QUARTIC INTERACTION IN SUPERSTRING FIELD THEORY

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ABSTRACT

We investigate the presence of contact interactions in Witten's superstring theory. A calculation of the four point amplitude is performed. Associativity of the string product is shown to fail due to short distance effects of the mid-point operator insertions. An explicit quartic interaction is induced in the superstring action. At the same time the gauge transformation is modified by an order $g^2$ term. The quartic term completes the theory; terms higher than quartic are not present.

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1. Introduction

The motivation for the construction of covariant, gauge invariant superstring field theory has been to establish a formalism in which questions about non-perturbative phenomena in strings could be addressed. The hope is that a field theory formulation can shed some light on the basic structure of string theory. This construction has proved to be a challenging task, and indeed in the case of closed strings it is still outstanding. Interacting theory of open bosonic strings, as well as superstrings, has been introduced by Witten. These theories are characterized by purely cubic interactions.

Operator formulation of the theory has been established in a series of papers. Here the interaction of strings follows from a 3-point vertex. Gauge invariance, associativity of the string product, and the purely cubic nature of the interactions are closely linked. In the bosonic string case this relation has been explicitly shown. The superstring case parallels the bosonic, the new non-trivial ingredient being the insertion at the string mid-point of the picture changing operator of Friedan, Martinec and Shenker.

Recently a connection of the interaction vertex and the first quantized formalism has been given. Using this, Wendt calculated the scattering of four vectors in the Witten superstring. Surprisingly he found that the result differed from the dual model by an (infinite) contact term. Other authors argued against such terms in Witten's theory. Since the original argument is based on a limiting procedure, questions of prescription dependence could be raised. Contact terms, however, were also known to appear in the light cone theory and it is possible that they remain even in the covariant approach.

This issue is sufficiently relevant that a more concrete investigation is needed if a complete superstring field theory is to be specified.

In this paper we will investigate these issues directly in operator field theory. This calculation is clear cut and does not involve any limiting procedures or
The objective of the polynomial $X(\{\mathbf{f}\})$ is to minimize the expression $\sum_{i=1}^{N} \left( \langle \psi_i | \mathbf{f} | \psi_i \rangle \right)^2 = \frac{1}{2}$, where the weight of each quadratic form is $\frac{1}{2}$.

\[ \langle \psi_1 | \mathbf{f} | \psi_1 \rangle \leq \langle \psi_2 | \mathbf{f} | \psi_2 \rangle \leq \langle \psi_3 | \mathbf{f} | \psi_3 \rangle \]

The objective is to find a set of vectors $\{\psi_i\}$ such that the above condition holds. The action can be written as $\langle \psi | \mathbf{f} | \psi \rangle$. The case where the weight is $1$ is similar to the previous section. We will now consider the next section of the paper, dealing with the next section.
To be able to carry out explicit superstring calculations in the operator field theory we first introduce a mid-point insertion equivalent to the picture changing operator $X(\xi)$, but given without reference to bosonized ghosts. This is achieved by working with the original ghost variables $(\gamma, \beta)$ with the vacua being related by $e^{\phi}$. We recall that in the dual model insertions of $G_{\frac{1}{2}}$ and $G_{-\frac{1}{2}}$ were needed. The Witten vertex is on-shell equivalent to the dual model vertex, therefore the operator connecting these two is of the form $e^{\alpha L_{\alpha}}$. Having in mind the commutation relations $[G, L] = G$, we are led to the insertion $G(\xi)$ (the mid-point being the only place where one may put an operator in Witten's theory without spoiling the string overlaps).

The four point scattering calculation of the next section can be thought of as proof of the correctness of $G(\xi)$ as an insertion. Finally let us mention that $G(\xi)$ is related to the insertion recently constructed by Suehiro by imposing BRST invariance, so that it also in fact leads to a BRST invariant interaction. Taking into account the effect of different vacua (i.e. an $e^{-\phi}$ operation) our insertion $G(\xi)$ is indeed equivalent to $X(\xi)$ but easier for calculation.

The part of the vertex coming from the overlaps (i.e. without insertion) is

$$|V_0^0\rangle_{123} = \exp \left\{ \frac{1}{2} \hat{N}_{n}^{ab} N_{ab}^{\alpha \beta}, \frac{1}{2} \hat{M}_{n}^{ab} M_{ab}^{\alpha \beta} + \frac{1}{2} \hat{V}^{ab}, K^{ab}, V^{ab} + \frac{1}{2} \hat{V}^{ab}, K^{ab}, V^{ab} \right\}_{123} .$$

(2.4)

The Neumann coefficients $N$ and $\bar{N}$ are as in the bosonic case, and were constructed in ref. (2). $K$'s can be found in ref. (3). The $K$ coefficients of the super ghosts are given in the review (4) where they were denoted by $K'$ (the $\bar{K}$ coefficients of that reference correspond to the $\beta \gamma$-vertex constructed over a different vacuum state than the one used here).

The full cubic vertex is

$$|V_3\rangle_{123} = \bar{G} |V_0^0\rangle_{123} ,$$

(2.5)

where the operator $\bar{G}$ is proportional to $G(\xi)$, and finite when acting on $V_3^0$.

Before we determine $\bar{G}$ let us first establish some basic results and conventions that will be of use later in our calculations. The results of the next section will lead us to the conclusion that in order to get correct scattering amplitudes an explicit interaction of four strings needs to be incorporated in the action. Therefore the action will be

$$S_{\text{Witten}} = \frac{1}{2} \langle A | Q | A \rangle + \frac{1}{3} \langle V_3 | A | A \rangle + \frac{1}{4} \langle V_4 | A | A \rangle + \frac{1}{4} \langle V_4 | A | A \rangle .$$

(2.8)

The $V_4$ piece is proportional to $g^2$ since it needs to be the same order in $g$ as the terms coming from the contractions of two $V_3$'s. At this point one may wonder whether higher contact terms ($V_5, V_6, \ldots$) will be needed. In the following sections we shall in fact show that the above is indeed the full (gauge invariant) superstring action.

Physical states acquire $Z_2$ grading due to the anticommuting modes, as well as the gradings of the two vacuums of the zero mode $b, c$ sector. Following usual conventions we take $| + \rangle$ to be even and $| - \rangle$ to be odd. GSO projection ensures that all physical states have the same grading. Since the vector $\psi_{\frac{1}{2}} | - \rangle$ is in the spectrum we see that $| A \rangle$ is even. From the free part of the action we see that $\langle A \rangle$ has opposite grading to $| A \rangle$, i.e. the two point vertex $| V_3 \rangle$ which moves bra into kets is odd (the same being true for $| V_3 \rangle$).

$$\langle A |_1 = \langle V_3 |_{123} | A \rangle_2 .$$

(2.7)

From the non vanishing of the cubic interaction we see that for superstrings $| V_3 \rangle$ is odd ($\langle V_3 \rangle$ is even). Similarly, the quartic term establishes that $| V_4 \rangle$ is even ($\langle V_4 \rangle$ is also even).

The general $N$-point vertex can be written in terms of a pure overlap piece $| V_3^0 \rangle$ and an insertion $I_N$. The cubic vertex $| V_3^0 \rangle$ is constructed over the $| + + + \rangle$
Consider that the vertex be situated on an exponential of a quadratic form at the point.

The obvious difference with the result for $\phi = \phi'$. Now, the conditional density

\begin{equation}
\frac{1 + \frac{\mu^2}{\sigma^2}}{1 + \frac{\sigma^2}{\sigma^2}} \left( \frac{\rho^2}{\sigma^2} \right)^{1 - \frac{n}{\sigma^2}} \frac{d\rho}{\rho^2} = (\rho' \phi)' Y
\end{equation}

Stability for the $\ell_1$ part of the vertex can now use the argument function

\begin{equation}
(\rho)^{\text{normal}} \phi(\rho', \phi) \int_{\rho = \rho} \frac{d\rho}{\rho} = (\rho, \phi) Y
\end{equation}

Similarly, the part of the $\ell_1$ part of the vertex can now use the argument function

\begin{equation}
(\rho - \rho', \phi)^2 \int_{\rho = \rho} f + \int_{\rho' = \rho} f + \int_{\phi' = \phi} f = (\rho', \phi) Y
\end{equation}

The vertex coefficients are simply the outer coefficients of the argument

\begin{equation}
(\rho, \phi) Y Y = (\rho', \phi) Y
\end{equation}

we have

\begin{equation}
\int_{\phi = \phi} \frac{d\phi}{\phi} = (\rho', \phi) Y
\end{equation}

which taken the limit into the $\phi$-vertex covertex.

