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BLACK HOLES IN A PERIODIC UNIVERSE

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Schwarzschild-like black hole solutions corresponding to periodic universes are presented and analyzed. In the case of one compact dimension an analytic solution is given. An interesting structure of horizons-within-horizons is discovered when the size of the periodic coordinate is smaller than the Schwarzschild radius. A variational method is also introduced and applied to the case of all spatial coordinates being compact.

1. Introduction

The principal motivation for looking at black hole solutions in periodic universes comes from a rather strange place – the new frontier between strings and cosmology. In a recent paper Brandenberger and Vafa[1] have looked at some striking implications of superstring theory on cosmology. Focusing on thermodynamics they argued that in order for the cosmological predictions to make sense strings must propagate on space-time where all the spatial directions are compact. Working on manifolds with toroidal compactification there arises a uniquely stringy symmetry $R \to R^2_{\text{Planck}}/R$ – a duality between "large" and "small". They found two consequences of this symmetry that are of immense interest to cosmology. First, duality gets rid of the initial big bang singularity at $R = 0$, starting the universe from Planck size. Second, the correct dimensionality of space-time is picked out, i.e., of all the $9 + 1$ dimensions only three spatial dimensions can expand significantly from Planck size and thus become visible.

Their thermodynamic results were prodded up by dynamical calculations[2] by generalizing the Friedmann-Robertson-Walker model to manifolds with manifest $R \to R^2_{\text{Planck}}/R$ duality. Here for every spatial coordinate of size R there is an orthogonal dual coordinate of inverse size. The model satisfies all the initial thermodynamic results.

We have used such manifolds to get rid of one singularity that arises in general relativity (the initial one). It is interesting now to see what is the effect of such manifolds on another singularity, namely, black holes.

The only black holes considered in periodic universes so far has been in the domain of Kaluza-Klein theories. That is of no use to us here because we do not postulate the kind of factoring of the metric between compact and non-compact dimensions that is done in Kaluza-Klein. Moreover, in the case of all dimensions being compact the Kaluza-Klein ansatz would lead to a constant and therefore trivial metric.

In this letter we present a brief introduction to two ways of dealing with black holes in periodic universes. The application of these techniques to dual manifolds will be taken up in a future work.

2. One Compact Dimension

We look at a (3 + 1)-dimensional universe, i.e., periodic along the z-axis with a period h. At the origin we put a point mass m. We are interested in calculating the gravitational field due to this mass. Conversely, instead of a periodic universe along z, we may think of the mass and all its image charges lying equidistant along the z-axis. This is now a static, axially symmetric problem. The general case of such a system was related by Weyl[3,4] to the solution of an underlying classical problem. The metric may in general be written as

$$ds^2 = e^{\nu-\lambda}(dr^2 + dz^2) + r^2 e^{-2\lambda} d\phi^2 - e^{2\lambda} dt^2 \tag{2.1}$$

In these coordinates Einstein's (vacuum) equations take on a very simple form

$$\frac{\partial \nu}{\partial r} = r\left\{\left(\frac{\partial \lambda}{\partial r}\right)^2 - \left(\frac{\partial \lambda}{\partial z}\right)^2\right\}, \quad \frac{\partial \nu}{\partial z} = 2r\,\frac{\partial \lambda}{\partial r}\,\frac{\partial \lambda}{\partial z}, \tag{2.2}$$

and their integrability gives

$$\nabla^2 \lambda \equiv \frac{\partial^2 \lambda}{\partial r^2} + \frac{1}{r}\frac{\partial \lambda}{\partial r} + \frac{\partial^2 \lambda}{\partial z^2} = 0\,. \tag{2.3}$$

It is convenient to introduce complex coordinates

$$\zeta = r + iz\,, \quad \bar{\zeta} = r - iz\,,$$

with the help of which Eqs. (2.2) may be written compactly as

$$\frac{\partial \nu}{\partial \zeta} = (\zeta + \bar{\zeta})\left(\frac{\partial \lambda}{\partial \zeta}\right)^2 .$$

Therefore ν follows by direct integration from λ.

As promised, the relativistic problem of finding the metric outside a given static, axisymmetric distribution of mass has been reduced to an associated problem in Newtonian gravity of finding a classical potential λ due to (in general) some other mass distribution, but of the same total mass.

