Fortschr. Phys. **51**, No. 2–3, 92 – 95 (2003) / **DOI** 10.1002/prop.200310008

An operator for simultaneous unsharp measurement of coordinate and momentum

Dragomir M. Davidović[1,*] and **Dušan Arsenović**[2]

[1] The Institute of Nuclear Sciences "Vinča", PO Box 522, 11001 Belgrade, Yugoslavia
[2] Institute of Physics, P.O. Box 57, Belgrade, Yugoslavia

Received 3 June 2002, accepted 19 June 2002
Published online 25 February 2003

PACS 03.65.Db

Starting from a complete but not overcomplete set of coherent states defined on a lattice in the phase space, we construct the orthonormal basis using the procedure due to Lödwin, in which all states to be normalized enter the procedure on equal footing. We show that our normalized states may be interpreted as eigenstates of the operator of simultaneous unsharp measurement of coordinate and momentum. In a sense we followed the classical idea of von Neumann but without drawbacks which in his construction of such an operator were due to the use of the Gram-Schmidt orthonormalization procedure which is somewhat inappropriate for this purpose. We discuss the obtained results.

1 Introduction and overview

It is a fundamental fact that all various macroscopic quantities that can be measured at all, can also be measured simultaneously. Von Neumann was the first who found the method by which two non-simultaneously measurable quantum mechanical quantities, such as for example coordinate and momentum, can be measured simultaneously with limited precision. He has also shown that all measurements with limited accuracy, or "unsharp" measurements in modern vocabulary, can be replaced by absolutely accurate measurements of other quantities, which are related to their non-simultaneously measurable originators, and which have discrete spectra. Von Neumann mathematically concretized this program for the case of coordinate and momentum. In this way he was able to answer the question: "What is the quantum mechanical probability to find a particle in a given region of phase space?" – or in other words to obtain information quantum mechanically about the position of a particle and its momentum at the same time. Such questions were until recently considered as senseless. This situation was somewhat unusual and strange because these results of von Neumann were published in early thirties of the last century in his book [1] which became classics; it seems however that they remain not broadly known. It is curious in this context that in the recent book of Omnes [2] about the interpretation of quantum mechanics, when analogous problems are treated these von Neumann results are not even mentioned.

Due to all this, we considered it worth-while to revisit the problem and to reaffirm this von Neumann idea in general, adding and improving some concrete particular points where it seemed desirable.

Von Neumann approach was this: it is well known that the system of coherent states is an overcomplete system and that in each coherent state the coordinate and momentum are defined at the same time with maximal accuracy possible in quantum mechanics. Now, von Neumann did the following. He partitioned the phase space in rectangular pieces of the size of the Planck constant h, and he chose one coherent

* Corresponding author E-mail: davidd@rt270.vin.bg.ac.yu

 0015-8208/03/2–302-0092 $ 17.50+.50/0

state from every piece. He orthogonalized this system using Gram-Schmidt procedure and mentioned that it was very easy to prove that this system is complete but not overcomplete. Then he defined these states to be eigenstates of quantum operators that reproduce the behavior of exact quantum operators of coordinate and momentum in an approximate fashion, i.e. the eigenstates of operators for simultaneous unsharp measurement of coordinate and momentum.

We found that the Gram-Schmidt procedure used in this context causes spurious asymmetry of various cells of phase space. We avoided this defect by using Lödwin's [3] orthonormalization procedure in which all vectors to be normalized are treated in a symmetric way. Apart from treating all cells in phase space from which we take vectors for orthonormalization on equal footing, we also achieved the following symmetry. Independently of the place in phase space, all normalized states are in exactly the same relation with their neighbors, so that all states obtained in orthogonalization have almost the same average values for coordinate and momentum which would have the coherent states from which they stem. In other words the average values of unsharp operators in their eigenstates are almost exactly equal to average values of exact coordinate and momentum operators in these states.

Von Neumann has not published his proof of completeness. His statement was reconfirmed only almost forty years later by Perelomov [4], with some subtleties added.

In the next section we present our results, and for completeness also reproduce the statement of Perelomov result.

