be done. Still we want to determine V_4 so that it cancels as much of the cubic piece as possible. We first calculate the piece known to us. The expression ${}_3\langle A|_2\langle A|_1\langle \delta_1 A|V_3\rangle_{123}$ simply becomes

$$ {}_3\langle A|_2\langle A|\langle V_2|_{14}\Big({}_5\langle A|_6\langle \Lambda| - {}_5\langle \Lambda|_6\langle A|\Big)|V_3\rangle_{456}|V_3\rangle_{123} \, . $$

By re-labeling the indices $\binom{123456}{541623}$ this is put in a more convenient form

$$ -{}_1\langle A|_4\langle A|\Big({}_2\langle A|_3\langle \Lambda| - {}_2\langle \Lambda|_3\langle A|\Big)\langle V_2|_{56}|V_3\rangle_{623}|V_3\rangle_{541} \, . \tag{5.9} $$

The contraction $\langle V_2|V_3\rangle|V_3\rangle$ has already been evaluated. When looking at asso-

ciativity we determined that

$$ \langle V_2|_{56}|V_3\rangle_{623}|V_3\rangle_{541} = i\,\frac{3^3\sqrt{3}}{2^4}\, e^{i\frac{\pi}{3}}\, \langle V_2|_{56}\, L^6\left(\frac{\pi}{2}+\epsilon\right)|V_3^0\rangle_{623}|V_3^0\rangle_{541} \, . \tag{5.10} $$

Moving the L insertion of the previous equation to the second leg we get

$$ \langle V_2|_{56}|V_3\rangle_{623}|V_3\rangle_{541} = i\,\frac{3^3\sqrt{3}}{2^4}\, e^{i\frac{\pi}{3}}\, \bar{\epsilon}\, L_+^2 \langle V_2|_{56}|V_3^0\rangle_{623}|V_3^0\rangle_{541} \, . $$

By virtue of equations (4.6) and (4.8) this can be written as

$$ \langle V_2|_{56}|V_3\rangle_{623}|V_3\rangle_{541} = i\,\frac{3^3\sqrt{3}}{2^4}\, e^{i\frac{\pi}{3}}\, \epsilon\, L_+^2|V_4^0\rangle_{1234} \, . \tag{5.11} $$

On the other hand we may also move the insertion onto the third leg. This gives us

$$ \langle V_2|_{56}|V_3\rangle_{623}|V_3\rangle_{541} = i\,\frac{3^3\sqrt{3}}{2^4}\, e^{i\frac{\pi}{3}}\, \epsilon\, L_+^3|V_4^0\rangle_{1234} \tag{5.12} $$

Note again the difference in phase for these two expressions. Using these last two results as well as the cyclicity of $|V_4^0\rangle$ enables us to write (9) as

$$ i\,\frac{3^3\sqrt{3}}{2^4}\, e^{i\frac{\pi}{3}}\, (\bar{\epsilon}-\epsilon)\; {}_4\langle A|_3\langle A|_2\langle A|_1\langle \Lambda|L_+^1|V_4^0\rangle_{1234} \, . \tag{5.13} $$

Using the explicit values of ϵ and $\bar{\epsilon}$ this can be made a bit more tidy. We have

$$ {}_3\langle A|_2\langle A|_1\langle \delta_1 A|V_3\rangle_{123} = -\bar{\epsilon}\left(\frac{3}{2}\right)^4\, {}_4\langle A|_3\langle A|_2\langle A|_1\langle \Lambda|L_+^1|V_4^0\rangle_{1234} \, . \tag{5.14} $$

As we see this is just proportional to the association. Loss of associativity thus indeed forces us to add g^2 order terms to our theory. The V_4 piece in equation

(5) is

$$\langle V_4|_{1234}|\delta_0 A\rangle_1|A\rangle_2|A\rangle_3|A\rangle_4 = \langle V_4|_{1234}Q_1|\Lambda\rangle_1|A\rangle_2|A\rangle_3|A\rangle_4 \, . \tag{5.15}$$

It is not very difficult to see that our only hope of getting (5) to work is if we choose

$$\langle V_4|_{1234} = \bar{e}\left(\frac{3}{2}\right)^4 \langle V_4^0|_{1234} b_+^1 \, . \tag{5.16}$$

Using the fact that $\{b, Q\} = L$ we find that

$$\begin{aligned}\langle V_4|_{1234}|\delta_0 A\rangle_1|A\rangle_2|A\rangle_3|A\rangle_4 = \bar{e}\left(\frac{3}{2}\right)^4 \langle V_4^0|_{1234}\, L_+^1|\Lambda\rangle_1|A\rangle_2|A\rangle_3|A\rangle_4 - \\ - \bar{e}\left(\frac{3}{2}\right)^4 \langle V_4^0|_{1234}\, Q_1 b_+^1|\Lambda\rangle_1|A\rangle_2|A\rangle_3|A\rangle_4\end{aligned} \, . \tag{5.17}$$

Therefore the V_3 and V_4 pieces of (5) almost cancel. What remains is simply

$$-\bar{e}\left(\frac{3}{2}\right)^4 \langle V_4^0|_{1234}\, Q_1 b_+^1|\Lambda\rangle_1|A\rangle_2|A\rangle_3|A\rangle_4 \, . \tag{5.18}$$

In order to proceed further let us say something about the BRST invariance of the "bare" quartic vertex V_4^0.