The monopole solution of the associated classical problem does not give the monopole solution of general relativity, i.e., the Schwarzschild solution. Rather, we recover what is known as the Curzon metric. It has been known for a long time that the Newtonian solution corresponding to Schwarzschild is that of a rod of uniform mass-per-length μ. From now on for convenience we choose the units

$2G = 1$. A rod of mass and length m (therefore $\mu = 1$) that is centered at the origin has the Newtonian potential

$$\lambda = \frac{1}{2} \ln \frac{r_+ + r_- - m}{r_+ + r_- + m} , \tag{2.4}$$

where $r_\pm \equiv \sqrt{r^2 + (z \pm m/2)^2}$. It is easy to see that the change of coordinates

$$r = R\sqrt{1 - m/R}\, \sin\theta\ ,\ \ z = (R - m/2)\cos\theta \tag{2.5}$$

gives us the Schwarzschild metric in the usual spherical coordinates.

Although general relativity is not a linear theory, in the simple case of a static, axisymmetric problem we have related it to a problem in Newtonian gravity, hence to a linear problem. The solution for any collection of n collinear point particles lying on the z-axis[5] is just a superposition of appropriate rods. In other words $\lambda = \Sigma\ \lambda_i$, where λ_i is the potential of a rod of mass and length m_i lying on the z-axis, centered at the point z_i.

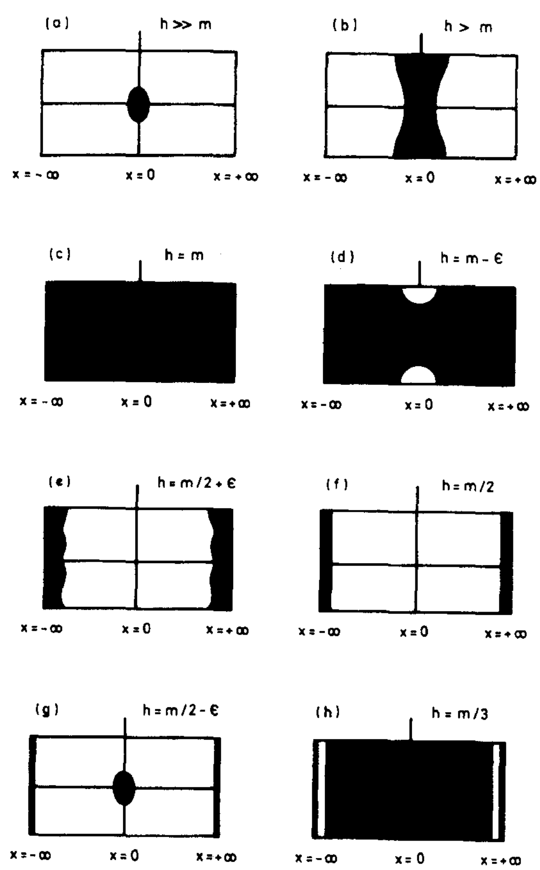

Fig. 1. The shapes of the horizon surfaces ($g_{00} = 0$) for various h/m ratios. Notice the "onion-like" structure of horizons appearing inside horizons as h/m decreases.

Even though this seems rather general, for more than one particle we immediately run into the problem of consistency. We have at the start assumed that we are dealing with a *static* system. It is, however, impossible to build a static gravitating system out of a finite number of particles. In our formalism this inconsistency manifests itself through the failure of the requirement of elementary flatness[8] of the manifold. Let us note that cosmic strings do not satisfy elementary flatness, i.e., they display a deficit angle. This comes about because of the negative pressure inside the string.

We, however, are looking at collections of point particles, hence there is no notion of pressure and elementary flatness must hold. Through the years these types of solutions have been investigated, but the only static solutions were obtained at the price of assuming the existence of negative masses.

The system that interests us is different. It *is* static because of the presence of an infinite number of equidistant particles (images). Our system is thus the simplest physical system (after the Schwarzschild particle) to be analyzed via the Weyl method. The difficulty we face is in summing the infinite series that determines λ. For $h >> m$ near the origin we feel just small perturbations to the Schwarzschild metric due to the presence of the image masses (Fig. 1). A systematic perturbation treatment may be developed in this region but it is not of much interest in what follows, and we do not present it here.

