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SYMMETRY BREAKING AS A CONSEQUENCE OF STABILIZATION OF A BOTTOMLESS THEORY

A. R. BOGOJEVIĆ

Niels Bohr Institute, Blegdamsvej 17, Copenhagen 2100, Denmark

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The Greensite–Halpern stabilization technique is applied to $d=0$ Euclidean field theories. The effective action is calculated using WKB as well as a variational approach. For a set of free theories with non-trivial measure the effective action is computed exactly, and a comparison is made with the results of approximate techniques. The effect of stabilization on the measure is to bring in a "centrifugal barrier" term that breaks parity invariance. A model with quartic interaction of wrong sign is also analyzed in the limit of large coupling.

1. Introduction

There has been a great burst of interest recently in the matrix model approach to gravity (see Refs. 2 and 3 and references therein). The success in extracting the Painlevé equation gave the possibility of understanding the non-perturbative contributions to the partition function. It is now known that acceptable solutions of Painlevé get complex non-perturbative contributions.[4,5] This is not very surprising since the equation follows from formal manipulations on an ill-defined partition function — coming from a theory whose action is not bounded from below. We refer to such actions as unstable. In strings each surface appears with positive weight and so we necessarily get a perturbation theory that is not Borel summable, and follows from an unstable action.

For this reason it is essential to understand how quantum theories will make sense based on unstable actions. The general approach to "stabilizing" such theories was developed by Greensite and Halpern[1] six years ago. It was recently rediscovered by Marinari and Parisi.[8] Here we shall study in detail certain $d = 0$ unstable theories where the stabilization program can be carried out in closed form. We shall also look at two approximate techniques that one may use: WKB and variational calculations. We shall compare their predictions to the exact results. For a set of unstable free theories with non-trivial measure we discover a symmetry breaking mechanism that comes about as a consequence of stabilization.

Stabilization of matrix models has been investigated numerically.[6,7] It was found that the stabilized theory correctly gives only real non-perturbative contributions. It is very important to get some analytical results about stabilized matrix models. An initial investigation in this direction has begun by Karliner and Migdal.[9] The results given here are used in Ref. 10 to give a derivation of a stabilized Painlevé equation.

2. Stabilization

The usual definition of an Euclidean partition function of a d-dimensional theory is

$$Z = \int [d\phi] e^{-\frac{1}{\hbar}S[\phi]} \ . \tag{2.1}$$

This expression is ill-defined if the action is not bounded from below.

Greensite and Halpern[1] have given a prescription of how to make sense of such unstable theories. They define the partition function as

$$Z = \langle \psi | \psi \rangle \ , \tag{2.2}$$

where $|\psi\rangle$ is the ground state wavefunction of an associated $(d+1)$-dimensional theory governed by the Fokker–Planck Hamiltonian

$$H = \int dx \left(-\frac{1}{2} \frac{\delta^2}{\delta\phi(x)^2} + V[\phi] \right) \ , \tag{2.3}$$

where the potential is given by

$$V[\phi] = \frac{1}{8\hbar^2} \left(\frac{\delta S}{\delta\phi(x)} \right)^2 - \frac{1}{4\hbar} \frac{\delta^2 S}{\delta\phi(x)^2} \ . \tag{2.4}$$

For the case of stable theories this is equivalent to the usual definition of the partition function. To show this note that we can write the Fokker–Planck Hamiltonian as

$$H = \int dx R(x)^+ R(x) \ , \tag{2.5}$$

where

$$R(x) = \frac{i}{\sqrt{2}} \left(\frac{\delta}{\delta\phi(x)} + \frac{1}{2} \frac{\delta S}{\delta\phi(x)} \right) \ . \tag{2.6}$$