In the spirit of (4.6) let us define yet another quartic vertex

$$|\tilde{V}_4\rangle_{1234} = \langle V_2|_{56}|V_3\rangle_{512}|V_3\rangle_{634} \, . \tag{5.19}$$

Proceeding as before with the insertion and OPE's we see that

$$|\tilde{V}_4\rangle \propto L|V_4^0\rangle \, .$$

On the other hand, simply from the fact that $\tilde{V}_4$ is a contraction of BRST invariant vertices it follows that it also is BRST invariant. As L commutes with Q it

follows that the bare vertex is also BRST invariant, *i.e.*

$$Q|V_4^0\rangle = 0 \, . \tag{5.20}$$

Note that by some arguments $K_n\tilde{V}_4 = 0$, but since $[K, L] \neq 0$ the bare vertex is *not* K_n invariant. Since cancellation of K_n anomalies depends only on the insertions through their conformal dimension, we have that $K_n b V_4^0 = 0$. The full vertex is K_n invariant, as it must be in order for our action to make sense. Using the BRST invariance of V_4^0 we write (18) as

$$\bar{e}\left(\frac{3}{2}\right)^4 \langle V_4^0|_{1234}\,(Q_2 + Q_3 + Q_4) b_+^1|\Lambda\rangle_1|A\rangle_2|A\rangle_3|A\rangle_4 \, .$$

This is easily rearranged to give

$$\begin{aligned}&\langle V_4|_{1234}|\Lambda\rangle_1 Q_2|A\rangle_2|A\rangle_3|A\rangle_4 \\ &+ \langle V_4|_{1234}|\Lambda\rangle_1|A\rangle_2 Q_3|A\rangle_3|A\rangle_4 \\ &+ \langle V_4|_{1234}|\Lambda\rangle_1|A\rangle_2|A\rangle_3 Q_4|A\rangle_4\end{aligned} \, . \tag{5.21}$$

Re-labeling the first by $\binom{1234}{3412}$, and second by $\binom{1234}{2341}$ and using cyclicity of $\langle V_4|$ gives us

$$\langle \delta_2 A|_4 = -\langle V_4|_{1234}\Big(|A\rangle_1|A\rangle_2|\Lambda\rangle_3 + |A\rangle_1|\Lambda\rangle_2|A\rangle_3 + |\Lambda\rangle_1|A\rangle_2|A\rangle_3\Big) \, . \tag{5.22}$$

We have now determined both the action (1) and the corresponding gauge transformation (2). In order for the action to be invariant under this transformation we need to show that equations (6) and (7) hold. Formula (7) is in some sense more restrictive since it consists of a single term. It is met since we have

$$\langle V_4|\delta A\rangle|A\rangle|A\rangle|A\rangle \;\propto\; \langle V_4|V_4\rangle = 0 \, . \tag{5.23}$$

The last step follows since $b^2 = 0$.

Lastly, we turn to (6). We may rewrite the quartic piece as

$$\langle V_4|_{1234}|\delta_1 A\rangle_1|A\rangle_2|A\rangle_3|A\rangle_4 = \\ \langle V_4|_{1234}\Big({}_5\langle A|_6\langle A| - {}_5\langle A|_6\langle A|\Big)|V_3\rangle_{156}|A\rangle_2|A\rangle_3|A\rangle_4$$

With a bit of rearrangement, taking care of the gradings of all of the objects, we find this to be equal to

$$\begin{aligned} &-\langle V_2|_{5x}\langle V_2|_{6y}\langle V_4|_{1234}|V_3\rangle_{156}|A\rangle_2|A\rangle_3|A\rangle_4|A\rangle_x|A\rangle_y \\ &-\langle V_2|_{5y}\langle V_2|_{6x}\langle V_4|_{1234}|V_3\rangle_{156}|A\rangle_2|A\rangle_3|A\rangle_4|A\rangle_x|A\rangle_y . \end{aligned} \tag{5.24}$$

The cubic piece, on the other hand, becomes

$$\begin{aligned} {}_6\langle A|_5\langle A|_4\langle \delta_2 A||V_3\rangle_{456} = {}_6\langle A|_5\langle A|\ \langle V_4|_{1234}|V_3\rangle_{456}\Big(&|A\rangle_1|A\rangle_2|A\rangle_3 + \\ &|A\rangle_1|A\rangle_2|A\rangle_3 + \\ &|A\rangle_1|A\rangle_2|A\rangle_3\Big) . \end{aligned} \tag{5.25}$$

We re-label $1 \leftrightarrow 4$ and then use cyclicity of $\langle V_4|$. This enables us to write the cubic piece of (6) as

$$\begin{aligned} \langle V_2|_{6x}\langle V_2|_{5y}\ \langle V_4|_{1234}|V_3\rangle_{156}\Big(&|A\rangle_2|A\rangle_3|A\rangle_4 \\ &+|A\rangle_2|A\rangle_3|A\rangle_4 \\ &+|A\rangle_2|A\rangle_3|A\rangle_4\Big)|A\rangle_x|A\rangle_y . \end{aligned} \tag{5.26}$$

