

Modified Coulomb gas construction of quantum Hall states from nonunitary conformal field theories

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Some fractional quantum Hall states observed in experiments may be described by first-quantized wavefunctions with special clustering properties such as the Moore-Read Pfaffian for filling factor $\nu=5/2$. This wavefunction has been constructed by constructing correlation functions of a two-dimensional conformal field theory (CFT) involving a free boson and a Majorana fermion. By considering other CFTs many other clustered states have been proposed as candidate fractional quantum Hall states under appropriate circumstances. It is believed that the underlying CFT should be unitary if one wants to describe an *incompressible*, i.e., gapped liquid state. We show that by changing the way one derives the wavefunction from its parent CFT it is possible to obtain an incompressible candidate state when starting from a nonunitary parent. The construction mimics a global change in parameters in the phase space of the electron system. We explicit our construction in the case of the so-called Gaffnian state (a state for filling factor $2/5$) and also for the Haldane-Rezayi state (a spin-singlet state at filling $1/2$). We note that there are obstructions along this path in the case of the permanent spin-singlet state of Read and Rezayi which can be characterized as a robust gapless state.

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I. INTRODUCTION

It is well known that two-dimensional electron gases in high magnetic field may form incompressible liquids with novel properties. This phenomenon called the fractional quantum Hall effect (FQHE) has been studied by various theoretical methods in the past twenty years. One successful approach is the use of explicit trial wavefunctions (WFs) explicitly written in the first-quantized language. The FQHE happens when the electrons occupy the low-lying Landau levels that are the quantized energy levels of a particle in a plane submitted to a perpendicular magnetic field. Its appearance also requires special commensurable ratios between the number of electrons and the number of states in the occupied Landau level. After the success of the Laughlin wavefunction,¹ the construction of so-called “composite fermion” (CF) wavefunctions has been very successful²⁻⁴ in describing many of quantum Hall incompressible liquids observed to date. However there are some states observed in experiments that do not fit easily in this scheme, the most prominent case being a state that forms at filling factor $\nu=5/2$. This state is beyond the reach of the previously mentioned theories because it has an even denominator which is forbidden in the CF constructions. There is an interesting proposal due to Moore and Read⁵ to capture the physics of this state by introducing the notion of pairing of composite fermions. The explicit wavefunction they propose, hereafter called the “Pfaffian,” has several desirable properties such as good overlap with the results of exact diagonalization of small systems and presumably has a gap to bulk density excitation. This state has also quasiparticle excitations with fractional charge $\pm e/4$ that have non-Abelian statistics, an unprecedented phenomenon in physics. There are some experimental evidences⁶ for these peculiar fractionally charged states. This special state has attracted attention in the context of quantum computing.⁷

It is known that many of the WFs proposed in the literature can be derived from two-dimensional massless quantum field theories possessing conformal invariance, i.e., conformal field theories (CFT). From a practical point of view the WFs can be written as correlation functions of some operators in a given CFT. The Laughlin wavefunction can be constructed from expectation values of exponentials of a free massless boson. The Pfaffian of Moore and Read can be constructed from a free massless boson and an additional massless fermion of Majorana type (the same type of fermion that appears in the critical theory of the classical two-dimensional Ising model). One can also, starting from a given CFT, deduce candidate WFs that have interesting properties inherited from its parent. For example, starting from parafermion CFT, Read and Rezayi^{8,9} have constructed WFs that have special vanishing properties: they are states that vanish when clusters of k (bosonic) particles are at the same point. This raises the following question: are there *a priori* restrictions on the CFTs from which one can derive WFs? Notably is unitarity of the CFT a necessary condition to derive candidate WFs for incompressible states? nonunitary CFT appear naturally in some physical systems, for example, usual percolation has a critical point which is described by a CFT with central charge $c=-2$, a nonunitary CFT. In the FQHE context, there is the so-called Haldane-Rezayi wavefunction¹⁰ with filling factor $\nu=1/2$ which is derived from a nonunitary CFT and it is known to be gapless.¹¹ So it may describe a critical point between different quantum Hall states but certainly not a bulk incompressible FQHE state. It has been argued by Read¹² that, generically, nonunitary CFTs lead to compressible gapless states. Recently many families of WFs with interesting algebraic properties have been constructed from nonunitary CFTs (Ref. 13) so it may very well be that they do not describe bulk gapped FQHE states.