Interesting behavior starts to show up as we put our mass m into a universe the same size or smaller than the Schwarzschild radius of that mass. It is therefore quite nice that for $h = m/n$, where $n = 1, 2, \ldots$, we can determine the metric in closed form. At these values of h the rods form a uniform mass distribution along the whole z-axis, with $\mu = n$. The value $h = m$ is particularly interesting in that it corresponds to the size of the universe when the rods first touch. The Newtonian potential is easily found to be $\lambda = -n \ln(r/m)$, and a further simple calculation of ν, using (4), gives us the metric

$$ds^2 = \left(\frac{r}{m}\right)^{2(n^2+1)} (dr^2 + dz^2) + \left(\frac{r}{m}\right)^{2n} r^2 d\phi^2 - \left(\frac{r}{m}\right)^{-2n} dt^2 \,. \qquad (2.6)$$

Let us stress again that we are not dealing here with a line singularity. Even at the critical h's the *physical system* is just a collection of collinear point masses. It is the associated classical system (a purely mathematical construct) that has in it the picture of rods, and hence the possibility of rods touching and overlapping. For this same reason nothing stops us from having the rods overlap.

Note that a very interesting thing has happened: at the critical sizes $h = m/n$ the metric $g_{\mu\nu}$ becomes translationally invariant along z. This is easy to visualize using the picture of rods touching; however, as a consequence of this we find that the gravitational field no longer carries information about the location of the mass that is its source. To analyze this in more detail we calculate the curvature invariants associated with (6). In 4 dimensions there are four such invariants, the first two are simply the square and cube of the Weyl tensor $C_{\lambda\mu\nu\kappa}$, i.e., $I \equiv C^{\lambda\mu\nu\kappa} C_{\lambda\mu\nu\kappa}$ and $J \equiv C_{\lambda\mu\nu\kappa} C^{\nu\kappa\rho\sigma} C^{\lambda\mu}_{\rho\sigma}$. A tedious but straightforward calculation gives

$$I = 16\,n^2(n+1)^2\,(n^2+n+1)\frac{1}{m^4}\left(\frac{m}{r}\right)^{4(n^2+n+1)},$$

$$J = 48\,n^4(n+1)^4\,\frac{1}{m^6}\left(\frac{m}{r}\right)^{6(n^2+n+1)}. \tag{2.7}$$

The other two invariants involve contractions with $\varepsilon_{\mu\nu\rho\sigma}$ and are identically zero.

We thus see that the gravitational field singularity is spread over the whole z-axis. Moreover, in the limit that n goes to infinity (size of compact dimension goes to zero) we find that all the invariants vanish outside the Schwarzschild radius $r = m$, and diverge inside it. The outside and inside are thus separated from each other by an infinite barrier. It is interesting to note that after we have compressed the compact coordinate to zero we are left with a (2 + 1)-dimensional space-time that (at least outside of $r = m$) behaves just like the usual 2 + 1 gravity in that all the curvature invariants vanish.

For general $h \in (m/(n+1), m/n]$ the Newtonian potential is written as $\lambda = -n \ln(r/m) + \delta\lambda$. It is a rather simple counting problem to determine that $\delta\lambda$ is the potential due to collinear non-overlapping rods of mass m and length nh, centered at $z_k = kh$ for n even, and $z_k = kh + h/2$ for n odd.

This can now serve as a basis for perturbative evaluation of the metric in the vicinity of the critical h's. For example, for $h = m/2 - \varepsilon$ and near the origin $\delta\lambda$ is approximately just a single Schwarzschild particle of mass m sitting at the origin. The details of the perturbative technique will be presented elsewhere. [9] We shall illustrate it here by looking at how the horizon changes with h.

The easiest way to determine the horizon is to remember the well-known theorem that for static space-time the surface of infinite redshift (i.e., $g_{00} = 0$) coincides with the horizon. By inspection we see that for the critical values of h this puts the horizon at $r = \infty$. Figure 1 depict how the horizon changes as h decreases. At $h = m$ (the first critical value) the whole universe is inside the horizon. For $h = m - \varepsilon$ there appears a spherical dimple at $z = h/2$ that is "outside". As we decrease h further (Figs. 1f and 1g) we see that at $h = m/2$ we get the whole universe outside. At $h = m/2 - \varepsilon$ another layer of the horizon forms near the origin, etc.

3. All Three Spatial Dimensions Compact

In this section we want to deal with a point mass located in a universe in which all the three spatial dimensions are compact with period h. Unlike the case of one compact dimension we now do not have the help of any continuous symmetries. We would thus have to be able to solve Einstein's equations in all generality. This obviously is not the way to proceed.

On the other hand, even before doing any calculations we know two important things about this system. First of all the metric must be periodic in x, y, z and independent of t. Second: when h becomes very large in the vicinity of the origin we should just get back the Schwarzschild solution.