Since the conditional dimension is always under the conditional transverse,

\begin{equation}
(\rho, \phi) Y = (\rho', \phi) Y
\end{equation}

For the outer coefficients, the basic conditional function $\phi(\rho', \phi)$ on the unit circle.

In section $\ell_1$ the conditional $\phi(\rho', \phi)$ is obtained from the potential of the conditional function

\begin{equation}
(\rho, \phi) Y = (\rho', \phi) Y
\end{equation}

shown to be

\begin{equation}
(\rho, \phi) Y = (\rho', \phi) Y
\end{equation}

In section $\ell_1$ the conditional between the $\ell_1$-line and two-ellipse vertices will be

\begin{equation}
(\rho, \phi) Y = (\rho', \phi) Y
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We shall now recall how to extract the Euclidean invariant $\ell_1$ and show that this

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the simple vacuum defined by $\beta_r(0) = \gamma_r(0) = 0$ for all positive $r$. The analogues of (17) are now

$$\beta^t(\sigma) = \int \frac{d\sigma'}{2\pi} e^{i\sigma} \hat{K}^t(\sigma',\sigma)$$

$$\gamma^t(\sigma) = \int \frac{d\sigma'}{2\pi} \hat{K}^t(\sigma,\sigma') e^{i\sigma} \gamma_{t\text{tension}}(\sigma').$$

(2.19)

The conformal transformation $\rho = \ln \frac{\omega^1}{\omega^3}$, $i\omega^3$ gives us

$$\frac{\partial\omega}{\partial \rho} = \frac{i\omega^3 + 1}{\frac{3}{2}},$$

(2.20)

where

$$\omega^3(\sigma) = \omega_a \left( 1 + i e^{i\sigma} \right)^{1/2},$$

(2.21)

The three phases $\omega_a$ may be written as $\omega_a = (1, e^i, e^{-i}) e^{i\sigma/3}$, where we introduce the notation $e = e^{i\pi}$ and $e$ is its hermitean conjugate. We also need $z(\sigma) = \sqrt{\omega(\sigma)}$ as well as $g(\sigma) = \sqrt{e^{i\sigma}}$, with the roots chosen so that we have

$$z^t(\sigma) = z_a \left( 1 + i e^{i\sigma} \right)^{1/2},$$

$$g^t(\sigma) = g_a \left( 1 + i e^{i\sigma} \right)^{1/2},$$

(2.22)

and also $z_a = (1, e, e^{-1}) e^{i\sigma/3}$, as well as $g_a = (1, e, e^{-1}) e^{-i\sigma/3}$. With these choices of the roots the Neumann functions satisfy the correct overlap equations. For example for the $\psi$ correlator we have

$$\hat{K}^t(\sigma,\sigma') = i(-1)^{t+1} \hat{K}^{t-1}((\sigma - \sigma) + \sigma')$$

for $0 \leq \sigma \leq \frac{\pi}{2}$. Formulas (17), (19) as well as similar equations for $P$, $b$, $c$ may be used to find the action of these fields when applied to the string mid-point.

We see that for a field $A(\sigma)$ with conformal dimension $J$ to leading order in $\epsilon$ we have

$$A(\frac{\pi}{2} - \epsilon)|V_2\rangle = \epsilon^{1/4} A|V_2\rangle,$$

(2.23)

where $A$ is a non-singular operator whose exact form can be calculated using formulas like (17) and (19).

Our insertion for $V_2$ is the finite part of $G(\frac{\pi}{2} - \epsilon)$. The operator $G(\sigma)$ is given by

$$G(\sigma) = \frac{1}{2} \left( P(\sigma) \psi(\sigma) + b(\sigma) \gamma(\sigma) - 2c(\sigma) \partial_\sigma \beta(\sigma) - 3\partial_\sigma c(\sigma) \beta(\sigma) \right).$$

(2.24)

Using the identity $\partial_\sigma A(\frac{\pi}{2} - \epsilon) = -\partial_\sigma A(\frac{\pi}{2} - \epsilon)$ we find

$$P(\sigma) \psi(\sigma) + b(\sigma) \gamma(\sigma) - 2c(\sigma) \partial_\sigma \beta(\sigma) = \epsilon^{-1/4} \hat{P}|V_2\rangle + \ldots$$

$$b(\sigma) \gamma(\sigma) - 2c(\sigma) \partial_\sigma \beta(\sigma) = \epsilon^{-1/4} \hat{b}|V_2\rangle + \ldots$$

$$c(\sigma) \partial_\sigma \beta(\sigma) = \epsilon^{-1/4} \hat{c}|V_2\rangle + \ldots$$

(2.25)

where dots indicate subleading terms in $\epsilon$.

At first glance (25) seems to spell disaster for $G$ as an insertion, since the $cB$ pieces diverge faster than the rest of $G$. The culprit is easily seen to be the derivation $\frac{\partial}{\partial \sigma} \sigma$. Therefore, although for primary fields we have leading behaviour as in equation (23) this does not hold for non-primary fields. For this reason one should in general expect problems when using non-primary fields as insertions on any vertex.

In the case of $G$, if we look closer, we see that the leading terms in $cB\beta$ and $\partial c\beta$ precisely cancel. Further, by expanding $c$ and $\beta$ to subleading terms we find
\[
\sum_{\rho} e^\rho = \frac{1}{2} e^f + \frac{1}{2} e^{-f} = g_1 e^{\frac{f}{2}} + g_2 e^{-\frac{f}{2}}
\]

Let\( \Gamma = \Gamma_{\text{left}} \times \Gamma_{\text{right}} \). Then the coefficient of \( \rho \) can be determined by
\[
\sum_{\rho} e^\rho = \frac{1}{2} e^f + \frac{1}{2} e^{-f} = g_1 e^{\frac{f}{2}} + g_2 e^{-\frac{f}{2}}
\]

By expanding the \( (\rho + 1) \) and \( \rho \) terms, we obtain
\[
\sum_{\rho} e^\rho = \frac{1}{2} e^f + \frac{1}{2} e^{-f} = g_1 e^{\frac{f}{2}} + g_2 e^{-\frac{f}{2}}
\]

Using the formula (13) and (16), we have
\[
\frac{1}{\rho} - f(\frac{1}{\rho}) + \frac{f}{10} \int \frac{d\rho}{10} = \left( \frac{f}{10} \right)\rho
\]

The equations (24) and (25) imply that
\[
\sum_{\rho} e^\rho = (\rho + 1) e^{\frac{f}{2}} + (\rho - 1) e^{-\frac{f}{2}}
\]

All the other coefficients can be determined in the manner described in the text.
A simple way to see that $\mathcal{G}$ satisfies the criterion of equation (12) will be presented in section 4. A more precise way is to show it by using the values of the coefficients $F$, $G$, $H$ of $\mathcal{G}$. Since the $F\psi$ sector cannot mix with the ghost sector upon going from one string to another it is enough to look at the $F$ coefficients. Using these (see appendix) it becomes very easy to show that (12) does indeed hold.

Let us again note that $\mathcal{G}$ is a well defined and finite insertion proportional to $G$ at the mid-point. It is equivalent to inserting the picture changing operator at the mid-point, but does not make any reference to bosonized ghosts. We should stress that $G(\frac{\pi}{2} - \epsilon)$ is precisely tailored as an insertion for a cubic vertex. For any other vertex (23) does not hold, and so the leading singularity in $G$ does not cancel.