From this it follows that the energy eigenstates satisfy $E \geq 0$. On the other hand, $\psi_0 = e^{-1/2\hbar S}$ satisfies $R\psi_0 = 0$, so that

$$H\psi_0 = 0 \ . \tag{2.7}$$

For a stable action S it follows that ψ_0 is normalizable, hence it is a state, and by virtue of (2.6) the ground state. Equation (2.2) then reduces to the usual definition of the partition function (2.1). Note, however, that the expression (2.2) is well-defined even when we are dealing with an unstable theory. In this case ψ_0 is not normalizable, hence not a state. The true ground state ψ has an energy slightly larger than zero. If we write it as

$$\psi = e^{-\frac{1}{2\hbar}S_{\rm eff}} , \tag{2.8}$$

then the partition function takes the more familiar form

$$Z = \int [d\phi] e^{-\frac{1}{\hbar}S_{\rm eff}[\phi]} , \tag{2.9}$$

given in terms of a (stable) effective action $S_{\rm eff}$. The problem of finding this effective action is just related to a $(d+1)$-dimensional quantum mechanics problem of finding the ground state of the Hamiltonian (2.3). Similarly for correlators we get the stabilized result

$$\langle Q[\phi]\rangle = \frac{1}{Z}\int [d\phi] Q[\phi] e^{-\frac{1}{\hbar}S_{\rm eff}[\phi]} . \tag{2.10}$$

As shown by Greensite and Halpern the stabilization technique we have outlined satisfies three very important requirements. First, S and $S_{\rm eff}$ give the same perturbation theory. Second, both have the same classical equations of motion. Third, the large-N expansions of the two theories are identical. The last two requirements are quite restrictive and there is no other stabilization technique known that satisfies them. It would be very important to prove that Greensite–Halpern stabilization is the unique one satisfying the above requirements. Another very important open question is the extension of this procedure to Minkowski theories.

3. Free Theory with a Wrong Sign

The simplest example of an unstable action is free theory with the wrong sign, i.e.,

$$S_0 = -\frac{1}{2}\phi_i K_{ij}\phi_j = -\bar{S}_0 . \tag{3.1}$$

By virtue of Eqs. (2.3) and (2.4) we have

$$H = \bar{H} + \frac{1}{2}K_{ii} , \tag{3.2}$$

where $H(\bar{H})$ represent Fokker–Planck Hamiltonians corresponding to the actions $S_0(\bar{S}_0)$. The two Hamiltonians differ by just a constant, so they share the same ground state wavefunction. Since $\bar{S}_0$ is a stable theory we know that we have $S_{\rm eff} = \bar{S}_0$. Therefore, the effect of stabilization is simply to flip the sign of the free action to

$$S_{\rm eff} = -S_0 = \bar{S}_0 . \tag{3.3}$$

This conclusion obviously holds for any Euclidean free field theory regardless of the number of dimensions.

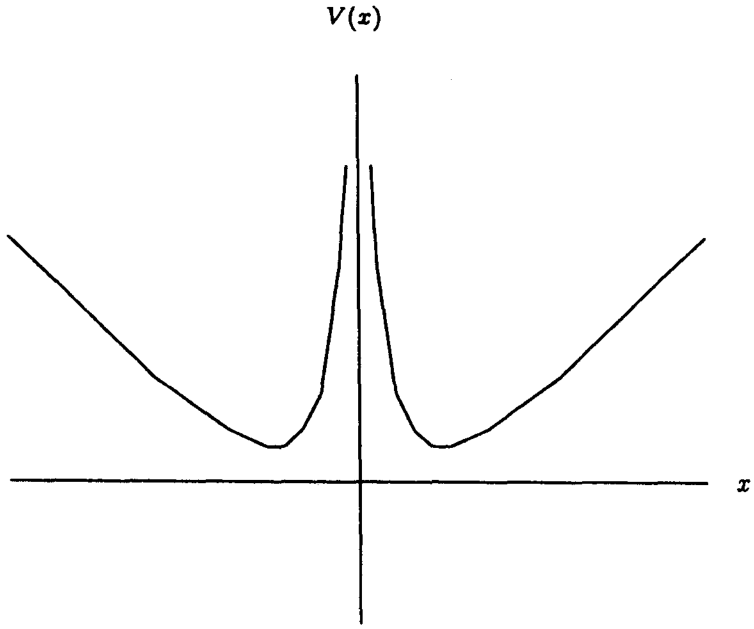

Fig. 1. Stabilizing potential.