As we see both (24) and (26) contain the contraction $\langle V_4|V_3\rangle$.

$$\langle V_4|_{1234}|V_3\rangle_{156} \propto \langle V_4^0|_{1234}\, b_+^1 G_-^1|V_3^0\rangle_{156} \propto \langle V_4^0|_{1234}\beta_-^1|V_3^0\rangle_{156} , \tag{5.27}$$

where in the last step we used the corresponding OPE. As with the calculation of associativity, the key in evaluating (6) is in carefully keeping track of phases that come about from moving β to the external legs. For the cubic piece we need to move the insertion onto legs 2,3,4 *i.e.* along V_4^0. For the quartic piece we move to 5,6 *i.e.* along V_3^0.

Let us recall that by equations (2.18) and (2.19) we have on the cubic vertex

$$\beta^a \sim \left(\frac{\partial \omega}{\partial \rho}\right)^{3/2} \sim \left(\sqrt{z_a}\right)^{-3} , \tag{5.28}$$

where we are just keeping the dependence on the index a. We found that the choice of roots $\sqrt{z_a} = g_a = (1, e, \bar{e})\ e^{i\frac{\pi}{12}}$ led to the Neumann functions with correct overlap equations. Using the fact $e^3 = \bar{e}^3 = 1$ gives us

$$\beta^1 = \beta^2 = \beta^3 , \tag{5.29}$$

on V_3. Similarly on V_4 we find

$$\beta^a \sim \left(\frac{\partial z}{\partial \rho}\right)^{3/2} \sim (\sqrt{z_a})^{-3} , \tag{5.30}$$

where the four string z's (fig.4) are

$$z_a = (1, -i, -1, i)\ e^{i\frac{\pi}{4}} . \tag{5.31}$$

The choice of the roots $\sqrt{z_a}$ must be consistent with the overlap equations. As we shall see at the end of this section the overlaps for the quartic vertex are *not cyclic*. The choice of roots consistent with the overlaps of $|V_4\rangle_{1234}$ (see appendix) is

$$\sqrt{z_a} = (1, e^{-i\frac{\pi}{4}}, e^{i\frac{\pi}{2}}, e^{-i\frac{3\pi}{4}})\ e^{i\frac{\pi}{8}} . \tag{5.32}$$

This gives us

$$\beta^1 = e^{i\frac{3\pi}{4}}\beta^2 = e^{i\frac{\pi}{2}}\beta^3 = e^{i\frac{\pi}{4}}\beta^4 , \tag{5.33}$$

acting on $|V_4\rangle_{1234}$. To get results for $|V_4\rangle_{2341}$ we just apply $\binom{1234}{2341}$. It follows that on $|V_4\rangle_{2341}$ we have

$$\beta^1 = e^{-i\frac{\pi}{4}}\beta^2 = e^{-i\frac{\pi}{2}}\beta^3 = e^{-i\frac{3\pi}{4}}\beta^4 . \tag{5.34}$$

If we write $\beta^1 = a\beta^2 = b\beta^3 = c\beta^4$ on V_4 then the cubic piece of equation (6) is proportional to $(a+b+c)\langle V_3^0|V_4^0\rangle$. Thus (33) gives $a+b+c = 1+\sqrt{2}i$, while (34) gives $-1-\sqrt{2}i$. These two results are consistent since $|V_4\rangle_{1234} = -|V_4\rangle_{2341}$ on GSO allowed states. One might think that this holds for all cyclic permutation of the legs of V_4. This is not the case precisely because of the non-cyclicity of the $\sqrt{z}$'s. On $|V_4\rangle_{3412}$ we have

$$\beta^1 = e^{-i\frac{\pi}{4}}\beta^2 = e^{-i\frac{\pi}{2}}\beta^3 = e^{i\frac{\pi}{4}}\beta^4 \; . \tag{5.35}$$

Here $a+b+c = -i+\sqrt{2}$, while $|V_4\rangle_{3412} = |V_4\rangle_{1234}$ on GSO allowed states. We seemingly get two contradictory answers for the cubic piece of (6). There in fact is no contaradiction, it simply implies that $\langle V_3^0|V_4^0\rangle$ gives zero when applied on GSO allowed states.

This contraction is nothing but $|V_5^0\rangle$. We have just shown that the quintic vertex *projects out* GSO allowed states. Note that the quartic piece of (6) is also proportional to $|V_5^0\rangle$, so that it also vanishes on the GSO allowed states. Equation (6) is therefore satisfied on such states.

The main results of this section should be reviewed. We have determined the complete action for the Witten superstring in the Neveu-Schwarz sector. The action contains a cubic and quartic interaction. Higher interactions are not present as a consequence of the gauge invariance of the action. Just as the action gets an extra g^2 piece, so does the gauge transformation law.

Although equations (16) and (4.6) completely determine the quartic vertex we really can't calculate the contraction in (4.6) explicitly. What can be done is the following. We may determine the overlaps satisfied by $|V_4^0\rangle$ (thus also by $|V_4\rangle$) from the known V_2 and V_3 overlaps. We can then construct the quartic vertex directly by Neumann function techniques just as was done for the cubic vertex.