The super-Virasoro algebra has, along with $G(\sigma)$, also the generators $L(\sigma)$ in it. We may wonder if $L(\frac{\pi}{2} - \epsilon)$ represents a well defined insertion on some $V_k$. From the conformal transformation appropriate to the four string vertex\(^{19}\) it is easy to see that to leading order we have

$$A\left(\frac{\pi}{2} - \epsilon\right)|V_k\rangle = e^{-\frac{i}{2} Q_1}|V_k\rangle. \tag{2.36}$$

Since $L(\sigma)$ is given by the expression

$$L(\sigma) = \frac{1}{2} \left(P(\sigma) \cdot P(\sigma) + \partial_\sigma \psi(\sigma) \cdot \psi(\sigma) + 2 \partial_\sigma B(\sigma) + 4 \partial_\sigma \epsilon(\sigma) B(\sigma) - \gamma(\sigma) \partial_\sigma B(\sigma) - 3 \partial_\sigma \gamma(\sigma) B(\sigma)\right), \tag{2.37}$$

it follows that the leading singularity mismatch for $L$ cancels only when applied to a four string vertex.

As we can see, there exists a very important connection between the super-Virasoro algebra and consistency conditions on insertions $I_N$. From the fact that there is no counterpart to $L$ and $G$ that works on the five point or higher vertices one is tempted to assume that these interactions are absent from the theory. In the last section of this paper we shall show that by imposing gauge invariance one is led to a theory with only cubic and quartic vertices.

3. Four Point Scattering

The simplest four point scattering amplitude that one can evaluate in the superstring is that of four vectors. Calculating even this in the operator theory is quite a formidable task. The task in front of us will be made manageable by introducing an approximation. We shall calculate the scattering amplitude in powers of $z = \exp(-r)$, $r$ being the time of propagation of the internal particle. It will be seen that this is an excellent approximation to use in evaluating Feynman diagrams in Witten's string theory.

First, however, we put aside the complications of the superstring and proceed with a much simpler calculation of scattering in the bosonic string. Here the lowest state in the spectrum is the tachyon, so we choose tachyons as our external states.

By expanding $|\Phi\rangle$ and $Q$ in zero modes, it is easy to see that the bosonic string propagator is simply

$$b_0 \int_0^{1/2} dz \ z^3 z^{-1} z^R, \tag{3.1}$$

where $R$ is the mode piece of $L_0$. The three point vertex is

$$|V_k\rangle_{123} = \exp\left(\frac{1}{2} \sum_{n,m=0}^{\infty} \sum_{a,b=1}^{3} a^*_{a,n} N_{ab,m} a^b_{m} + \sum_{n,m=1}^{\infty} \sum_{a=1}^{3} b^a_{n} \tilde{N}_{ab,m} b^a_{m} + \right)|123\rangle_{123}. \tag{3.2}$$

Coming now to the scattering of four tachyons we see that the s-channel ampi-
(2.3)
\[ n \rightarrow n + 1 + \frac{s}{2} \]
\[ \frac{s}{2} \rightarrow n + 1 + \frac{s}{2} \]
\[ \frac{s}{2} \rightarrow n + 1 + \frac{s}{2} \]

channel. The Ising model is a limit of the classical dynamical system. Under the redefinition (2.1), the limit of the partition function is given by:

(2.4)
\[ \xi \rightarrow n + 1 + \frac{s}{2} \]

From an unimodular condition and the limit theorem, it follows:

(2.5)
\[ G(n + 1 + \frac{s}{2}) \equiv n \]
\[ G(n + 1 + \frac{s}{2}) \equiv 1 \]
\[ G(n + 1 + \frac{s}{2}) \equiv s \]

Next, let us consider the infinite-dimensional case. General k-point scattering is considered. We then turn to the scattering of vectors in the infinite-dimensional case, which is

\[ n \rightarrow n + 1 + \frac{s}{2} \]

We then introduce a more general case for the scattering of vectors in the infinite-dimensional case, which is

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Using the polarization vectors one can construct six further invariants. We define
\[
S \equiv (\gamma_1 \cdot \gamma_0)(\gamma_3 \cdot \gamma_4) \\
T \equiv (\gamma_1 \cdot \gamma_0)(\gamma_5 \cdot \gamma_7) \\
U \equiv (\gamma_1 \cdot \gamma_3)(\gamma_5 \cdot \gamma_7) 
\]
along with
\[
S = \gamma_1 \cdot P \gamma_0 \cdot P \gamma_5 \cdot \gamma_4 + \gamma_1 \cdot P \gamma_0 \cdot P \gamma_5 \cdot \gamma_7 \\
+ \gamma_1 \cdot P \gamma_5 \cdot P \gamma_0 \cdot \gamma_4 + \gamma_1 \cdot P \gamma_5 \cdot P \gamma_0 \cdot \gamma_7 \\
\]
\[
T = \gamma_1 \cdot P \gamma_0 \cdot P \gamma_3 \cdot \gamma_4 + \gamma_1 \cdot P \gamma_0 \cdot P \gamma_3 \cdot \gamma_7 \\
+ \gamma_1 \cdot P \gamma_3 \cdot P \gamma_0 \cdot \gamma_4 + \gamma_1 \cdot P \gamma_3 \cdot P \gamma_0 \cdot \gamma_7 \\
\]
\[
U = \gamma_1 \cdot P \gamma_3 \cdot P \gamma_5 \cdot \gamma_4 + \gamma_1 \cdot P \gamma_3 \cdot P \gamma_5 \cdot \gamma_7 \\
+ \gamma_1 \cdot P \gamma_5 \cdot P \gamma_3 \cdot \gamma_4 + \gamma_1 \cdot P \gamma_5 \cdot P \gamma_3 \cdot \gamma_7 
\]
Under the action of \(P\) these transform into each other just like \(s, t, u\).

As in the bosonic case, the \(s\)-channel amplitude here will be given in terms of integrals of the form \(\int_0^1 dx \left(1 - x^{1/2}\right)^{3 - 1/2}\). Three of these transform simply under \(P\).
\[
\beta_s \equiv \int_0^1 dx \left(1 - x^{1/2}\right)^{3 - 1/2} \\
\beta_t \equiv \int_0^1 dx \left(1 - x^{1/2}\right)^{3 - 1/2} \\
\beta_u \equiv \int_0^1 dx \left(1 - x^{1/2}\right)^{3 - 1/2} 
\]
It is easy to see that
\[
\beta_s \rightarrow \overline{\beta}_s \\
\beta_t \rightarrow \overline{\beta}_t \\
\beta_u \rightarrow \overline{\beta}_u \]
where \(\overline{\beta}\) denotes the same integrand as \(\beta\) but integrated over the complementary range, i.e. from \(1\) to \(1\).

We proceed with the calculation. The propagator is now
\[
\frac{1}{l_0 - 1} 
\]
while the vertex is \(\left|V_3\right| = \left|G\right|V_3^0\). Expressions for \(G\) and \(\left|V_3^0\right|\) were given in the previous section.

The vertex with vectors on two of its legs is given by
\[
|V_{13}(3)\rangle = \left(1 - |\psi_{1}^3 \cdot \gamma_1\right)\left(1 - |\psi_{2}^3 \cdot \gamma_2\right)\langle V_3|_{13} \\
= (P_{13}(3)Q_{13}(3) + R_{13}(3))\exp(Z_{13}(3)) \left|\bar{\psi}_{3}^1, \bar{\psi}_{3}^2\right\rangle 
\]
where we have employed the following notation
\[
P_{13}(3) = c_{\alpha}^3 N^\alpha_{\alpha} \delta^3 + b_{\alpha}^3 K^\alpha_{\alpha} \delta^3, \\
Q_{13}(3) = K_{\alpha \beta}^1 \gamma_1 \cdot \gamma_3 + K_{\alpha \beta}^2 \gamma_2 \cdot \gamma_3 \cdot \psi_{3}^1, \\
R_{13}(3) = (K_{\alpha \beta}^1 \gamma_1 \cdot \gamma_2 + K_{\alpha \beta}^2 \gamma_2 \cdot \gamma_1) \cdot \psi_{3}^1 + \\
+ (K_{\alpha \beta}^1 \gamma_1 \cdot \gamma_2 + K_{\alpha \beta}^2 \gamma_2 \cdot \gamma_1) \cdot \alpha \cdot \gamma_3 K_{\alpha \beta}^3 \psi_{3}^2 + \\
+ (\alpha \cdot \gamma_3 K_{\alpha \beta}^3 + \gamma_3 \cdot K_{\alpha \beta}^3 \alpha) K_{\alpha \beta}^4 \psi_{3}^1, \\
Z_{13}(3) = \frac{1}{2} c_{\alpha}^3 N^\alpha_{\alpha} c_{\alpha}^3 + \frac{1}{2} b_{\alpha}^3 K^\alpha_{\alpha} b_{\alpha}^3 + \frac{1}{2} \psi_{3}^1 K_{\alpha \beta}^3 \psi_{3}^2 + b_{\alpha}^3 K^\alpha_{\alpha} c_{\alpha}^3 + \beta_{\alpha}^3 K_{\alpha \beta}^3 \beta_{\alpha}^3. 
\]

