Solitary states in arrays of excitable FitzHugh-Nagumo units

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***Abstract*—While coherence-incoherence patterns have been exhaustively studied in systems of coupled oscillators, their mechanisms of emergence and their relationship en route from complete coherence to incoherence in coupled *excitable* systems remain as yet unresolved. Here we disclose two types of solitary states in arrays of non-locally coupled excitable FitzHugh-Nagumo units with dominant repulsive over attractive interactions. While the prevailing type of solitary states is shown to derive its dynamical features from unbalanced two-cluster states in globally coupled networks, the minority type is fundamentally a consequence of non-locality of interactions. Apart from the states whose local structure is based on successive spiking of units, we also find solitary states where local excitability and slow-fast dynamics give rise to *leap-frog* activity characterized by an alternating order of units' spiking. The main impact of noise on system's behavior is shown to be the reduction of its multistability, whereby the solitary states are suppressed in favour of patched patterns.**

***Index Terms*—Excitable systems, clustering, solitary states, leap-frog activity, noise, noise-induced preference of attractors**

I. INTRODUCTION

The discovery of chimera states [1], [2] has fundamentally affected our understanding of emergent phenomena in coupled oscillators. Instead of studying the onset of synchronization and the appearance of collective modes, the main focus shifted toward the onset of and the links between the states with symmetry breaking of synchrony [3], such as cluster states [4], [5], chimeras [6], [7], or solitary states [8], [9], where populations of identical units with symmetrical couplings spontaneously split into groups admitting different dynamics. In contrast to coupled oscillators, these two problems still remain unresolved for the class of coupled excitable systems [10], where the isolated dynamics is stationary rather than oscillatory, but oscillations may be triggered by sufficiently strong perturbations from interactions and/or noise. Excitability is one of the building blocks of physics of life, shaping the activity of neuronal, endocrine and cardiac tissues, but is also pervasive in other fields, including climate dynamics, chemical reactions and lasers. One cannot expect beforehand that the results on coherence-incoherence patterns from coupled oscillators would trivially extend to the realm of coupled excitable systems. Apart from theory, addressing fundamental problems concerning the onset and relationship between synchrony-broken states in coupled excitable systems is relevant for different applications, including neuroscience, especially in relation to achieving efficient information transmission, working memory or desynchronization supposed to prevent setting in of pathological states. Here we disclose the mechanisms of emergence of solitary states in systems of excitable units with prevalent repulsive over attractive interactions and the slow-fast local dynamics, the paradigm often found in neuronal systems [11].

Our model concerns an array of N identical, non-locally coupled FitzHugh-Nagumo units, whose dynamics is given by [12], [13]

$$\begin{aligned}\varepsilon \dot{u_k} &= u_k - \frac{u_k^3}{3} - v_k + \frac{\kappa}{2R}\sum_{j=k-R}^{k+R}[g_{uu}(u_j - u_k) + g_{uv}(v_j - v_k)]\\ \dot{v_k} &= u_k + b + \frac{\kappa}{2R}\sum_{j=k-R}^{k+R}[g_{vu}(u_j - u_k) + g_{vv}(v_j - v_k)].\end{aligned} \tag{1}$$

The individual slow-fast dynamics is prototypical for type II excitability [11], and comprises fast activator variables u_k and slow recovery variables v_k with a timescale separation due to the smallness of parameter $\varepsilon = 0.05$. All the indices are periodic modulo N. An isolated unit undergoes a singular Hopf bifurcation at $b = 1$ which mediates between the excitable ($b \gtrsim 1$) and the oscillatory regime ($b < 1$). Above Hopf bifurcation, there is a further canard transition at $b \approx 1 - \varepsilon/8$ [15], where the harmonic subthreshold (low-amplitude) oscillations transform to relaxation (large-amplitude) oscillations. Here, the value $b = 1.001$ is set so that uncoupled units are in the excitable regime. Each unit interacts with R nearest neighbors to its left and to its right, rendering the coupling radius $r = R/N$. Interactions between a pair of units involve not only direct terms between two activator or two recovery variables, but also the cross terms, which may be compactly written via the rotational coupling matrix [12] $G = \begin{pmatrix} g_{uu} & g_{uv} \\ g_{vu} & g_{vv} \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$. Note that parameter φ modifies the prevalence of attractive and repulsive interactions [13] by affecting the sign of the interaction terms. Coupling strength $\kappa = 0.4$ is considered to be uniform over the array.