The ghost number of $|V_4\rangle$ is 2, so it is easier to construct than $|V_4^0\rangle$, since it can be given over the $|+++ +\rangle$ vacuum. In fact over this vacuum the full vertex V_4 is a pure exponential, *i.e.* no insertions are needed, as can be shown by

evaluating the corresponding K_n anomalies. In the appendix we have calculated the diagonal coefficients of V_4 in this way. Only the diagonal ones contribute to the K_n anomalies.

As we have seen, the vertex V_4^0 was defined through the contraction of two cubic vertices without insertions

$$|V_4^0\rangle_{1234} = \langle V_2|_{56}|V_3^0\rangle_{126}|V_3^0\rangle_{345} \; .$$

The overlaps of the fields on the quartic vertex may be found from the known two and three string overlaps. The only non-trivial quartic overlaps will turn out to be the ones for the ψ field. The ψ overlaps on V_2 and V_3^0 are given by

$$\begin{aligned} \langle V_2|_{65}\big(\psi^6(\sigma) + i\psi^5(\pi-\sigma)\big) &= 0 \\ \langle V_2|_{65}\big(\psi^5(\sigma) - i\psi^6(\pi-\sigma)\big) &= 0 \; , \end{aligned} \tag{5.36}$$

as well as

$$\begin{aligned} \big(\psi^1(\sigma) - i\psi^3(\pi-\sigma)\big)|V_3^0\rangle_{123} &= 0 \\ \big(\psi^2(\sigma) - i\psi^1(\pi-\sigma)\big)|V_3^0\rangle_{123} &= 0 \\ \big(\psi^3(\sigma) - i\psi^2(\pi-\sigma)\big)|V_3^0\rangle_{123} &= 0 \; . \end{aligned} \tag{5.37}$$

All of these are valid for $\sigma \in (0, \frac{\pi}{2})$. It follows that for V_4^0 we have

$$\begin{aligned} \big(\psi^1(\sigma) + i\psi^4(\pi-\sigma)\big)|V_4^0\rangle_{1234} &= 0 \\ \big(\psi^2(\sigma) - i\psi^1(\pi-\sigma)\big)|V_4^0\rangle_{1234} &= 0 \\ \big(\psi^3(\sigma) - i\psi^2(\pi-\sigma)\big)|V_4^0\rangle_{1234} &= 0 \\ \big(\psi^4(\sigma) - i\psi^3(\pi-\sigma)\big)|V_4^0\rangle_{1234} &= 0 \; , \end{aligned} \tag{5.38}$$

for $\sigma \in (0, \frac{\pi}{2})$. V_4^0 obviously is not cyclic as we have already indicated. This is in fact not a problem. One should bear in mind that the V_2 overlap for $\psi(\sigma)$ is itself not cyclic. Consistency of the theory is saved however since V_2 is cyclic when acting on GSO-allowed states. As a consequence, the same can be said for V_4^0 when acting on GSO-allowed states.

Having derived the overlaps the vertex can be constructed in the usual way (appendix). We should also mention that this construction determines V_4 up to an overall normalization constant. Therefore we have

$$|V_4\rangle_{1234} = C\frac{1}{\epsilon}\exp\Big(\frac{1}{2}\alpha^a_{-n}\mathcal{N}^{ab}_{nm}\alpha^b_{-m}+\frac{1}{2}\psi^a_{-r}K^{ab}_{rs}\psi^b_{-s}+ \\ +b^a_{-n}\tilde{\mathcal{N}}^{ab}_{nm}mc^b_{-m}+\beta^a_{-r}\tilde{K}^{ab}_{rs}\gamma^b_{-s}\Big)|+\rangle_{1234}\,. \tag{5.39}$$

C is the finite normalization constant, and the $\frac{1}{\epsilon}$ dependence comes from (16) and equation (2.36) which states that (since b has conformal dimension 2) we have

$$b|V_4\rangle \;\propto\; \frac{1}{\epsilon}\bar{b}|V_4\rangle\,. \tag{5.40}$$

We see that as in Wendt's paper the extra four point scattering term diverges as $\frac{1}{\epsilon}$. Let us finally note that the constant C can be determined by looking (*a lá* Wendt) at the scattering of four vectors and choosing C so that the V_4 piece cancels the extra term in the V_3V_3 contraction making the total amplitude equal to the result of the dual model. The quartic piece gives rise to

$${}_1\langle\varsigma_1|_2\langle\varsigma_2|_3\langle\varsigma_3|_4\langle\varsigma_4|V_4\rangle_{1234}\,.$$

It is rather easy to show that by using the properties of the Neumann coefficients of V_4 given in the appendix this is equal to

$$C\frac{1}{\epsilon}\Big(\frac{1}{2}\Big)^{-s/2}\Big(\frac{1}{2}\Big)^{-t/2}(\varsigma_1\cdot\varsigma_2\;\varsigma_3\cdot\varsigma_4+\varsigma_2\cdot\varsigma_3\;\varsigma_4\cdot\varsigma_1-\varsigma_1\cdot\varsigma_3\;\varsigma_2\cdot\varsigma_4)= \\ =C\frac{1}{\epsilon}\Big(\frac{1}{2}\Big)^{-s/2}\Big(\frac{1}{2}\Big)^{-t/2}\Big(S+T-U\Big)\,. \tag{5.41}$$

On the other hand, our result $A^{extra}_s + A^{extra}_t$ may be written as

$$\Big(\frac{1}{2}\Big)^{-s/2}\Big(\frac{1}{2}\Big)^{-t/2}\Big((1+1+\ldots)(S+T)-(1+\ldots)U\Big)\,.$$

The two are obviously of the same form. The divergence $\frac{1}{\epsilon}$ in the perturbative calculation comes through a divergent sum $1+1+\ldots$. If our calculation had been exact then it would be possible to read off the value of C.