The amplitude in the \(s\)-channel may be written as
\[
A_s = \int_0^1 dz \left(1 - z^{1/2}\right)^{3 - 1/2}\langle V_{4}\rangle (0) z^a \left|V_{13}(3)\right\rangle. 
\]
The matrix equation has been evaluated in the bosonic calculation. The result is 

\[
\sum_{ij} (\mathbf{B} \cdot \mathbf{B} - \mathbf{B} \cdot \mathbf{B}) \mathbf{B} = \mathbf{B}
\]

with components given by 

\[
X(i) (\mathbf{1} + \frac{\mathbf{Y}}{\mathbf{X}}) \mathbf{X}^2 = \mathbf{1}
\]

Expanding in powers of \( \mathbf{X} \), we find 

\[
\mathbf{X} = \mathbf{1} + \mathbf{X} + \mathbf{X}^2 + \mathbf{X}^3 + \ldots
\]

and the subscripts denote the order, so we should take only that sector of the expression 

\[
\mathbf{X} + \mathbf{X}^2 + \mathbf{X}^3 + \mathbf{X}^4 + \ldots
\]

By definition, we mean 

\[
\mathbf{X}^n = \mathbf{X} \cdot \mathbf{X} \cdot \ldots \cdot \mathbf{X}
\]

terms of \( \mathbf{X} \) and \( \mathbf{Y} \). We find that 

\[
\mathbf{X} = \mathbf{1}
\]

We discover the above matrix element as \( \mathbf{A} \) or the alternative given us. In
\[
\int_0^{1/2} dz \, z^{-1/2 - 1} (1 - z)^{-1/2} (1 + \frac{1}{2} z + \cdots) (K_0(1) + \frac{27}{16} K_1 + \cdots) = \\
= \int_0^{1/2} dz \, z^{-1/2 - 1} (1 - z)^{-1/2} \left( K_0 + z \left( \frac{5}{4} K_0 + \frac{27}{16} K_1 \right) + \cdots \right).
\]

(3.27)

The amplitude \( A_s \) consists of two types of terms. The first are terms of the form \( \langle \zeta \cdot \zeta | \langle \zeta \cdot p | \langle \zeta \cdot p \rangle \) In terms of the invariants \( (8,11,12) \), as well as the integrals \( (13) \), we find this piece to be just

\[
A_s = (T - U) \beta_u + (S - U) \beta_v.
\]

(3.28)

The second set of terms is of the form \( \langle \zeta \cdot \zeta | \langle \zeta \cdot \zeta \rangle \). It may be written in the following way

\[
A_s^v = \frac{t}{2} T \beta_u + \frac{2}{S} S \beta_v + \left( 1 + \frac{u}{2} \right) (T - U) \beta_u + A_s^{extra},
\]

(3.29)

where we have

\[
A_s^{extra} = (s - 2) \beta_u (U - \frac{3}{2} T - 2 S) - \frac{2}{S} \beta_u S + \frac{t}{2} \beta_v S + \ldots,
\]

(3.30)

and dots indicate terms that come from higher powers of \( z \). Collecting the two pieces, and using the transformation properties of the invariants and integrals under \( P \) we have

\[
A(s,t) = A_s + A_v = (T - U + \frac{t}{2} T) (\beta_u + \beta_v) + (S - U + \frac{u}{2} S) (\beta_u + \beta_v) + \\
+ \left( 1 + \frac{u}{2} \right) (T - U) (\beta_u + \beta_v) + A(s,t)^{extra}.
\]

(3.31)

The integrals \( \beta + \bar{\beta} \) are now given in terms of Euler beta functions, so that

\[
A(s,t) = \left( \frac{1}{4} u (U - S - T) - \frac{1}{4} t (T - U + \frac{t}{2} T) \right) \frac{\Gamma(-s/2) \Gamma(-t/2)}{\Gamma(1 - s/2 - t/2)} + A(s,t)^{extra}.
\]

(3.32)

Using (9) this can be cast in a more symmetric looking form

\[
A(s,t) = -\left( \frac{1}{2} u S + \frac{1}{2} T + \frac{1}{2} u U - \frac{1}{4} u S T \right) \frac{\Gamma(-s/2) \Gamma(-t/2)}{\Gamma(1 - s/2 - t/2)} + A(s,t)^{extra}.
\]

(3.33)

Except for the extra piece this is the correct dual model result.

We see that an extra four point term is indeed present. In order to be able to cancel it with an added contact term it must not have any poles. To the order that we have calculated we see that the first poles do cancel. Extending this we can get the exact extra four point term by always adding pieces in such a way that order by order we cancel all poles. Obviously this extension is not unique. In the next section we shall find the exact expression for the contact term. The main reason for the previous calculation is that it exhibits the existence of an extra term in a straightforward way and does not involve any limiting procedures.

4. Associativity

One of the central axioms of Witten's string field theory is associativity of his string product \( \ast \), i.e. the assumption that

\[
A \ast (B \ast C) = (A \ast B) \ast C.
\]

Associativity is crucial in showing gauge invariance of the string action. The operator representation of Witten's theory gives a concrete representation for \( \ast \).

We must now check if the assumed associativity indeed holds. We have

\[
| A \ast (B \ast C) \rangle = \langle A |_{15} (B \ast C) | V_s \rangle_{121}.
\]

Writing out \( B \ast C \), and using cyclicity of \( V_s \), we find

\[
| A \ast (B \ast C) \rangle = \langle A |_{15} (B_{15} (C | V_s \rangle_{411} | V_s \rangle_{411}.
\]

(4.1)
Having both integrations as factors, to the higher leg and performing the OPE we write

\[ \tag{14} \zeta_{\text{eff}}(\phi_{\text{eff}}(\phi_{\text{eff}}(\phi_{\text{eff}})_{\text{eff}})_{\text{eff}})_{\text{eff}})_{\text{eff}} = t'(0 \cdot \partial + \nu) \]

We find that the calculation proceeds in the same way as the previous one.

It is easy to continue onward that the final result is independent of this order. It is a simple expression for the calculation of the OPE in different sections. It is well known that the OPE is a convolution of two objects.

We will have more to say about this in the next section. Let us note here that

\[ \tag{15} \zeta_{\text{eff}}(\phi_{\text{eff}}(\phi_{\text{eff}}(\phi_{\text{eff}})_{\text{eff}})_{\text{eff}})_{\text{eff}} = \zeta_{\text{eff}}(\phi_{\text{eff}}) \]

where we have dropped (as assumed in section 2) the 'source' four-point vertex by

\[ \tag{16} \zeta_{\text{eff}}(\phi_{\text{eff}}(\phi_{\text{eff}}(\phi_{\text{eff}})_{\text{eff}})_{\text{eff}})_{\text{eff}} = t'(0 \cdot \partial + \nu) \]

Finally, having made the integration in the section we may now do the integration in the leg readout.

\[ \tag{17} \zeta_{\text{eff}}(\phi_{\text{eff}}(\phi_{\text{eff}}(\phi_{\text{eff}})_{\text{eff}})_{\text{eff}})_{\text{eff}} = \zeta_{\text{eff}}(\phi_{\text{eff}}) \]

where \( d \) is dropped. The result is the same.

\[ \tag{18} \zeta_{\text{eff}}(\phi_{\text{eff}}(\phi_{\text{eff}}(\phi_{\text{eff}})_{\text{eff}})_{\text{eff}})_{\text{eff}} = \zeta_{\text{eff}}(\phi_{\text{eff}}) \]

That is the final result. That is, the result is the same.

\[ \tag{19} \zeta_{\text{eff}}(\phi_{\text{eff}}(\phi_{\text{eff}}(\phi_{\text{eff}})_{\text{eff}})_{\text{eff}})_{\text{eff}} = \zeta_{\text{eff}}(\phi_{\text{eff}}) \]

where we have used the result from the previous section. From this we get the operator product expansion expression in the section of (9).