2 Theory and results

One of the usual representations of coherent states is [5]

$$|\alpha\rangle = \mathrm{e}^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{1}$$

where α is complex number and $|n\rangle$ are eigenstates of the harmonic oscillator Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{q}^2}{2}$. Average values of operators of coordinate $\hat{q}$ and momentum $\hat{p}$ in these states are [5]

$$\langle\hat{q}\rangle_\alpha = \langle\alpha|\hat{q}|\alpha\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\alpha + \alpha^*) \tag{2}$$

$$\langle\hat{p}\rangle_\alpha = \langle\alpha|\hat{p}|\alpha\rangle = -\mathrm{i}\sqrt{\frac{m\hbar\omega}{2}}(\alpha - \alpha^*). \tag{3}$$

Defining $\Delta\hat{q} = \hat{q} - \langle\hat{q}\rangle$, $\Delta\hat{p} = \hat{p} - \langle\hat{p}\rangle$ one obtains

$$\langle(\Delta\hat{q})^2\rangle_\alpha = \frac{\hbar}{2m\omega}, \quad \langle(\Delta\hat{p})^2\rangle_\alpha = \frac{\hbar m\omega}{2} \tag{4}$$

so that

$$\langle(\Delta\hat{q})^2\rangle_\alpha \langle(\Delta\hat{p})^2\rangle_\alpha = \frac{\hbar^2}{4}. \tag{5}$$

In this way coherent states minimize the Heisenberg uncertainty relations. This is the main reason why they are the best candidates for construction of eigenstates of operators for unsharp measurement of coordinate and momentum. Coherent states form an overcomplete system. Von Neumann [1] gave strong physical arguments, which we will not repeat here, for which the complete orthonormal set constructed from coherent states, by partitioning the phase space, may be considered as a set of eigenstates of operators for the best inaccurate simultaneous measurement of coordinate and momentum of a quantum particle. Perelomov [4] rigorously proved that coherent states of the form

$$|\alpha_{kl}\rangle = |k\omega_1 + l\omega_2\rangle, \tag{6}$$

where k and l are integers and ω_1 and ω_2 are such complex numbers which in complex plane represent sides of a parallelogram with the surface S equal to π

$$S = \mathrm{Im}(\omega_2\omega_1^*) = \pi, \tag{7}$$

form a complete set when from all possible combinations (k, l) one and only one arbitrarily chosen, is excluded. If $S < \pi$, the system is overcomplete; if $S > \pi$ – noncomplete. Points $k\omega_1 + l\omega_2$ form a lattice in complex plane, and the surface of an elementary cell is $S = \mathrm{Im}(\omega_2\omega_1^*)$. Note that to the cell of surface $S = \pi$ in α plane corresponds the cell in phase plane of surface h.

In our considerations we found convenient to choose the square elementary cell so that we have

$$|\alpha_{kl}\rangle = |\sqrt{\pi}(k + \mathrm{i}l)\rangle. \tag{8}$$

We excluded the vector sufficiently far away, practically at infinity. Now we applied to this set the Löwdin [3] procedure of orthogonalization. From the normalized but nonorthogonal set $|\alpha_{kl}\rangle$ we obtained the orthonormal basis $|\tilde{\alpha}_{kl}\rangle$. For simplification of notations instead of pair of indices (i, j) we will sometimes use one capital letter. Let N be the matrix of scalar products

$$N_{IJ} = \langle\alpha_I|\alpha_J\rangle \tag{9}$$

then, by Löwdin's procedure we have

$$|\tilde{\alpha}_I\rangle = \sum_J |\alpha_J\rangle (N^{-1/2})_{JI}; \tag{10}$$

$$\langle\tilde{\alpha}_K|\tilde{\alpha}_L\rangle = \delta_{K,L}, \tag{11}$$

$$N = I + N', \quad N^{-1/2} = I - \frac{1}{2}N' + \frac{3}{8}N'^2 - \dots \tag{12}$$

and the quantity

$$\Delta^2 = \sum_I [\langle\tilde{\alpha}_I| - \langle\alpha_I|][|\tilde{\alpha}_I\rangle - |\alpha_I\rangle] \tag{13}$$

is minimized. In this way all $|\tilde{\alpha}_I\rangle$ differ from their originators $|\alpha_I\rangle$ in a symmetric and minimal way.