The paper is organized as follows. In Sec. II we characterize the prevalent type of solitary states, which have their dynamical counterparts in unbalanced two-cluster states in globally coupled networks. Section III concerns the minority type of solitary states, which mainly emerges due to non-locality of interactions and shows some peculiar features, such as self-organization based on leap-frog (leader-switching) dynamics [14] between pairs of units. Sec. IV addresses the impact of noise on solitary states, while Sec. V provides the summary of our main results.

II. PREVALENT TYPE OF SOLITARY STATES

In the following, we demonstrate that the main dynamical features of the prevailing types of solitary states in arrays, called SS1 and SS2, derive from the corresponding unbalanced two-cluster states in globally coupled networks, where an assembly is split into two groups of *identically* synchronized elements. In particular, the features inherited by the solitary states are the frequency locking of typical and solitary units as well as the respective orbits of the units. Note that the coupling parameter φ for all the relevant solutions is such that three out of four interaction terms between two units are repulsive.

Let us first consider the unbalanced two-cluster states in networks with global coupling. Classification of two-cluster states and mechanisms of their emergence may be analyzed by studying the dynamics of the reduced system, which essentially has the same form as the system describing two interacting nonidentical excitable units. Combining the results of two approaches [13], one based on the method of evaporation exponents and the other involving probe oscillators, we have been able to construct the stability diagram for the unbalanced two-cluster states in a network with global coupling, see Fig. 1(a). The results are presented in the (φ, p) parameter plane, where p is the cluster partition parameter, indicating that for the given solution, the two clusters comprise pN and $(1-p)N$ units. Domains of stability of cluster solutions are shown in orange, while the black and green lines outlining their boundaries respectively denote period-doubling curves and curves of branching points. One finds six distinct regimes with 1:1 (domains IV-VI), 1:2 (I, II) or 2:3 (III) frequency locking, all featuring the so-called *mixed-mode oscillations* [16] where relaxation oscillations are combined with subthreshold oscillations, see Fig. 1(b)-(e). For certain types of cluster states, p can become rather small, but nevertheless does not tend to zero. This implies that there can exist only periodic two-cluster states with a sufficiently unbalanced partition, similar to what has already been found for systems of type I excitable elements [17].

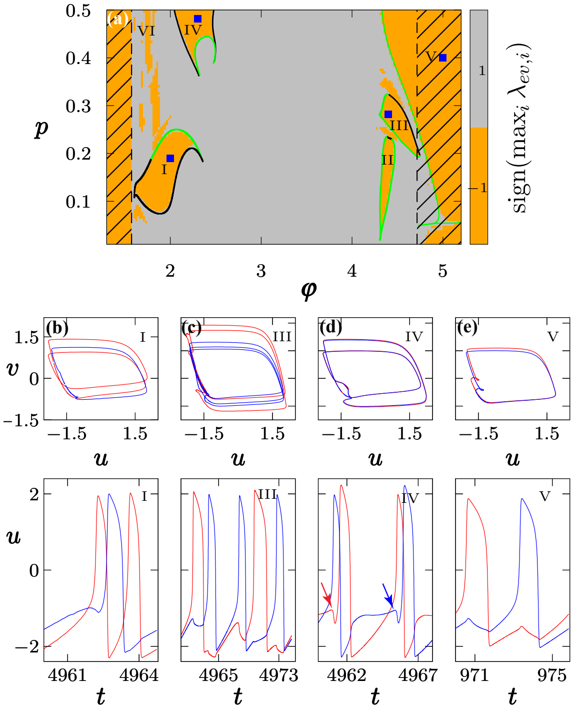

Fig. 1. Unbalanced periodic two-cluster states in networks with global coupling. (a) Stability diagram in (φ, p) plane. Orange/grey: stable/unstable solutions; black solid lines: period-doubling bifurcations, green lines: curves of branching points; dashed black lines: destabilization/reappearance of homogeneous stationary state. (b)-(e) Time series $u_i(t), i \in \{A, B\}$ and phase portraits associated with (φ, p) values (blue squares) from (a). Parameters are: $b = 1.001$, $\varepsilon = 0.05$, $\kappa = 0.4$.