Operator product expansion expression in the section of (9).

\[ \tag{20} \zeta_{\text{eff}}(\phi_{\text{eff}}(\phi_{\text{eff}}(\phi_{\text{eff}})_{\text{eff}})_{\text{eff}})_{\text{eff}} = \zeta_{\text{eff}}(\phi_{\text{eff}}) \]

We may now move back to the same stage, so that

\[ \tag{21} \zeta_{\text{eff}}(\phi_{\text{eff}}(\phi_{\text{eff}}(\phi_{\text{eff}})_{\text{eff}})_{\text{eff}})_{\text{eff}} = \zeta_{\text{eff}}(\phi_{\text{eff}}) \]

where we have

\[ \tag{22} \zeta_{\text{eff}}(\phi_{\text{eff}}(\phi_{\text{eff}}(\phi_{\text{eff}})_{\text{eff}})_{\text{eff}})_{\text{eff}} = \zeta_{\text{eff}}(\phi_{\text{eff}}) \]
find
\[ \frac{3^2}{2} \cdot i \cdot \frac{\sqrt{3}}{2} \cdot i \cdot (A^2 \cdot B \cdot C) \left| V_2 \right| L^2 \left| V_2 \right| \left| V_2 \right| \left| V_2 \right| , \]

We next take the L insertion to the external leg with the help of equation (8).

Using the fact that
\[ \left| V_2 \right| \left| V_2 \right| \left| V_2 \right| \left| V_2 \right| = \left| V_2 \right| \left| V_2 \right| \left| V_2 \right| \left| V_2 \right| , \]

(4.8)

which has been proven in the bosonic case, and just represents associativity for that string, we arrive at the final expression for \((A \ast B) \ast C\), namely
\[
\frac{3^2 \sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot e \cdot (A^2 \cdot B \cdot C) L \left| V_2 \right| L \left| V_2 \right| L \left| V_2 \right| , \]

(4.9)

The reason for the difference in the phase (\(e \neq \frac{1}{2}\)) of these two results is that in moving L from the internal line where it is created via the OPE to our fixed external line we once proceed clockwise and the other time counter-clockwise along the legs of a vertex.

We thus confirm that associativity does not hold, but rather that due to the effect of the insertions, it suffers a slight generalization
\[
\frac{3^2 \sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot e \cdot (A \ast (B \ast C)) . \]

(4.10)

To recapitulate, associativity fails because L (arising through the short distance expansion of two G's), unlike the G's themselves, acquires a phase while cycling on the legs of the cubic vertex. As we have seen, G is tailor-made for the cubic vertex, but L does not have this property. In fact, the only vertex on which L makes sense as an insertion is a quartic one. This in fact represents a non-trivial check of the consistency of these calculations.

5. Gauge Invariance

The failure of associativity has great repercussions on the theory. The central property of the action in (2.1) was invariance under the gauge transformation (2.2). Proof of gauge invariance relies on associativity. Thus the action, as well as the gauge transformation, may have to be changed a bit. From the result of four point scattering we are already resigned to adding an explicit four string interaction
\[
\mathcal{S}_{\text{written}} = \frac{1}{2} \langle A | Q | A \rangle + \frac{1}{3} \delta \langle V_2 | A | A \rangle + \frac{1}{4} \delta^2 \langle V_1 | A | A \rangle A \rangle . \]

(5.1)

We shall now show that imposing gauge invariance completely determines both the action and the gauge transformation. Both acquire explicit order \(g^2\) terms.

One easily sees that invariance of the above action under the transformation
\[
\delta A = (\delta_0 A) + \theta (\delta_1 A) + g^2 (\delta_2 A) \]

(5.2)

implies the following equations (in orders of \(g\))
\[
(A | Q | \delta_0 A) = 0 \]

(5.3)
\[
\langle V_2 | \delta_0 A \rangle A \rangle A \rangle A \rangle = \langle A | Q | \delta_0 A \rangle = 0 \]

(5.4)
\[
\langle V_2 | \delta_0 A \rangle A \rangle A \rangle A \rangle = \langle V_2 | \delta_0 A \rangle A \rangle A \rangle A \rangle = \langle A | Q | \delta_0 A \rangle = 0 \]

(5.5)
\[
\langle V_2 | \delta_0 A \rangle A \rangle A \rangle A \rangle = \langle V_2 | \delta_0 A \rangle A \rangle A \rangle A \rangle = \langle A | Q | \delta_0 A \rangle = 0 \]

(5.6)
\[
\langle V_2 | \delta_0 A \rangle A \rangle A \rangle A \rangle = \langle A | Q | \delta_0 A \rangle = 0 \]

(5.7)

Equations (3) and (4) are as before, i.e. without \(g^2\) modifications. They determine \(\delta_0 A\) and \(\delta_1 A\). (5) is for the central equation and it determines the
Indeed forces us to add a new term to our theory. This is a fact proportional to the association. Loss of associativity this
as we see this is just proportional to the association. Loss of associativity this

(9.1)
\[ \psi_i \beta_i | i, j, k | V_i V_j V_k \]  

\[ \sum_{\beta} \psi_i \beta_i = \psi_i \beta_i | i, j, k | V_i V_j V_k \]

Determine the explicit values of \( \beta \) and \( \gamma \) this can be made a bit more tidy.

\[ \psi_i \beta_i | i, j, k | V_i V_j V_k \]  

\[ \sum_{\beta} \psi_i \beta_i = \psi_i \beta_i | i, j, k | V_i V_j V_k \]

This is what we make it more tidy.

(9.2)
\[ \psi_i \beta_i | i, j, k | V_i V_j V_k \]  

\[ \sum_{\beta} \psi_i \beta_i = \psi_i \beta_i | i, j, k | V_i V_j V_k \]

To use the expression (9.2), simply become.

\[ \psi_i \beta_i | i, j, k | V_i V_j V_k \]  

\[ \sum_{\beta} \psi_i \beta_i = \psi_i \beta_i | i, j, k | V_i V_j V_k \]

On the other hand we may also move the association onto the third left.

\[ \psi_i \beta_i | i, j, k | V_i V_j V_k \]  

\[ \sum_{\beta} \psi_i \beta_i = \psi_i \beta_i | i, j, k | V_i V_j V_k \]

By virtue of equations (9.1) and (9.2) these can be written as

\[ \psi_i \beta_i | i, j, k | V_i V_j V_k \]  

\[ \sum_{\beta} \psi_i \beta_i = \psi_i \beta_i | i, j, k | V_i V_j V_k \]

Having the inversion of the previous equation to the second right we get

\[ \psi_i \beta_i | i, j, k | V_i V_j V_k \]  

\[ \sum_{\beta} \psi_i \beta_i = \psi_i \beta_i | i, j, k | V_i V_j V_k \]

Activity we determined that
\begin{equation}
\langle V_4|_{1234}|0\rangle^2|A\rangle_3|A\rangle_3|A\rangle_4 = \langle V_4|_{1234}Q_1|A\rangle_3|A\rangle_3|A\rangle_4|A\rangle_4 \, . \tag{5.15}
\end{equation}

It is not very difficult to see that our only hope of getting (5) to work is if we choose

\begin{equation}
\langle V_4|_{1234} = \tilde{z} \left(\frac{3}{2}\right)^4 \langle V_4^0|_{1234} \delta^2 \, . \tag{5.16}
\end{equation}

Using the fact that \(\{k, Q\} = L\) we find that

\begin{equation}
\langle V_4|_{1234}|0\rangle^2|A\rangle_3|A\rangle_3|A\rangle_4 = \tilde{z} \left(\frac{3}{2}\right)^4 \langle V_4^0|_{1234} L^1_1|A\rangle_1|A\rangle_2|A\rangle_3|A\rangle_4 - \tilde{z} \left(\frac{3}{2}\right)^4 \langle V_4^0|_{1234} Q_1 \delta^4_1|A\rangle_1|A\rangle_2|A\rangle_3|A\rangle_4 \, . \tag{5.17}
\end{equation}

Therefore the \(V_3\) and \(V_4\) pieces of (3) almost cancel. What remains is simply

\begin{equation}
-\tilde{z} \left(\frac{3}{2}\right)^4 \langle V_4^0|_{1234} Q_1 \delta^4_1|A\rangle_1|A\rangle_2|A\rangle_3|A\rangle_4 \, . \tag{5.18}
\end{equation}

In order to proceed further let us say something about the BRST invariance of the "bare" quartic vertex \(V_4^0\).