Using this let us develop a variational approach to the problem. We will construct a family of metrics (depending on a certain number of parameters) that in the limit of large h goes over into Schwarzschild. In Cartesian coordinates the Schwarzschild solution is simply the block diagonal matrix

$$g^S_{\mu\nu} = \begin{pmatrix} -1 + \dfrac{m}{r} & \\ & 1 + \dfrac{m}{r^2(r-m)}\mathbf{A} \end{pmatrix}, \tag{3.1}$$

where we have introduced the expressions $\mathbf{A}_{ij} \equiv x_i x_j$, and $r \equiv \sqrt{x^2 + y^2 + z^2}$.

We construct our k-parameter family of metrics by "making the Cartesian coordinates periodic", i.e., by the substitution of $\hat{x}_i$ for x_i in the above expression, where

$$\propto_i \equiv \sum_{a=1}^{k} A_a \frac{h}{2\pi a} \sin\left(\frac{2\pi a}{h} x_i\right), \tag{3.2}$$

and the parameters A_a sum up to one, guaranteeing that for large h we get the correct limiting behavior.

The ansatz of using $\hat{g}_{\mu\nu}(A_a)$ in our variational calculation will be justified if we find that results do not change much as we increase the number of variational parameters k. Also, this kind of variational scheme can be applied to the case of one compact dimension making contact with the analytical results of the first section.

A straightforward and very long calculation gives us the action

$$\hat{S}(A_a) \quad \int_{-h/2}^{h/2} dxdydz \, \frac{3\hat{r} - 2m}{2\hat{r}^8(\hat{r} - m)} f(x,y,z), \tag{3.3}$$

where we have introduced

$$f(x,y,z) = [(\hat{\mathbf{x}} \cdot \hat{\mathbf{T}} \cdot \hat{\mathbf{x}})^2 - \hat{r}^2 \hat{\mathbf{x}} \cdot \hat{\mathbf{T}}^2 \cdot \hat{\mathbf{x}}],$$

as well as the vector $\hat{\mathbf{x}}$ with components $\hat{x}_i$ and the (diagonal) matrix $\hat{\mathbf{T}}$ with components $\partial_i \hat{x}_j$.

Two regions contribute the most to this. One is at $\hat{r} = 0$, which is just a pole at the origin, while the other is at $\hat{r} = m$. The origin just contributes a piece that does not depend on the variational parameters and is thus of no interest to us. From the $\hat{r} = m$ surface we get the contribution

$$\frac{1}{2m^7} \int dxdydz \, \frac{1}{\hat{r} - m} f(x,y,z).$$

For a large enough value of the ratio h/k the quantity $f(x,y,z)$ is fairly constant over the $\hat{r} = m$ surface, and hence the above integral is just proportional to the area of that surface.

We have just related the problem of minimizing the action to the simpler one of minimizing an area. Now this can be easily tackled numerically with the help of

Monte-Carlo techniques. In the case of very large h the surface is very nearly a sphere of radius m, and the area may be calculated analytically. Figure 2 illustrates how the parameters that minimize the area depend on the value of the mode cut-off k. Details of the numerical calculation will be given in a later publication.[9]

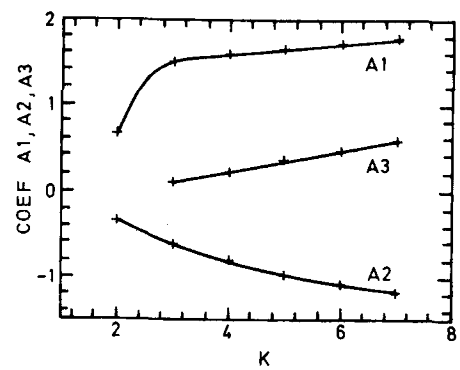

Fig. 2. The Fourier coefficients minimizing the action for various cutoffs k and assuming large h/m.

4. Conclusions

In this letter we have given a brief introduction into two calculation schemes tailored to the analysis of Schwarzschild black holes in universes with one or more periodic spatial dimensions. In the analytical calculations presented in the first section we have uncovered an interesting onion-like structure of horizons-within-horizons. We have also found a spreading of the gravitational singularity at the center of the black hole that is due to the compact dimension. In the second section a variational technique was given. For certain values of the size of the compact dimensions h the calculation was related to the problem of minimizing an area of a surface.

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