Making a comparison to the results in Fig. 1, we have established that the prevailing type of solitary states in non-locally coupled arrays is in fact a dynamical counterpart of unbalanced two-cluster states in globally coupled networks. In particular, the state SS1 in Fig. 2 is associated with two-cluster states from region I, whereas SS2 (not shown) is related to the two-cluster states from region V. The SS1 and SS2 types of states emerge within the same φ intervals as their two-cluster state counterparts and maintain the associated locking of clusters' frequencies. However, the nonlocal character of interactions and the associated fluctuations in mean-fields felt by individual units cause the clusters of solitary and typical units to become fuzzy [8] instead of exact, cf. Fig. 2(b). Put differently, under nonlocal coupling $r < 1/2$, the unbalanced cluster states inherent to globally coupled networks lose permutation symmetry, so that the solitary and typical clusters comprise frequency locked but not identically synchronized units. The spatial profile of average spiking frequencies $\omega_k = 2\pi N_k/\Delta$, where N_k is the number of spikes recorded within a long interval Δ, indicates a 2:1 subharmonic frequency locking of solitary to typical units. Qualitative similarity with unbalanced cluster states found in region I of Fig. 1(a), both in terms of local phase portraits and the time series $u_k(t)$, is demonstrated in Fig. 2(d) and Fig. 2(f).

The intrinsic structure of SS1 is further described via the cross-correlation matrix $C_{kj} = \frac{\langle \hat{u}_k(t)\hat{u}_j(t)\rangle_T}{\sqrt{\langle \hat{u}_k(t)^2\rangle_T \langle \hat{u}_j(t)^2\rangle_T}}$, where $\langle \cdot \rangle_T$ denotes temporal averaging, while $\hat{u}_j(t) = u_j(t) - \langle u_j(t)\rangle_T$ present the deviations of $u_j(t)$ from the corresponding means, see Fig. 2(e).

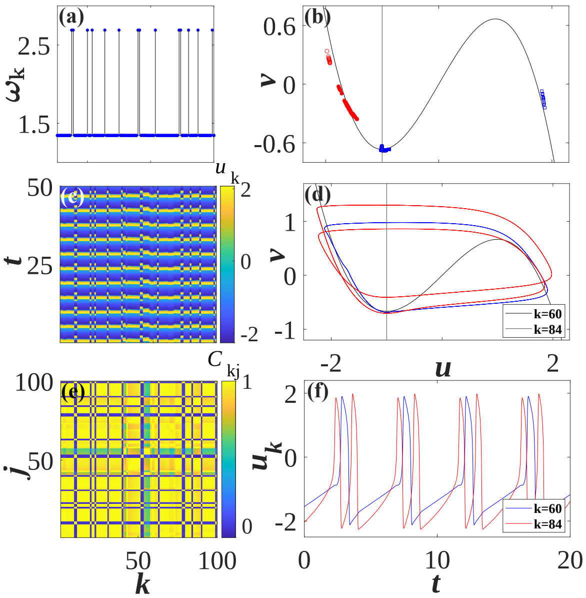

Fig. 2. Solitary state SS1. (a) Spatial profile of ω_k; (b) Red and blue: two snapshots of local variables (u_k, v_k), black: nullclines of an isolated unit; (c) Spatiotemporal dynamics of $u_k(t)$; (d) Phase portraits $(u_k(t), v_k(t))$ of solitary ($k = 84$) and typical unit ($k = 60$); (e) Cross-correlation matrix C_{kj}; (f) Time series $u_k(t)$ for units from (d). System parameters are: $\varphi = 1.85$, $r = 0.2$, $\kappa = 0.4$, $N = 100$.