In the spirit of (4.6) let us define yet another quartic vertex

\begin{equation}
|\tilde{V}_4|_{1234} = \langle V_4|_{1234}|V_4^0|_{1234}|V_4^0|_{1234} \, . \tag{5.19}
\end{equation}

Proceeding as before with the insertion and OPE's we see that

\begin{equation}
|\tilde{V}_4| \propto L|V_4^0| \, .
\end{equation}

On the other hand, simply from the fact that \(\tilde{V}_4\) is a contraction of BRST invariant vertices it follows that it also is BRST invariant. As \(L\) commutes with \(Q\) it follows that the bare vertex is also BRST invariant, i.e.

\begin{equation}
Q|\tilde{V}_4^0| = 0 \, . \tag{5.20}
\end{equation}

Note that by some arguments \(K_c \tilde{V}_4 = 0\), but since \([K, L] \neq 0\) the bare vertex is not \(K_c\) invariant. Since cancellation of \(K_c\) anomalies depends only on the insertions through their conformal dimension, we have that \(K_c \delta V_4^0 = 0\). The full vertex is \(K_c\) invariant, as it must be in order for our action to make sense. Using the BRST invariance of \(V_4^0\) we write (18) as

\begin{equation}
\tilde{z} \left(\frac{3}{2}\right)^4 \langle V_4^0|_{1234} (Q_2 + Q_3 + Q_4) \delta^4_1|A\rangle_1|A\rangle_2|A\rangle_3|A\rangle_4 \, .
\end{equation}

This is easily rearranged to give

\begin{equation}
\langle V_4|_{1234}|A\rangle_1|Q_2|A\rangle_2|A\rangle_3|A\rangle_4 \, ,
\end{equation}

\begin{equation}
\langle V_4|_{1234}|A\rangle_1|Q_3|A\rangle_2|A\rangle_3|A\rangle_4 \, ,
\end{equation}

\begin{equation}
\langle V_4|_{1234}|A\rangle_1|Q_4|A\rangle_2|A\rangle_3|A\rangle_4 \, . \tag{5.21}
\end{equation}

Re-labeling the first by \((1234)\), and second by \((1342)\) and using cyclicity of \(\langle V_4|\) gives us

\begin{equation}
\langle 5|A\rangle_1 = -\langle V_4|_{1234}(|A\rangle_1|A\rangle_2|A\rangle_3 + |A\rangle_1|A\rangle_3|A\rangle_2 + |A\rangle_2|A\rangle_3|A\rangle_1) \, . \tag{5.22}
\end{equation}

We have now determined both the action (1) and the corresponding gauge transformation (2). In order for the action to be invariant under this transformation we need to show that equations (6) and (7) hold. Formula (7) is in some sense more restrictive since it consists of a single term. It is met since we have

\begin{equation}
\langle V_4|5|A\rangle_1|A\rangle_2|A\rangle_3|A\rangle_4 \propto \langle V_4|5|4\rangle = 0 \, . \tag{5.23}
\end{equation}

The last step follows since \(\delta^4 = 0\).
\[\frac{1}{\sqrt{\lambda}}(\delta L(\delta A)) - \delta \beta = \delta \theta = \delta \theta = \delta \theta = \delta \theta \]

\[\text{If we take the cubic piece of equation (g) along } x = 0 \text{ then the cubic piece we need is:} \]

\[\text{where in the last step we need the corresponding OPF as with the calculation of} \]

\[\text{We see both (g) and (22) contain the connection} \]

\[\text{We recall } 1 \mapsto A \text{ and then use cyclicity of } (\delta A) \text{ to write the} \]

\[\text{The cubic piece on the other hand, becomes} \]

\[\text{With } \beta \text{ of equation (g), taking care of the restrictions of all the derivatives, we have} \]

\[\text{Finally, we turn to (g), we may rewrite the cubic piece as} \]

\[\text{Let us recall that by equation (22)} \text{ and (21)(2) we have on the cubic vertices} \]

\[\text{To get results for } \text{we have } (\delta A) \text{ on the right} \]

\[\text{The choice of tools connected with the overlaps of } \]

\[\text{The choice of tools must be consistent with the overarching} \]

\[\text{where the four vertices are} \]

\[\text{on } \text{Similarly on } A \text{ we find} \]

\[\text{over the equations. Taking the last equation} \]

\[\text{choice of tools to the remaining functions with correct} \]

\[\text{where we are just keeping the dependence on the index } A. \text{ We found that the} \]

\[\text{The final result is (g). We copy the cubic piece as} \]
gives \(-1 - \sqrt{2}i\). These two results are consistent since \(|V_{4}\rangle_{1314} = -|V_{4}\rangle_{1314}\) on GSO allowed states. One might think that this holds for all cyclic permutation of the legs of \(V_{4}\). This is not the case precisely because of the non-cyclicality of the \(\sqrt{2}i\)'s. On \(|V_{4}\rangle_{3412}\) we have

\[
\beta^{1} = e^{i\pi/2} = e^{-i\pi/2} = e^{i\pi/2}.
\]  \hspace{1cm} (5.35)

Here \(a + b + c = -i + \sqrt{2}\), while \(|V_{4}\rangle_{3412} = |V_{4}\rangle_{1314}\) on GSO allowed states. We seemingly get two contradictory answers for the cubic piece of (6). There is no contradiction, it simply implies that \(<V_{3}^{0}|V_{4}\rangle^{0}\) gives zero when applied on GSO allowed states.

This contraction is nothing but \(|V_{4}\rangle^{0}\). We have just shown that the quintic vertex projects out GSO allowed states. Note that the quartic piece of (6) is also proportional to \(|V_{4}\rangle^{0}\), so that it also vanishes on the GSO allowed states. Equation (6) is therefore satisfied on such states.

The main results of this section should be reviewed. We have determined the complete action for the Witten superstring in the Neveu-Schwarz sector. The action contains a cubic and quartic interaction. Higher interactions are not present as a consequence of the gauge invariance of the action. Just as the action gets an extra \(g^{2}\) piece, so does the gauge transformation law.

Although equations (16) and (4.6) completely determine the quartic vertex we really can't calculate the contraction in (4.6) explicitly. What can be done is the following. We may determine the overlaps satisfied by \(|V_{4}\rangle^{0}\) (thus also by \(|V_{4}\rangle\)) from the known \(V_{2}\) and \(V_{3}\) overlaps. We can then construct the quartic vertex directly by Neumann function techniques just as was done for the cubic vertex.

The ghost number of \(|V_{4}\rangle\) is 2, so it is easier to construct than \(|V_{4}\rangle^{0}\), since it can be given over the \(|+ + + +\rangle\) vacuum. In fact over this vacuum the full vertex \(V_{4}\) is a pure exponential, i.e. no insertions are needed, as can be shown by evaluating the corresponding \(K_{\alpha}\) anomalies. In the appendix we have calculated the diagonal coefficients of \(V_{4}\) in this way. Only the diagonal ones contribute to the \(K_{\alpha}\) anomalies.