Let us next concentrate on a less trivial example when a non-trivial measure term present in (2.1). We shall look at $d = 0$ theories given by

$$Z = \int_{-\infty}^{+\infty} dx x^{2n} e^{-\frac{1}{\hbar}S_0} , \tag{3.4}$$

where $n \in \mathbb{Z}$ and $S_0 = -1/2\ x^2$. We apply the stabilization technique to the action

$$S = -\frac{1}{2}x^2 - n\hbar \ln x^2 . \tag{3.5}$$

This action is obviously unstable for all values of n. Dropping a trivial constant term, the stabilizing potential is simply

$$V(x) = \frac{1}{8\hbar^2}x^2 + \frac{n^2 - n}{2}\frac{1}{x^2} . \tag{3.6}$$

Note that $V(x)$ is invariant under $n \to 1 - n$, so that

$$S_{\text{eff}}(x, n) = S_{\text{eff}}(x, 1 - n) . \tag{3.7}$$

As an example of this we have $S_{\text{eff}}(x, 1) = S_{\text{eff}}(x, 0) = 1/2\ x^2$. We therefore need only look at the cases $n = 2, 3, 4, \ldots$. The stabilizing potential $V(x)$ is shown in Fig. 1. Note that if we write $n - 1 = l$, then we see that stabilization has brought about a "centrifugal barrier" term. For $n = 2, 3, 4, \ldots$ the "orbital angular momentum" takes on the values $l = 1, 2, 3, \ldots$ and it becomes impossible to tunnel through the barrier at the origin. We thus have a symmetry breaking. Assuming hereafter that we are in the right well the ground state wavefunction is as in Fig. 2. The corresponding effective action is shown in Fig. 3.

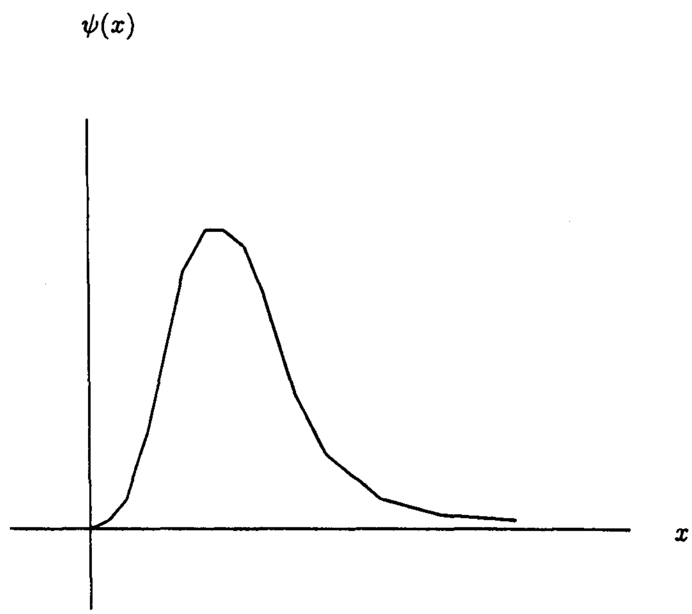

Fig. 2. Ground state wavefunction.

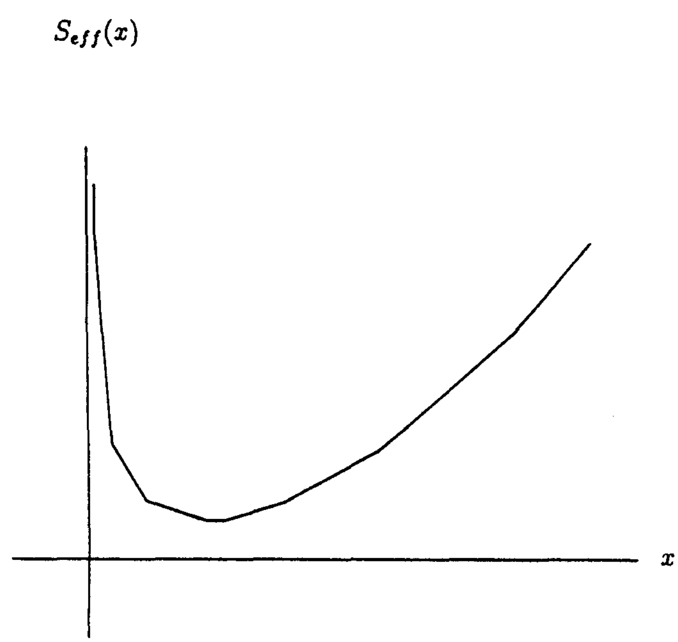

Fig. 3. Effective action.