6. Conclusion

Let us recapitulate the results of this work. In the first part of the paper we have presented an insertion for the cubic interaction of Witten's superstring that is equivalent to the picture changing operator but is given in terms of ghosts in their natural (not bosonized) form. This simplifies explicit calculations and also makes the connection to dual models more manifest. We then gave a straightforward calculation of the tree scattering of four vectors in the superstring. The existence of a (divergent) contact term is confirmed. The derivation is free of non trivial limiting procedures. Though tedious, this calculation shows that in Witten's theory the expansion in $x = e^{-\tau}$ converges rapidly. This may prove useful in calculating other diagrams (tadpole perhaps).

In the second part of the paper we showed that in the superstring the Witten string product does not obey associativity as one would naively assume. This result comes about through analyzing operator product expansions of mid-point insertions. Failure of associativity necessitates the modification of the superstring action, as well as of the gauge transformation. By imposing gauge invariance we show that quintic and higher interactions are not present in the theory. The quartic vertex is determined, and it is shown that this precisely gets rid of the extra term in our four point scattering calculation. Thus by imposing gauge invariance we recover correct dual model amplitudes.

Even though the string product fails to associate, it does this in a particularly simple way by the appearance of a phase. The mathematical structure that emerges seems interesting in its own right. Another indication of the existence of an underlying structure not yet understood is the connection between the super Virasoro algebra and the severe constraints on insertions of (non primary) operators at vertex mid-points.

Due to the fact that both the action and gauge transformation are modified by extra terms the question of gauge fixing should again be addressed. As we have seen, it is strongly indicated that vertices higher than quartic can't be made to satisfy all the properties required of them if they are to be a part of the action. This would seem to imply that the usual trick of simply substituting the full fields $|\Phi\rangle$ for the physical fields $|A\rangle$ in the action works here as well.

One of us (A. R. B.) would like to thank Sanjay Jain and Richard Woodard for many useful discussions.

APPENDIX

A.1. Neumann Coefficients of the Cubic Vertex V_3

N^{11}_{nm}	$m=1$	$m=2$	$m=3$
$n=0$	0	$-\frac{2}{3^3}$	0
$n=1$	$-\frac{5}{3^3}$	0	$\frac{2^5}{3^4}$
$n=2$	0	$\frac{13}{2\cdot 3^5}$	0

N^{12}_{nm}	$m=1$	$m=2$	$m=3$
$n=0$	$-\frac{2}{3\sqrt{3}}$	$\frac{1}{3^3}$	$\frac{2\cdot 11}{3^5\sqrt{3}}$
$n=1$	$\frac{2^4}{3^3}$	$\frac{2^5}{3^5}$	$\frac{2^4}{3^6}$
$n=2$	$-\frac{2^3}{3^3\sqrt{3}}$	$-\frac{2^6}{3^5}$	$-\frac{2^5\cdot 5}{3^6\sqrt{3}}$

$\tilde{N}^{11}_{nm}$	$m=1$	$m=2$	$m=3$
$n=0$	0	$\frac{2^3}{3^3}$	0
$n=1$	$\frac{11}{3^3}$	0	$-\frac{2^4\cdot 5}{3^6}$
$n=2$	0	$-\frac{19}{2\cdot 3^5}$	0

$\tilde{N}^{12}_{nm}$	$m=1$	$m=2$	$m=3$
$n=0$	$-\frac{2^2}{3\sqrt{3}}$	$-\frac{2^2}{3^3}$	$\frac{2^2\cdot 17}{3^5\sqrt{3}}$
$n=1$	$\frac{2^3}{3^3}$	$\frac{2^3\cdot 5}{3^4\sqrt{3}}$	$\frac{2^3\cdot 5}{3^6}$
$n=2$	$-\frac{2^3\cdot 5}{3^4\sqrt{3}}$	$-\frac{2^3\cdot 7}{3^5}$	$\frac{2^3\cdot 19}{3^6\sqrt{3}}$

K^{11}_{rs}	$s=\frac{1}{2}$	$s=\frac{3}{2}$	$s=\frac{5}{2}$
$r=\frac{1}{2}$	0	$\frac{6}{2\cdot 3^3}$	0
$r=\frac{3}{2}$	$-\frac{5}{2\cdot 3^3}$	0	
$r=\frac{5}{2}$	0		0