As we have seen, the vertex \(V_{4}^{0}\) was defined through the contraction of two cubic vertices without insertions

\[
|V_{4}^{0}\rangle_{1314} = |V_{2}\rangle_{1315}|V_{2}\rangle_{1546}^{0}.
\]

The overlaps of the fields on the quartic vertex may be found from the known two and three string overlaps. The only non-trivial quartic overlaps will turn out to be the ones for the \(\psi\) field. The \(\psi\) overlaps on \(V_{2}\) and \(V_{4}^{0}\) are given by

\[
\langle V_{1315}|\psi^{0}(\sigma) + i \psi^{0}(\pi - \sigma)|V_{1315}\rangle = 0
\]
\[
\langle V_{1315}|\psi^{0}(\sigma) - i \psi^{0}(\pi - \sigma)|V_{1315}\rangle = 0,
\]  \hspace{1cm} (5.36)

as well as

\[
\langle V_{1315}|\psi^{0}(\sigma) - i \psi^{0}(\pi - \sigma)|V_{1315}\rangle = 0
\]
\[
\langle V_{1315}|\psi^{0}(\sigma) - i \psi^{0}(\pi - \sigma)|V_{1315}\rangle = 0.
\]  \hspace{1cm} (5.37)

All of these are valid for \(\sigma \in (0, \frac{\pi}{2})\). It follows that for \(V_{4}^{0}\) we have

\[
\langle V_{1315}|\psi^{0}(\sigma) + i \psi^{0}(\pi - \sigma)|V_{1315}\rangle = 0
\]
\[
\langle V_{1315}|\psi^{0}(\sigma) - i \psi^{0}(\pi - \sigma)|V_{1315}\rangle = 0
\]
\[
\langle V_{1315}|\psi^{0}(\sigma) - i \psi^{0}(\pi - \sigma)|V_{1315}\rangle = 0
\]
\[
\langle V_{1315}|\psi^{0}(\sigma) - i \psi^{0}(\pi - \sigma)|V_{1315}\rangle = 0,
\]  \hspace{1cm} (5.38)

for \(\sigma \in (0, \frac{\pi}{2})\). \(V_{4}^{0}\) obviously is not cyclic as we have already indicated. This is in fact not a problem. One should bear in mind that the \(V_{2}\) overlap for \(\psi(\sigma)\) is itself not cyclic. Consistency of the theory is saved however since \(V_{4}\) is cyclic when acting on GSO-allowed states. As a consequence, the same can be said for \(V_{4}^{0}\) when acting on GSO-allowed states.
6. Conclusion

In conclusion, we have presented an alternative to the current method of calculating the Hessian. Our approach has several advantages over the traditional method. Firstly, it allows for a more efficient computation by breaking down the calculation into smaller, more manageable parts. Secondly, it provides a clearer understanding of the underlying structure of the network, enabling better insights into the learning process. Finally, our method is more scalable and can be applied to larger networks without significant computational overhead.

Acknowledgments

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References


Appendix

Proof of Theorem 1: Let $f(x)$ be a smooth function with $n$ variables. We aim to show that the gradient of $f(x)$ can be computed efficiently using our method. The gradient of $f(x)$ is defined as $\nabla f(x) = \frac{\partial f}{\partial x}$. Our method involves computing the Hessian of $f(x)$, which is the matrix of second-order partial derivatives.

Let $\mathbf{H}(x)$ be the Hessian of $f(x)$, and let $\mathbf{A}(x)$ be a matrix of auxiliary variables. We have

$$
\mathbf{A}(x) = \begin{bmatrix} a_1(x) & a_2(x) & \cdots & a_n(x) \end{bmatrix}
$$

where $a_i(x)$ is a function of $x$ that depends on the specific application. The gradient of $f(x)$ can be computed as

$$
\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\
\frac{\partial f}{\partial x_2} \\
\vdots \\
\frac{\partial f}{\partial x_n} 
\end{bmatrix} = \mathbf{H}(x) \mathbf{A}(x)
$$

To solve for $\mathbf{A}(x)$, we can express $\mathbf{H}(x)$ as

$$
\mathbf{H}(x) = \begin{bmatrix} h_{11}(x) & h_{12}(x) & \cdots & h_{1n}(x) \\
h_{21}(x) & h_{22}(x) & \cdots & h_{2n}(x) \\
\vdots & \vdots & \ddots & \vdots \\
h_{n1}(x) & h_{n2}(x) & \cdots & h_{nn}(x) 
\end{bmatrix}
$$

and solve for $\mathbf{A}(x)$ by using the singular value decomposition (SVD) of $\mathbf{H}(x)$.

$$
\mathbf{H}(x) = \mathbf{U} \Sigma \mathbf{V}^T
$$

where $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices and $\Sigma$ is a diagonal matrix containing the singular values of $\mathbf{H}(x)$. The solution for $\mathbf{A}(x)$ is then

$$
\mathbf{A}(x) = \mathbf{V} \Sigma^{-1} \mathbf{U}^T \nabla f(x)
$$

This method allows for efficient computation of the gradient of $f(x)$, which is essential for many applications in machine learning and optimization.

Experiments

We conducted experiments on several datasets to evaluate the performance of our method compared to the traditional method. Our results show a significant improvement in terms of computational efficiency and accuracy.

Conclusion

Our method provides a novel approach to computing the gradient of a function, which is particularly useful in the context of deep learning and optimization problems. Further research is needed to explore the potential of this method in other applications.
to satisfy all the properties required of them if they are to be a part of the action. This would seem to imply that the usual trick of simply substituting the full fields $|\Phi\rangle$ for the physical fields $|\mathcal{A}\rangle$ in the action works here as well.

One of us (A. R. B.) would like to thank Sanjay Jain and Richard Woodard for many useful discussions.

### APPENDIX

#### A.1. Neumann Coefficients of the Cubic Vertex $V_3$

- **$N^{11}_{\mathcal{A}A}$**

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- **$N^{11}_{\mathcal{A}A}$**

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<td>$\frac{11}{3\sqrt{3}}$</td>
</tr>
<tr>
<td>$m=2$</td>
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<td>$\frac{1}{3}$</td>
<td>$\frac{11}{3\sqrt{3}}$</td>
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<tr>
<td>$m=3$</td>
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<td>$-\frac{1}{3\sqrt{3}}$</td>
<td>$-\frac{11}{3\sqrt{3}}$</td>
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</tbody>
</table>

- **$\tilde{N}^{11}_{\mathcal{A}A}$**

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- **$\tilde{N}^{11}_{\mathcal{A}A}$**

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<tr>
<td>$m=3$</td>
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<td>$-\frac{1}{3\sqrt{3}}$</td>
<td>$-\frac{11}{3\sqrt{3}}$</td>
</tr>
</tbody>
</table>
\[
\frac{1}{\sqrt{2}}(2)\frac{\sqrt{2}}{\sqrt{2}} = \frac{\alpha_d}{v_d}
\]
\[
\frac{1}{\sqrt{2}}(2)\frac{\sqrt{2}}{\sqrt{2}} = \frac{\alpha_d}{v_d}
\]
\[
\alpha_f \frac{1}{\sqrt{2}}(2)\frac{\sqrt{2}}{\sqrt{2}} = \frac{\alpha_d}{v_d}
\]

For the coefficients of the \( F(2) \) piece we have the formula:

\[
\mathcal{H}^{X^+}X^+ + \mathcal{H}^{X^-}X^- + \mathcal{H}^{X^0}X^0 + \mathcal{H}^{X^+}X^- + \mathcal{H}^{X^-}X^+ + \mathcal{H}^{X^0}X^0 = 0
\]

The insertion has been written as

A.2. Coefficients of the Insertion

\[
\mathcal{H}^{X^+}X^+ = \mathcal{H}^{X^-}X^+
\]
\[
\mathcal{H}^{X^-}X^- = \mathcal{H}^{X^+}X^-
\]
\[
\mathcal{H}^{X^0}X^0 = \mathcal{H}^{X^0}X^0
\]
\[
\mathcal{H}^{X^+}X^- = \mathcal{H}^{X^-}X^+
\]
\[
\mathcal{H}^{X^-}X^+ = \mathcal{H}^{X^0}X^0
\]

and lower indices

Finally, the coefficients are (anti) symmetric under interchange of both upper

\[
\mathcal{H}^{X^+}X^- = \mathcal{H}^{X^-}X^+
\]
\[
\mathcal{H}^{X^-}X^+ = \mathcal{H}^{X^+}X^-
\]
\[
\mathcal{H}^{X^0}X^0 = \mathcal{H}^{X^0}X^0
\]
\[
\mathcal{H}^{X^+}X^- = \mathcal{H}^{X^-}X^+
\]
\[
\mathcal{H}^{X^-}X^+ = \mathcal{H}^{X^0}X^0
\]

from the (\ref{eq:4}) the \( \gamma \) and \( \tau \) are

\[
\mathcal{H}^{X^+}X^- = \mathcal{H}^{X^-}X^+
\]
\[
\mathcal{H}^{X^-}X^+ = \mathcal{H}^{X^+}X^-
\]
\[
\mathcal{H}^{X^0}X^0 = \mathcal{H}^{X^0}X^0
\]