III. MINORITY TYPE OF SOLITARY STATES

Here we present the minority type of solitary states that do not have counterparts among the unbalanced two-cluster states in networks with all-to-all coupling. A typical example is a state called SS3 illustrated in Fig. 3. In contrast to the prevalent solitary states, such as SS1 and SS2, self-organization of SS3 rests on leap-frog dynamics [14] of pairs of units, including both solitary–typical pairs of units and pairs of only solitary or only typical units, see Fig. 3(b),(d) and (f) which illustrate the alternating dynamics between the different pairs of units. States like SS3 are an immediate consequence of nonlocal interactions that give rise to self-localized excitations [18] at the interfaces between adjacent domains with different dynamics. The four unlocked units actually appear on the background of an alternating (spatially modulated) wave, which likely derives from selecting the parameter values in vicinity of a Turing-Hopf codimension-two bifurcation point. The spatial profile of ω_k still features two clusters, but their frequency ratio is different than in the SS1 state, cf. Fig. 2(a). The solitary units have distinct average frequencies from the bulk due to events where they show two successive spikes instead of a single spike and a subthreshold oscillation, an example of which is denoted by an arrow in Fig. 3(d). Apparently, SS3 is characterized by a correlation structure of local dynamics more complex than the SS1 state, see Fig. 3(e) and Fig. 2(e). Also, unlike the prevalent solitary states, the maximal Lyapunov exponent $\lambda_{max} \approx 2 \times 10^{-5}$ for the minority type of solitary states suggests weakly chaotic solutions.

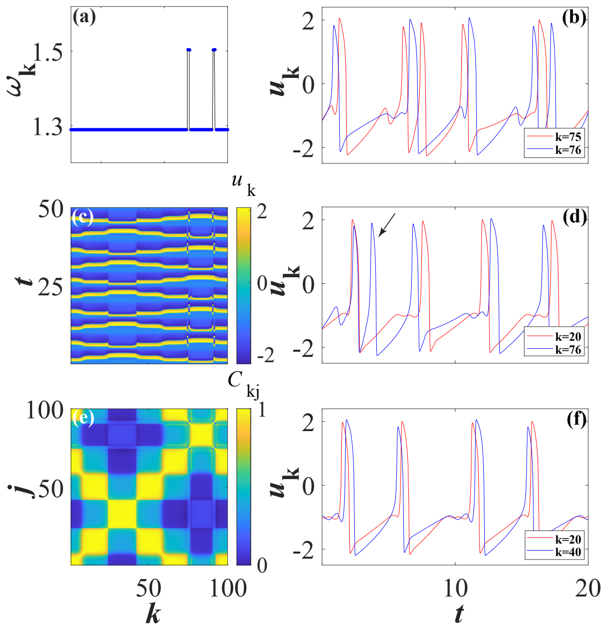

Fig. 3. Solitary state SS3. (a) Spatial profile of ω_k; (b) Time traces $u_k(t)$ for solitary units $k = 75$ and $k = 76$; (c) Spatiotemporal dynamics of $u_k(t)$; (d) Time series $u_k(t)$ for solitary unit ($k = 76$) and typical unit ($k = 20$); (e) Cross-correlation matrix C_{kj}; (f) Time traces $u_k(t)$ illustrate leap-frog dynamics within the majority cluster (units $k = 20$ and $k = 40$). System parameters are: $\varphi = 1.788$, $r = 0.2$, $\kappa = 0.4$, $N = 100$.

IV. PERSISTENCE UNDER NOISE

In general, the impact of noise on deterministic dynamics may be twofold. For one, it may be quantitative, in the sense that the noise may enhance/suppress some features of deterministic dynamics, while it may also be qualitative, such that the noise may induce new types of behavior, associated with crossing of thresholds or separatrices, or with a stabilization of deterministically unstable states. In neuronal media, the phenomena witnessing the constructive role of noise are abundant and have become known by the umbrella term of stochastic facilitation. Some of the classical examples are resonant phenomena, like coherence resonance [10], and the ability of triggering spontaneous switching between the coexisting metastable states. In *locally* coupled excitable systems, the noise can qualitatively impact pattern formation, including the onset of waves, spiral dynamics and pacemaking [10].