K^{12}_{rs}	$s=\frac{1}{2}$	$s=\frac{3}{2}$	$s=\frac{5}{2}$
$r=\frac{1}{2}$	$-\frac{2^2}{3\sqrt{3}}$	$-\frac{2^4}{3^3}$	$\frac{2\cdot 7}{3^4\sqrt{3}}$
$r=\frac{3}{2}$	$\frac{2^4}{3^3}$		
$r=\frac{5}{2}$	$-\frac{2\cdot 7}{3^4\sqrt{3}}$		

$\tilde{K}^{11}_{rs}$	$s=\frac{1}{2}$	$s=\frac{3}{2}$	$s=\frac{5}{2}$
$r=\frac{1}{2}$	$-\frac{2}{3}i$	$\frac{17}{2\cdot 3^3}$	$\frac{1}{3}i$
$r=\frac{3}{2}$	$-\frac{17}{2\cdot 3^3}$	0	$\frac{167}{2^3\cdot 3^5}$
$r=\frac{5}{2}$	$\frac{1}{3}i$		

$\tilde{K}^{12}_{rs}$	$s=\frac{1}{2}$	$s=\frac{3}{2}$	$s=\frac{5}{2}$
$r=\frac{1}{2}$	$\frac{2}{3\sqrt{3}}-\frac{2}{3}i$		
$r=\frac{3}{2}$			
$r=\frac{5}{2}$			

We list here some important symmetries of the Neumann coefficients

$$N^{ab}_{nm} = N^{a+1\ b+1}_{nm}$$

Same cyclicity property holds for $\tilde{N}$'s, K's and $\tilde{K}$'s. The (13) coefficients follow from the (12)'s *via*

$$\begin{aligned}
N^{12}_{nm} &= (-)^{n+m} N^{13}_{nm} \\
\tilde{N}^{12}_{nm} &= (-)^{n+m} \tilde{N}^{13}_{nm} \\
K^{12}_{rs} &= (-)^{r+s} K^{13}_{rs} \\
\tilde{K}^{12}_{rs} &= (-)^{r+s} \tilde{K}^{13}_{rs} \, .
\end{aligned}$$

Finally, the coefficients are (anti) symmetric under interchange of both upper and lower indices

$$\begin{aligned}
N^{ab}_{nm} &= N^{ba}_{mn} \\
\tilde{N}^{ab}_{nm} &= \tilde{N}^{ba}_{mn} \\
K^{ab}_{rs} &= -K^{ba}_{sr} \\
\tilde{K}^{ab}_{rs} &= -\tilde{K}^{ba}_{sr} \, .
\end{aligned}$$

A.2. Coefficients of the $\overline{G}$ insertion

The insertion has been written as

$$\overline{G} = \alpha^a_{-n}\psi^b_{-r}F^{ab}_{nr} + p^a\psi^b_{-r}F^{ab}_{0r} + c^a_{-n}\beta^b_{-r}G^{ab}_{nr} + b^a_{-n}\gamma^b_{-r}H^{ab}_{nr} \, .$$

For the coefficients of the $P(\frac{\pi}{2})\psi(\frac{\pi}{2})$ piece we have the formulas

$$\begin{aligned}
F^{ab}_{0r} &= f(r)e^{-i\frac{\pi}{4}}f^{ab} \\
F^{ab}_{1r} &= -i\frac{2^2}{3}f(r)e^{-i\frac{\pi}{4}}f^{ab} \\
F^{ab}_{2r} &= -\frac{2^3}{3^2}f(r)e^{-i\frac{\pi}{4}}f^{ab} \\
&\ldots
\end{aligned}$$

where

$$f(\tfrac{1}{2}) = \frac{1}{3}$$
$$f(\tfrac{3}{2}) = i\frac{2}{3^2}$$
$$f(\tfrac{5}{2}) = -\frac{13}{2\cdot 3^2}$$
$$\dots$$

as well as

$$f^{11} = 1$$
$$f^{12} = e^{-\frac{2\pi}{3}i} \equiv \bar{e}$$
$$f^{13} = e^{\frac{2\pi}{3}i} \equiv e\,.$$

As we have seen the ghost coefficients are harder to calculate since they come both from leading and subleading terms. The only ones that were used in the four point scattering were

$$H^{11}_{1\frac{1}{2}} = \frac{2}{3}$$
$$G^{11}_{1\frac{1}{2}} = \frac{20}{3}i\,.$$

A.3. Neumann Coefficients of the Quartic Vertex V_4

We proceed here to give an outline of the construction of the quartic vertex *via* Neumann function techniques. For a detailed introduction as well as explanation of notation see [2,3]. The conformal transformation that takes the four string overlap into the unit circle[2] is

$$\rho = \ln\frac{z^2 - i}{z^2 + i} - i\frac{\pi}{2}\,. \tag{A.1}$$

It follows that $\frac{\partial z}{\partial \rho} = \frac{z^4+1}{4iz}$. Inverting (1) we find

$$z^a(\sigma) = z_a\left(\frac{1+ie^{i\sigma}}{1-ie^{i\sigma}}\right)^{1/2},$$

where $z_1^2 = z_3^2 = -z_2^2 = -z_4^2 = i$. In fact the roots must be chosen so that