The Schrödinger equation reads

$$\frac{d^2\psi}{dx^2}+\frac{2}{\hbar^2}\left(\hbar^2E-\frac{1}{8}x^2-\frac{n^2-n}{2}\frac{\hbar^2}{x^2}\right)\psi=0\ . \tag{3.8}$$

This may be solved exactly. We shall do this at the end of this section. Let us first determine $S_{\rm eff}$ by the use of approximation techniques at our disposal. These are essential once we look at models with interactions. The exact solution will then allow us to gauge the validity of various approximations.

Let us use WKB to determine the ground state. The classical turning points are given (for $x>0$) by

$$x_{1/2}=\sqrt{a^2\mp D}\ , \tag{3.9}$$

where $a^2=4\hbar^2E, b^4=4\hbar^2(n^2-n)$ and $D^2=a^4-b^4$ are all positive quantities. For $x<0$ the turning points are $-x_{1/2}$. It is easy to see that there is no tunneling through the barrier at $x=0$, since the integral $\int_{-x_1}^{x_1}\sqrt{2V-2E}\ dx$ is infinite.

The ground state energy is determined by the condition

$$\int_{x_1}^{x_2}\sqrt{2E-2V}\ dx=\frac{\pi}{2}\ . \tag{3.10}$$

The integral on the right may easily be evaluated in terms of elementary functions. We get

$$\frac{1}{4\hbar}\left\{\sqrt{2a^2t-t^2-b^4}-a^2\arcsin\left(\frac{a^2-t}{D}\right)-b^2\arcsin\left(\frac{a^2-\frac{b^4}{t}}{D}\right)\right\}\ , \tag{3.11}$$

evaluated between the limits $t_{1/2}=a^2\mp D$. This leaves us with the simple result $1/4\hbar\ (a^2-b^2)\pi$, and so the ground state energy is given by

$$E=\frac{1}{2\hbar}\left(1+\sqrt{n^2-n}\right)\ . \tag{3.12}$$

For $x>x_2$ the ground state wavefunction is just

$$\psi(x)\simeq\frac{1}{\sqrt{\kappa(x)}}\ e^{-\int_{x_2}^{x}\kappa(y)dy}\ , \tag{3.13}$$

with $\kappa(x)=\sqrt{2V(x)-2E}$. Again, this integration may easily be performed. Rather than write this out it is more illustrative to give the result for large x. We have

$$\int_{x_2}^{x}\kappa(y)dy\approx\frac{1}{2\hbar}\int_{\sqrt{\hbar}A}^{x}\sqrt{y^2-\hbar A^2}dy\ , \tag{3.14}$$

where $A^2=4(1+\sqrt{n^2-n})$. Integrating we get

$$\frac{1}{4\hbar}x\sqrt{x^2-\hbar A^2}-\frac{A^2}{4}\ln\left(\frac{x+\sqrt{x^2-\hbar A^2}}{x}\right)\ .$$

Expanding in $\hbar/x^2$ we find

$$\psi(x) \propto \sqrt{\mu(x)} e^{\frac{1}{2\hbar} S_0} , \tag{3.15}$$

where we have

$$\mu(x) = \frac{1}{x}\left(1 + \frac{A^2}{2}\frac{\hbar}{x^2} + \dots\right) \exp\left\{\frac{A^4}{8}\frac{\hbar}{x^2} + \dots\right\} . \tag{3.16}$$

Dots indicate terms that are $\mathcal{O}(\hbar^2/x^4)$. Therefore, for large (positive) x, the effect of stabilization is as before just to flip the sign of the free action. Now there is also a change in the measure according to $x^{2n} \to \mu(x)$.

From (1.4) we see that in the case of a general theory we get the asymptotic result

$$S_{\text{eff}} \simeq |S| . \tag{3.17}$$

At the same time stabilization always induces a measure term.

Returning to the models governed by (3.5), we may similarly use WKB to determine S_{eff} in the region $x_1 < x < x_2$. As we have seen in (3.11) the answer will be given in terms of arc sines and logs. We can then use this to obtain an expansion of S_{eff} around the minimum. We shall present here another, simpler derivation of this expansion by the use of a variational approach.