Same cyclic property holds for \( \gamma, \tau, \) and \( \alpha \). The (\ref{eq:12}) coefficients follow

\[
\mathcal{H}^{X^+}X^- = \mathcal{H}^{X^-}X^+
\]

We list here some important symmetries of the Manton coefficients:

\[
\begin{array}{ccc}
\mathcal{H}^{X^+}X^- & = & \frac{1}{2} \\
\mathcal{H}^{X^-}X^+ & = & \frac{1}{2} \\
\mathcal{H}^{X^0}X^0 & = & \frac{1}{2} \\
\end{array}
\]
where

\[
\begin{align*}
    f_1^{(1)} & = \frac{1}{3} \\
    f_2^{(1)} & = \frac{2}{3} \\
    f_3^{(1)} & = -\frac{13}{2 - 3^2} \\
    \ldots
\end{align*}
\]

as well as

\[
\begin{align*}
    f_1^{(1)} & = 1 \\
    f_2^{(1)} & = e^{-\frac{\pi i}{3}} = e \\
    f_3^{(1)} & = e^{\frac{\pi i}{3}} = e .
\end{align*}
\]

As we have seen the ghost coefficients are harder to calculate since they come both from leading and subleading terms. The only ones that were used in the four point scattering were

\[
\begin{align*}
    H_1^{(1)} & = \frac{2}{3} \\
    G_1^{(1)} & = \frac{20}{3} .
\end{align*}
\]

A.3. Neumann Coefficients of the Quartic Vertex \( V_4 \)

We proceed here to give an outline of the construction of the quartic vertex using Neumann function techniques. For a detailed introduction as well as explanation of notation see [3,3]. The conformal transformation that takes the four string overlap into the unit circle is

\[
\rho = \ln \frac{z^2 - i}{z^2 + i} - \frac{\pi}{2} . \tag{A.1}
\]

It follows that \( \frac{\partial z^4}{\partial z} = \frac{z^4}{d\rho} \). Inverting (1) we find

\[
z^4(\rho) = z_4 \left( \frac{1 + i e^{\rho}}{1 - i e^{\rho}} \right)^{1/4} ,
\]

where \( z_4 = z_3 = -z_2 = -z_1^2 = i \). In fact the roots must be chosen so that

\[
z_4 = (1, -i, -1, i) e^{i\frac{\pi}{4}} \text{ as in figure 4, since this gives}
\]

\[
z^4(\rho) = z_4^{-1} (\rho - \rho_0) , \tag{A.2}
\]

for \( \rho \in (0, \frac{\pi}{2}) \). This is a necessary condition for the Neumann functions to satisfy appropriate overlap equations. Let us focus on the \( \psi \) correlator. It is simply

\[
K(\rho, \rho') = \left( \frac{\partial z^4}{\partial \rho} \right)^{1/4} \frac{1}{\pi - z^4} \left( \frac{\partial z^4}{\partial \rho} \right)^{1/4} , \tag{A.3}
\]

where \( \alpha_1 = \alpha_3 = -\alpha_2 = -\alpha_4 = 1 \) are the "charges" corresponding to the \( z_4 \)'s. Keeping only the dependence on the \( \sigma \) index we see that

\[
K^\sigma(\rho) \sim \left( \frac{\partial z^4}{\partial \rho} \right)^{1/4} \sim \left( \frac{\alpha_4}{\alpha_3} \right)^{1/4} .
\]

Since \( z_3 = -z_4 \) we have

\[
\begin{align*}
    K^1(\rho) & \sim \frac{1}{\sqrt{z^4(\rho)}} \\
    K^3(\rho) & \sim \frac{1}{\sqrt{z^4(\rho)}} \\
    K^4(\rho) & \sim \frac{1}{\sqrt{z^4(\rho)}} \\
    K^s(\rho) & \sim \frac{1}{\sqrt{z^4(\rho)}} .
\end{align*} \tag{A.4}
\]

On the other hand the \( \psi \) correlator needs to have the same overlaps as \( \psi \). Using the overlaps derived in section 5 and equations (4) we see that

\[
\begin{align*}
    \sqrt{z^4(\rho)} & = i \sqrt{z^4(\rho - \rho_0)} \\
    \sqrt{z^4(\rho)} & = -i \sqrt{z^4(\rho - \rho_0)} \\
    \sqrt{z^4(\rho)} & = -i \sqrt{z^4(\rho - \rho_0)} \\
    \sqrt{z^4(\rho)} & = -i \sqrt{z^4(\rho - \rho_0)} .
\end{align*} \tag{A.5}
\]

As promised the overlaps of \( \sqrt{z^4} \) are not cyclic. From the overlaps it follows that we must choose the roots \( \sqrt{z_4} \) to be \( \sqrt{z_4} = (1, e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}) e^{i\frac{\pi}{4}} \). Figure 5 depicts these roots.
Expanding in $x$ and $y$ we easily get the needed vertex corrections. Let us first
\[
\left((x-\delta(x))e^x(x+1)\right)_{x=0} = \frac{(x-\delta(x))e^{x+1} - (x-\delta(x))e^x}{x+1} = \frac{e^{x+1} - e^x}{x+1} = \sum_{n=1}^{\infty} \frac{e^{x+1} - e^x}{x+1} \frac{x^n}{n!}
\]

Similarity for $y$ we find
\[
\left((y-\delta(y))e^y(y+1)\right)_{y=0} = \frac{(y-\delta(y))e^{y+1} - (y-\delta(y))e^y}{y+1} = \frac{e^{y+1} - e^y}{y+1} = \sum_{n=1}^{\infty} \frac{e^{y+1} - e^y}{y+1} \frac{y^n}{n!}
\]

Adding these two give us
\[
\psi' + \psi \Psi' \text{ where } \psi \text{ and } \Psi \text{ are the two parts }
\]

and similarly for $\chi$.

Similarly, for $\chi$ we have
\[
\left((x-\delta(x))e^x(x+1)\right)_{x=0} = \frac{(x-\delta(x))e^{x+1} - (x-\delta(x))e^x}{x+1} = \frac{e^{x+1} - e^x}{x+1} = \sum_{n=1}^{\infty} \frac{e^{x+1} - e^x}{x+1} \frac{x^n}{n!}
\]

Again only the diagonal part of the matrix is non-zero, and we can write it as
\[
\frac{1}{1+y} \left( \frac{\Delta_x}{\Delta_y} \frac{\Delta_y}{\Delta_x} \right) \frac{x-z}{1} = \left( \frac{\Delta_x}{\Delta_y} \frac{\Delta_y}{\Delta_x} \right) \chi
\]

Similarity for $\psi$ we have in complete analogy with the above
\[
\psi' = \chi \psi
\]

where we have introduced the position $\chi$. The coarse grid is simply
\[
\left(\chi(x)\right)_{x=0} = \frac{e^{x+1} - e^x}{x+1} \frac{x^n}{n!}
\]

Equation (4) is not very difficult to put $\chi$ into the form

We now turn to evaluating the diagonal coefficients of the quartic vertex...

We can find $\chi$ and can be found in [1]...

and the quartic vertex. The coefficients of the $x$ and $y$ sections are not listed...

Even simpler:

Note that calculating the non-diagonal coefficients (just as in the cubic vertex) is...
REFERENCES

Figure 1. s-channel amplitude

Figure 2. t-channel amplitude

Figure 3. three string w plane

Figure 4. four string z plane

Figure 5. four string √z plane