$z_a = (1, -i, -1, i)\ e^{i\frac{\pi}{4}}$ as in figure 4, since this gives

$$z^a(\sigma) = z^{a-1}(\pi - \sigma)\,, \tag{A.2}$$

for $\sigma \in (0, \frac{\pi}{2})$. This is a necessary condition for the Neumann functions to satisfy appropriate overlap equations. Let us focus on the ψ correlator. It is simply

$$K(\sigma,\sigma') = \left(\alpha\frac{\partial z}{\partial \rho}\right)^{1/2}\frac{1}{z - z'}\left(\alpha'\frac{\partial z'}{\partial \rho'}\right)^{1/2}, \tag{A.3}$$

where $\alpha_1 = \alpha_3 = -\alpha_2 = -\alpha_4 = 1$ are the "charges" corresponding to the z_a's. Keeping only the dependence on the a index we see that

$$K^a(\sigma) \sim \left(\alpha_a\frac{\partial z^a}{\partial \rho}\right)^{1/2} \sim \left(\frac{\alpha_a}{z^a}\right)^{1/2}.$$

Since $z_2 = -z_4$ we have

$$\begin{aligned} K^1(\sigma) &\sim \frac{1}{\sqrt{z^1(\sigma)}}\\ K^2(\sigma) &\sim \frac{1}{\sqrt{z^4(\sigma)}}\\ K^3(\sigma) &\sim \frac{1}{\sqrt{z^3(\sigma)}}\\ K^4(\sigma) &\sim \frac{1}{\sqrt{z^2(\sigma)}}\,. \end{aligned} \tag{A.4}$$

On the other hand the ψ correlator needs to have the same overlaps as ψ. Using the overlaps derived in section 5 and equations (4) we see that

$$\begin{aligned} \sqrt{z^1(\sigma)} &= i\sqrt{z^2(\pi-\sigma)}\\ \sqrt{z^2(\sigma)} &= -i\sqrt{z^3(\pi-\sigma)}\\ \sqrt{z^3(\sigma)} &= -i\sqrt{z^4(\pi-\sigma)}\\ \sqrt{z^4(\sigma)} &= -i\sqrt{z^1(\pi-\sigma)}\,. \end{aligned} \tag{A.5}$$

As promised the overlaps of $\sqrt{z}$ are not cyclic. From the overlaps it follows that we must choose the roots $\sqrt{z_a}$ to be $\sqrt{z_a} = (1, e^{-i\frac{\pi}{4}}, e^{i\frac{\pi}{2}}, e^{-i\frac{3\pi}{4}})\ e^{i\frac{\pi}{8}}$. Figure 5 depicts these roots.

We now turn to evaluating the diagonal coefficients of the quartic vertex. Using equation (3) it is not very difficult to put K into the form

$$K^{11}(\sigma,\sigma') = \frac{1}{2}\frac{x^{1/2}x'^{1/2}}{x-x'}\Big(g(x)g(-x') + g(x')g(-x)\Big)\,, \qquad \text{(A.6)}$$

where we have introduced the notation $x = ie^{i\sigma}$. The four string g is simply $g(x) = \left(\frac{1+x}{1-x}\right)^{1/4}$.

Similarly, for the $\beta\gamma$ correlator we have, in complete analogy with the three string case

$$\tilde{K}(\sigma,\sigma') = \left(\alpha\frac{\partial z}{\partial \rho}\right)^{-1/2}\frac{1}{z-z'}\left(\alpha'\frac{\partial z'}{\partial \rho'}\right)^{3/2}\frac{z^4+1}{z'^4+1}\,.$$

Again only the diagonal part of this interests us here, and we can write it as

$$\tilde{K}^{11}(\sigma,\sigma') = \frac{1}{2}\frac{x^{1/2}x'^{1/2}}{x-x'}\Big(g(x)g(-x') + \frac{1+x}{1-x}\,\frac{1-x'}{1+x'}\,g(x')g(-x)\Big)\,. \qquad \text{(A.7)}$$

The Fourier coefficients of the scattering part of K^{11} and $\tilde{K}^{11}$ are the vertex coefficients. We thus have

$$K^{11}(\sigma,\sigma') = \sum_{r,s>0} e^{ir\sigma}e^{is\sigma'}K^{11}_{rs} + \sum_{r>0} e^{-ir(\sigma-\sigma')}\,, \qquad \text{(A.8)}$$

and similarly for $\tilde{K}$. The easiest way to read off the K^{11}_{rs}'s is to apply $(\partial_\rho + \partial_{\rho'})$ on (6) and (8). Equating these two gives us

$$\sum_{n,m=0}^{\infty}\frac{n+m+1}{i^{n+m+1}}K^{11}_{n+\frac{1}{2},m+\frac{1}{2}}x^n x'^m = \frac{1}{4}\frac{1+xx'}{(1-x^2)(1-x'^2)}\Big(g(x)g(-x') - g(x')g(-x)\Big)\,. \qquad \text{(A.9)}$$

Similarly for $\tilde{K}$ we find

$$\sum_{n,m=0}^{\infty}\frac{n+m+1}{i^{n+m+1}}\tilde{K}^{11}_{n+\frac{1}{2},m+\frac{1}{2}}x^n x'^m = \frac{1}{4}\frac{1+xx'}{(1-x^2)(1-x'^2)}g(x)g(-x') - \frac{3}{4}\frac{1+xx'}{(1-x)^2(1+x')^2}g(x')g(-x)\Big)\,. \qquad \text{(A.10)}$$

Expanding in x and x' we easily get the needed vertex coefficients. Let us just

note that calculating the non-diagonal coefficients (just as in the cubic vertex) is even simpler.