As we have seen, for large positive x we have $S_{\text{eff}} \approx -S_0$, while at the origin we have $S_{\text{eff}} = \infty$. The simplest trial wavefunction to take is therefore

$$\psi = e^{-\frac{1}{4\hbar}x^2 + \frac{\lambda}{2}\ln x^2} , \tag{3.18}$$

with the variational parameter λ satisfying $\lambda > 0$. We determine λ so that it minimizes

$$E(\lambda) = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} . \tag{3.19}$$

A simple calculation gives

$$2\hbar E(\lambda) = \lambda - \frac{\lambda(\lambda-1) - n(n-1)}{2\lambda - 1} , \tag{3.20}$$

where we have thrown away an unimportant constant. The minimum of $E(\lambda)$ for $n = 2, 3, 4, \dots$ where $\lambda = n$, and so our effective action is

$$S_{\text{eff}} = \frac{1}{2}x^2 - n\hbar \ln x^2 . \tag{3.21}$$

The minimum effective action is at

$$x_{\min} = \sqrt{2n\hbar} . \tag{3.22}$$

By expanding around this minimum we get

$$S_{\text{eff}} = \frac{1}{2}z^2 - \frac{1}{12\sqrt{n\hbar}}\, z^3 + \dots \,, \tag{3.23}$$

where we have introduced $z = \sqrt{2}(x - x_{\text{min}})$.

We turn now to the exact solution of (3.8). From the equation we see that for large x we have asymptotically $\psi \simeq \exp(-1/4\hbar\, x^2)$, while near the origin $\psi \approx x^n$. We thus write the wavefunction as

$$\psi(y) = e^{-\frac{1}{2}y} y^{\frac{n}{2}} u(y) \,, \tag{3.24}$$

where $y = x^2/2\hbar$, and $u(y)$ is easily seen to satisfy the equation

$$yu'' + \left(n + \frac{1}{2} - y\right)u' - \left(\frac{n}{2} + \frac{1}{4} - \hbar E\right)u = 0 \,. \tag{3.25}$$

This is a confluent hypergeometric differential equation (see, for example, Ref. 11). The general solution may be written as

$$u(y) = A_1F_1\left(\frac{n}{2}+\frac{1}{4}-\hbar E, n+\frac{1}{2}; y\right) + By^{\frac{1}{2}-n}\ {}_1F_1\left(-\frac{n}{2}+\frac{3}{4}-\hbar E, \frac{3}{2}-n; y\right) \,. \tag{3.26}$$

From the requirement that $u(y)$ does not diverge at the origin we see that we must have $B = 0$. At the same time for $\psi(y)$ to vanish at large y it must be true that ${}_1F_1$ is a polynomial. We have

$${}_1F_1(a, b; x) = 1 + \frac{a}{b}\frac{x}{1!} + \frac{a(a+1)}{b(b+1)}\frac{x^2}{2!} + \cdots \tag{3.27}$$

hence this is a polynomial for $a = 0, -1, -2, \dots$. This determines the energy levels of our model to be

$$\hbar E_m = m + \frac{n}{2} + \frac{1}{4} \,, \tag{3.28}$$

i.e., they are equally spaced just as in the case of the harmonic oscillator. The wavefunctions are

$$\psi_m(x) = e^{-\frac{1}{4\hbar}x^2} x^n\ {}_1F_1\left(-m, n + \frac{1}{2}; \frac{x^2}{2\hbar}\right) \,. \tag{3.29}$$

For the ground state $m = 0$, and by (3.27) we see that ${}_1F_1$ is then just 1. The effective action is thus exactly what we found in (3.21) by using the variational approach, and so for example (3.22) correctly gives the VEV

$$\langle x \rangle = \sqrt{2n\hbar} \,. \tag{3.30}$$

The WKB approximation, while giving (3.17) correctly is not of much use in determining the integration measure $\mu(x)$. On the other hand, we have checked numerically that the correlator $\langle x \rangle$ calculated by WKB turns out to be in good agreement with the exact result.

4. Quartic Interaction with Wrong Sign

Let us now apply the stabilization technique to an unstable interacting theory. We look at the action

$$S = \frac{1}{2}x^2 - gx^4 , \tag{4.1}$$

where the coupling constant g is positive. Following the general prescription we may easily construct the stabilizing Hamiltonian for this model. However, in this case we do not get an exactly solvable theory for any value of g, and have to settle for an approximate derivation of $S_{\rm eff}$. As we shall show it is possible to get a closed result for $g \to \infty$ in a way that parallels the work of the previous section.