Now we can define the operators for unsharp measurement of coordinate and momentum in the form

$$\hat{\tilde{Q}} = \sum_K q_K |\tilde{\alpha}_K\rangle\langle\tilde{\alpha}_K|, \quad \hat{\tilde{P}} = \sum_K p_K |\tilde{\alpha}_K\rangle\langle\tilde{\alpha}_K|, \tag{14}$$

where q_K and p_K are coordinates in phase plane of corresponding knots of the lattice.

We have found numerically that with a very high accuracy

$$q_K \approx \langle\tilde{\alpha}_K|\hat{q}|\tilde{\alpha}_K\rangle, \quad p_K \approx \langle\tilde{\alpha}_K|\hat{p}|\tilde{\alpha}_K\rangle. \tag{15}$$

The accuracy is independent of K. This relation would be exact if we were not obliged to exclude one vector in order to avoid overcompleteness.

In Gram-Schmidt procedure the situation is somewhat different. The above relation is fulfilled for the initial vector in orthogonalization exactly, and for the rest of them approximately with accuracy which is dependent of the order of the vectors in the process of orthogonalization.

Let us consider now two subsets $\{|\alpha_{kl}\rangle\}$ and $\{|\alpha'_{kl}\rangle = |\alpha_{kl} + \Delta_{mn}\rangle\}$ where Δ_{mn} is fixed. These subsets represent regions of the same shape and size but in different parts of the α-plane. Let us assume that these parts are sufficiently far from each other. We are interested to see when the process of orthonormalization will give the same corresponding coefficients in both subsets. Obviously this will be so if the matrix of scalar products N is the same in both cases. As $|\alpha_{kl}\rangle = |\sqrt{\pi}(k + \mathrm{i}l)\rangle$, $|\alpha_{mn}\rangle = |\sqrt{\pi}(m + \mathrm{i}n)\rangle$ and

$$\langle\alpha|\beta\rangle = \exp\left(-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta^2| + \alpha^*\beta\right) \tag{16}$$

we have

$$\langle\alpha_{kl}|\alpha_{pq}\rangle = \exp\left(-\frac{\pi}{2}[k^2 + l^2 + p^2 + q^2] + \pi(k - \mathrm{i}l)(p + \mathrm{i}q)]\right) \tag{17}$$

$$= \exp\left(-\frac{\pi}{2}[(k-p)^2 + (l-q)^2]\right) \cdot (-1)^{kq-lp}. \tag{18}$$

Now

$$\langle\alpha'_{kl}|\alpha'_{pq}\rangle = \langle\alpha_{kl} + \Delta_{mn}|\alpha'_{pq}\rangle = \langle\alpha_{kl}|\alpha_{pq}\rangle \cdot (-1)^{(k-p)n+(q-l)m}. \tag{19}$$

These scalar products will be equal whenever the integers m and n are even. This means that different parts of phase space are equivalent and the phase space is homogeneous, because we can always choose m and n to be even, as the shift for only one cell is macroscopically negligible. Finally in our approach, based on the von Neumann idea, the regions of phase space and corresponding sets of projectors $|\tilde{\alpha}_I\rangle\langle\tilde{\alpha}_I|$ are in one to one correspondence. The problem to which part of phase space projects the given projector, which is present in some other approaches [2], in our approach can not arise.

References

[1] J. von Neumann, Mathematische Grundlagen der Quantenmechanik. Die Grundlagen der mathematischen Wissenschaften, Band XXXVIII (Springer, Berlin, 1932).

[2] R. Omnès, The Interpretation of Quantum Mechanics, Princeton Series in Physics (1994).

[3] P. O. Löwdin, J. Chem. Phys. **18**, 365 (1950).

[4] A. M. Perelomov, Teor. Mat. Fiz. **6**, 213 (1971).

[5] C. Cohen-Tannoudji, B. Diu, and F. Laloë, Mechanique Quantique, Vol. 1 (Hermann, Paris, 1973).