We finish by giving a brief list of the first few (diagonal) Neumann coefficients of the full quartic vertex. The coefficients of the X and bc sectors are not listed and can be found in [2].

K^{11}_{rs}	$s=\frac{1}{2}$	$s=\frac{3}{2}$	$s=\frac{5}{2}$	$s=\frac{7}{2}$
$r=\frac{1}{2}$	0	$\frac{1}{2^3}$	0	$-\frac{11}{2^7}$
$r=\frac{3}{2}$		0	$\frac{1}{2^7}$	

$\tilde{K}^{11}_{rs}$	$s=\frac{1}{2}$	$s=\frac{3}{2}$	$s=\frac{5}{2}$	$s=\frac{7}{2}$
$r=\frac{1}{2}$	i	$\frac{5}{2^3}$		$-\frac{89}{2^6}$
$r=\frac{3}{2}$	$-\frac{5}{2^3}$	$\frac{i}{2^3}$	$\frac{41}{2^7}$	
$r=\frac{5}{2}$		$-\frac{41}{2^7}$		
$r=\frac{7}{2}$	$\frac{89}{2^6}$			

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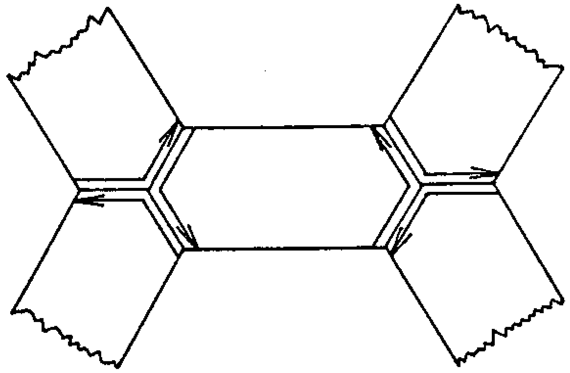

Figure 1.

s-channel amplitude

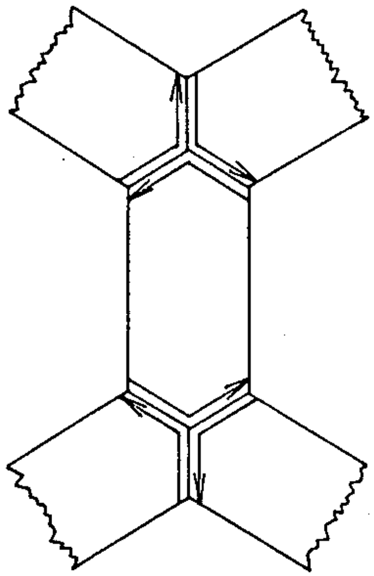

Figure 2.

t-channel amplitude

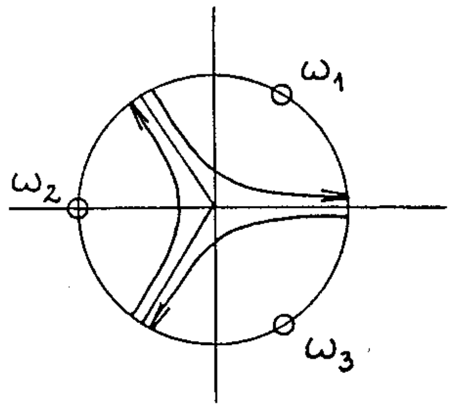

Figure 3.

three string ω plane

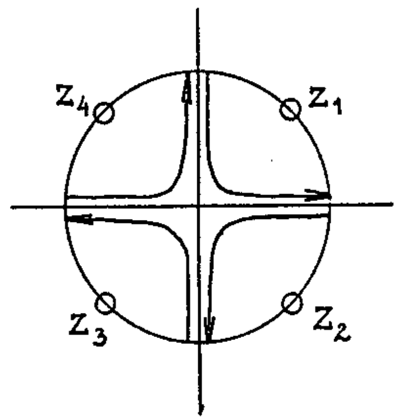

Figure 4.

four string z plane

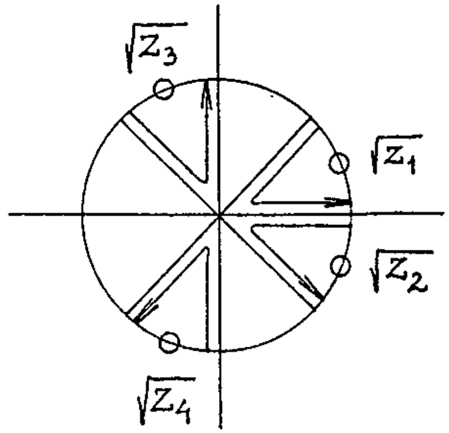

Figure 5.

four string $\sqrt{z}$ plane