We may introduce an auxiliary field σ and cast (4.1) in the form

$$S = \frac{1}{2}x^2 + \frac{1}{2}\sigma^2 - \sqrt{2g}\, x^2\sigma . \tag{4.2}$$

This makes the action quadratic in x. The partition function is

$$Z = \int d\sigma\; e^{-\frac{1}{2}\sigma^2} \int dx\; e^{-\frac{1}{2}(1-\sqrt{2g}\,\sigma)x^2} . \tag{4.3}$$

For the rest of this section we set $\hbar = 1$. The x integration is easily performed. For $\sigma > 1/\sqrt{2g}$ this is the Gaussian of the wrong sign, but we may then use (3.3) to get

$$Z = \int d\sigma\; e^{-\Sigma(\sigma)} , \tag{4.4}$$

where $\Sigma(\sigma) = 1/2\ \sigma^2 + 1/2 \ln|\sigma - 1/\sqrt{2g}|$, and where we have dropped a trivial constant term. Taking $g \to \infty$ gives us

$$\Sigma(\sigma) = \frac{1}{2}\sigma^2 + \lambda \ln \sigma^2 \tag{4.5}$$

with $\lambda = 1/4$. This action is not bounded from below, but this cannot be the case since we started in (4.1) with an unstable action. Note the similarity to the action in (3.5). The stabilizing potential is now

$$V(\sigma) = \frac{1}{8}\sigma^2 + \frac{\lambda(\lambda+1)}{2}\frac{1}{\sigma^2} . \tag{4.6}$$

This is just as we had before with the substitution of $n-1$ for λ. We may thus directly write

$$\Sigma_{\rm eff}(\sigma) = \frac{1}{2}\sigma^2 - (\lambda+1)\ln\sigma^2 . \tag{4.7}$$

Setting $\lambda = 1/4$ we find

$$Z = \mathcal{N} \int_0^\infty d\sigma\ \sigma^{\frac{5}{2}}\ e^{-\frac{1}{2}\sigma^2}\ . \tag{4.8}$$

Treatment of unstable quartic theory for large but finite coupling will be presented elsewhere. We are able to proceed with this perturbative calculation (in $1/g$) due to the fact that (as we have seen in the previous section) the complete set of eigenstates of the Fokker–Planck Hamiltonian corresponding to (4.5) is known.

5. Conclusions

The aim of this paper is to give examples of unstable $d = 0$ field theories for which it is possible to complete the Greensite–Halpern stabilization scheme in closed form. We find that even in the simple case of unstable free theories with non-trivial measure terms, stabilization gives rise to interesting physics — this case giving rise to a "centrifugal barrier" term that breaks parity invariance. The exact solutions enable us to check the validity of approximation techniques for calculating $S_{\rm eff}$ that are at our disposal, and that must be used for all but the simplest few models. At the end we have dealt with the more interesting model of a wrong sign quartic interaction. In the case of infinite coupling this model also becomes exactly solvable. This enables us to construct a perturbation theory in $1/g$.

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References

1. J. Greensite and M. Halpern, *Nucl. Phys.* **B242** (1984) 167.
2. M. R. Douglas and S. H. Shenker, *Nucl. Phys.* **B335** (1990) 635.
3. D. Gross and A. Migdal, *Phys. Rev. Lett.* **643** (1990) 127.
4. F. David, Saclay preprint SPhT/90-090.
5. F. David, Saclay preprint SPhT/90-043.
6. J. Ambjørn, J. Greensite, and S. Varsted, Niels Bohr preprint NBI-HE-90-39, July, 1990.
7. J. Ambjørn and J. Greensite, Niels Bohr preprint NBI-HE-90-67, September, 1990.
8. E. Marinari and G. Parisi, *Phys. Lett.* **B240** (1990) 375.
9. M. Karliner and A. Migdal, Princeton preprint PUPT-1191, July, 1990.
10. A. R. Bogojević, Niels Bohr preprint, NBI-HE-90-67, November, 1990.
11. G. Arfken, *Mathematical Methods for Physicists* (Academic Press, 1970).