Here, we consider the persistence of solitary states under noise by including independent Gaussian white noise terms to act on both activator and recovery variables in (1). Since in our model the deterministic dynamics of an array involves extensive multistability, the noise affects its behavior in a way qualitatively different from the one in locally coupled excitable systems. Indeed, the noise here reduces the multistability of system dynamics, in particular by causing suppression of solitary states. This is a manifestation of noise-induced preference of attractors [19], an effect that may be explained

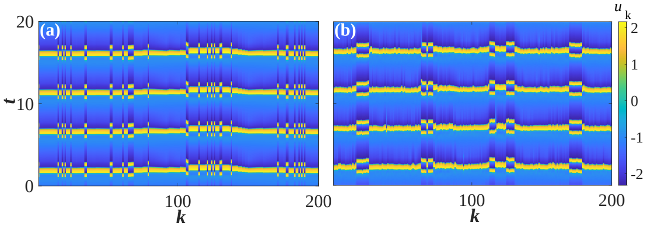

Fig. 4. Vanishing of solitary state of SS1 type for small noise. (a) Typical SS1 state in the absence of noise; (b) Patched pattern emerging from SS1 under small noise $\sigma = 0.0011$. Parameters are: $\varphi = 2.0$, $\kappa = 0.4$, $r = 0.2$, $N = 200$.

in the following way: in highly multistable systems, the noise smears out the stability boundaries of attractors, and only the attractors with sufficiently large attraction basins remain physically accessible. Such an effect has so far been observed in systems of coupled oscillators, coupled chaotic maps and multistable fiber lasers. From another point of view, this effect may also be seen as emerging from a strongly biased switching process [20], [21], where the system displays a regime shift to a coexisting state never to regain the initial state. Smaller noise tends to shift the system from a solitary state to a coexisting state that may be described as a patched pattern [23], featuring a piecewise constant profile of average spiking frequencies. There, an array self-organizes into majority (bulk) and minority patches, maintaining a mutual 1:2 subharmonic frequency locking, the same as in SS1 solitary state. Figure 4 shows how under a small noise, an initial SS1 state gets transformed to a patched pattern. Introducing intermediate noise promotes rotating waves rather than patched patterns, whereas large noise just results in turbulence.

V. CONCLUSION

We have reported on the emergence mechanisms of two types of solitary states in arrays of excitable FitzHugh-Nagumo elements with non-local coupling and prevailing repulsive over attractive interactions. The prevalent type of solitary states features periodic dynamics, and derives its main features, such as the frequency locking between the typical and solitary units and the associated local dynamics, from unbalanced cluster states in all-to-all coupled networks. This qualitative picture to a certain degree resembles the scenario in globally coupled Stuart-Landau oscillators, where clustering is found to be a necessary symmetry-breaking step en route to the onset of chimeras [22]. The minority type of solitary states is fundamentally a consequence of non-locality of interactions, and has no counterpart among unbalanced cluster states in networks with a global coupling. In contrast with the prevalent type, their dynamics is weakly chaotic and features a peculiar form of pattern self-organization involving leap-frog activity between the pairs of units. Leap-frogging emerges due to multiscale character of the system and the phase-sensitive excitability of relaxation oscillations, underlying the system's high sensitivity to perturbations close to the canard transition [14], [24]. Together with [14], [25], our results indicate the importance of the interplay between canard transition and repulsive interactions for pattern formation in systems of coupled excitable units.

Concerning the influence of noise, it has been demonstrated that it may be used to control the multistability of system dynamics via noise-induced preference of attractors, an effect where the attractors having larger basin of attraction are favoured over the ones with a smaller basin. In this way, small noise turns out to be capable of suppressing solitary states by promoting spatially homogeneous patched patterns with a similar split of average spiking frequencies.

Solitary states we discovered conform to the definition of weak chimeras, if indeed one extends it beyond coupled oscillator systems. Given that solitary states in systems of coupled oscillators are already known to mediate the desynchronization transition from full synchrony to chimeras [9], an important point for future study is to investigate whether a similar scenario holds for systems of coupled excitable elements.

ACKNOWLEDGMENT

I.F. acknowledges funding from the Institute of Physics Belgrade through grant by the Ministry of Education, Science and Technological Development of Republic of Serbia.

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