When formulated in first-quantized language, most of the quantum Hall WFs have expressions that do not translate

easily in the Fock basis of second quantization. In general, a true FQHE WF has components on all Fock basis states allowed by symmetry. This is the case, for example, of the exact eigenstates of the Coulomb problem in the lowest Landau level (LLL) as obtained by exact diagonalization. However some of the model trial WFs have simpler expressions. It has been known for a long time¹⁴ that the celebrated Laughlin wavefunction has nonzero components only in a restricted set of the Fock basis. Indeed one can define a partial order relation onto the Fock states and there is a special set of occupation numbers, i.e., a special basis element that is “greater” than all the other terms appearing in the expansion of the Laughlin state. This special element is called the dominant partition in the language of polynomials of several variables (our convention in this paper) and is also called the root partition in the literature. This means in practice that these WFs are simpler than a generic state and also that some of their properties are encoded/can be read off the dominant partition. So the contemplation of the dominant partition may be a tool to uncover previously unknown relationship between quantum Hall state, to be proved by other methods for the whole WF.

In this paper we show that the analysis of the dominant partition suggests that some quantum Hall WFs, constructed from nonunitary CFTs, may be “cured” by addition of extra quasihole-quasiparticle excitations to produce presumably *bonafide* gapped Abelian quantum Hall states. This is in line with what we expect from a critical theory located right at the boundary of a gapped phase: some perturbation/modification of it has to do with the bulk gapped phase. Here we point out such a mechanism for two special quantum Hall states, the Gaffnian state and the Haldane-Rezayi state. Even this is suggestive, it remains to prove that it holds for the full quantum Hall state. We show that a modification in the Coulomb-gas formulation precisely allows to prove that our identification holds for the complete state, i.e., not only one (important) term of the Fock basis. This can be done by using special background charges as introduced some time ago in the CFT literature.

In Sec. II we show that by boundary insertions we can transform the Gaffnian (bosonic) state at filling factor $\nu = 2/3$ into the Jain state of bosons at $\nu = 2/3$ (this implies that fermionic cases at $\nu = 2/5$ are related in the same manner) and we transform the Haldane-Rezayi spin-singlet state by boundary insertions of charged excitations into a (331) multicomponent Halperin state. The states we obtain through these transformations are all Abelian incompressible states. In Sec. III we give a general prescription in the Coulomb-gas language to show that the correspondence we found holds not only for the dominant partition but for the full quantum Hall WFs. We also point out at least one case for which this scheme is probably more complex: the permanent state which cannot be transmuted by this mechanism into a gapped state. In Sec. IV we apply the boundary insertion construction to unitary Pfaffian state to find out whether it would transform into an Abelian state. It is shown that the Pfaffian remains stable under these transformations. Section V contains our conclusions. In Appendix A, we discuss briefly the neutral excitations on top of the Haldane-Rezayi state. In Appendix B, we explicitly derive the Moore-Read Pfaffian WF from the Coulomb-gas CFT.

II. FROM GAPLESS TO GAPFULL STATES VIA BOUNDARY INSERTIONS

A. Quantum Hall states

We consider WFs for electrons residing in the LLL. In the symmetric gauge, one-body orbitals are given by

$$\phi_m(z) = \frac{1}{\sqrt{2\pi m!} 2^m} e^{-|z|^2/4}, \quad (1)$$

where $z = x + iy$ is the complex coordinate in the plane where electrons are confined and the positive integer m gives the angular momentum of the state: $L_z = m\hbar$ (we have set the magnetic length to unity). A general N -body LLL quantum state is thus of the form

$$\Psi(z_1, \dots, z_N) = f(z_1, \dots, z_N) e^{-\sum_i |z_i|^2/4}. \quad (2)$$

In the remainder of the paper we will always omit the (universal) exponential factor. The physics of two-dimensional electrons in the LLL is governed by the following Hamiltonian:

$$\mathcal{H} = \sum_i \frac{1}{2m_b} \left(\mathbf{p}_i + \frac{e}{c} \mathbf{A}_i \right)^2 + \sum_{i < j} \frac{e^2}{\epsilon r_{ij}}, \quad (3)$$

where m_b is the band mass of the electron, ϵ is the dielectric constant of the host semiconductor and the distance between electrons i , and j is $r_{ij} = |r_i - r_j|$. In the LLL the kinetic energy is quenched and, in principle, one has to diagonalize the interaction potential in Eq. (3) in the Fock space constructed from products of one-body states in Eq. (1). Several different schemes have been developed to understand the physics of this problem since no general analytical solution is possible. It is feasible to diagonalize numerically the Hamiltonian above Eq. (3) if one considers a small number of electrons so that the Fock space is not enormously large. This method has the advantage of being unbiased, i.e., there is no *a priori* hypothesis on the form the many-body states but it is limited to a small number of electrons on the order of 12–15, depending on the filling factor of the LLL one wants to study. Exact diagonalization gives the low-lying levels as a function of the conserved quantum numbers allowed by the geometry of the system. For example, in the unbounded plane and using the symmetric gauge for the vector potential $\mathbf{A} = \frac{1}{2} \mathbf{B} \times \mathbf{r}$ the only conserved quantity is the angular momentum along the axis perpendicular to the plane (i.e., the \mathbf{B} axis). Another successful approach is to construct explicit trial wavefunctions. Originally this was pioneered by Laughlin who wrote down an explicit formula¹ for the wavefunction of N electrons when the filling factor of the LLL is precisely $1/3$. This Laughlin wavefunction is not an exact eigenstate of Hamiltonian Eq. (3), however it was shown very soon by Haldane that it encompasses all the physics of the exact ground state. This demonstration was done by comparison with data from exact diagonalization.¹⁵ This approach has been extended by Jain^{2–4} to many (if not all) fractions displaying the FQHE. The wavefunctions constructed in this approach are known under the name of “composite fermion” wavefunctions. Similarly they are not exact eigenstates of the full many-body problem but comparison

with exact diagonalization results show that they capture the essential physics. A detailed account is given in Jain's book.⁴ The composite fermion wavefunctions are built from Jastrow-type correlation factors in a way that generalizes the usual notion of Slater determinant. It is also feasible to use these wavefunctions as an alternate basis set and to diagonalize the Hamiltonian Eq. (3) in this basis. This has proved useful to describe, for example, the fate of electrons in quantum dots.^{16,17} The same exact diagonalization techniques have been employed also in the context of bosonic systems, motivated by the developments of experiments on ultracold gases. This allows, for example, for studies of the crystalline structures^{18,19} that form on small systems analogous to quantum dots in electronic systems.

Some trial wavefunctions have been also obtained by arguments based upon conformal field theories. In this approach one constructs wavefunctions by computing expectation values of a product of operators of a definite CFT. This is a way to reproduce the Laughlin wavefunction and it leads to many interesting proposals, the most physically relevant so far being the Pfaffian wavefunction of Moore and Read. In the CFT approach, it is not known from the beginning if the wavefunction is relevant to a given physical situation, one has to compare its predictions with exact diagonalization and/or experimental facts.

B. Quantum Hall polynomials

All the physics is contained in the analytic function f . This function can be expanded in powers of the z_i coordinates and a general term in the expansion is characterized by the set of occupation numbers of the one-body orbitals $\{n_m, m=0, 1, 2, \dots\}$. We will consider also bosonic quantum Hall states for which one can have $n_m > 1$. If we start from a fermionic state Ψ_F then antisymmetry and LLL means that necessarily one can factor out a Jastrow-type factor:

$$\Psi_F = \prod_{i < j} (z_i - z_j) \Psi_B, \quad (4)$$

with Ψ_B a bosonic, i.e., symmetric LLL wavefunction. So it is enough to consider bosonic wavefunctions. The filling factors of these two states are then related by $1/\nu_F = 1 + 1/\nu_B$. A given configuration of occupation numbers (n_0, n_1, n_2, \dots) fully characterizes each term of the expansion of f . The set of occupation numbers can be regarded as giving a partition of N since $N = \sum_m n_m$. Alternatively one can also specify the same configuration by giving all the m values that appear with nonzero occupation numbers $(m_1 \dots m_1 m_2 \dots m_2 \dots)$ where each m is repeated n_m times. This set of numbers then defines equivalently a partition of the total angular momentum $L_z = \sum_m m n_m$. In the physics literature it is common to specify the set of occupation numbers while the mathematical literature²⁰ on symmetric polynomials uses instead the partitioning of L_z . A partition λ defines also a unique symmetric monomial m_λ given by

$$m_\lambda = z_1^{k_1} \dots z_N^{k_N} + \text{permutations}. \quad (5)$$

This can be considered as a (unnormalized) wavefunction for N bosons in the LLL where the quantum numbers k_i of oc-

cupied orbitals can be associated in a one-to-one correspondence to a set of occupation numbers $\{n_m\}$. For example, the monomial for $N=3$ $m=z_1^2 z_2 z_3 + \text{perm}$ is defined by the partition (0210...) since there are two bosons in the $m=1$ orbital and one boson in the $m=2$ orbital. An arbitrary bosonic WF in the LLL can be expanded in terms of such monomials, each of them being indexed by a partition

$$f = \sum_{\lambda} c_{\lambda} m_{\lambda}, \quad (6)$$

where c_{λ} are some coefficients. For a given f it may happen that not all partitions appear in the expansion above. Indeed there is a partial ordering on partitions called the dominance ordering: let λ and μ two partitions then $\lambda \geq \mu$ if $\lambda_1 + \dots + \lambda_i \geq \mu_1 + \dots + \mu_i$ for all i . This is only a partial order: it may happen that the relation above does not allow comparison of two partitions. Some of the trial wavefunctions proposed in the FQHE literature have the property that there is a dominant partition with respect to (wrt) this special order and all partitions appearing in the expansion Eq. (6) are dominated by a leading one

$$\Psi = \sum_{\mu \leq \lambda} c_{\mu} m_{\mu}. \quad (7)$$

This was first noted by Haldane and Rezayi¹⁴ in the case of the Laughlin wavefunction. This property of dominance is also shared by many of the special orthogonal polynomials in several variables.²⁰ It was realized after the work of Feigin *et al.* in Ref. 21 that the Read-Rezayi (RR) trial wavefunctions^{8,9} are all particular cases of the so-called Jack polynomials. These symmetric polynomials noted J_{λ}^{α} are a family indexed by a partition λ and depend upon one parameter α . In fact we have

$$\Psi_{\text{RR}}^{(k)} = \mathcal{S} \prod_{i_1 < j_1} (z_{i_1} - z_{j_1})^2 \dots \prod_{i_k < j_k} (z_{i_k} - z_{j_k})^2 \propto J_{\lambda_k}^{-(k+1)}(\{z_i\}), \quad (8)$$

where the first equality defines the Read-Rezayi states, one divides the particles into k packets and \mathcal{S} means symmetrization the product of partial Jastrow factors. In the case of the RR states we have $\alpha = -(k+1)$ and $\lambda_k = (k0k0k0\dots)$. The usual bosonic Laughlin wavefunction is the special subcase when there is only one packet $k=1$ and the Moore-Read Pfaffian corresponds to the case $k=2$. In general the filling factor of the order- k RR state is $\nu = k/2$. Such WFs may describe some incompressible liquids of rapidly rotating bosons or, after due multiplication by a Jastrow factor, some elusive quantum Hall states in the second Landau level of electrons such as the $\nu=12/5$ state.

It is convenient also to study the FQHE in the spherical geometry which has no boundaries and possesses the full rotation symmetry. In this case the LLL is finite dimensional since the sphere has a finite area. Basis (unnormalized) functions of the LLL can be taken as: $\Phi_S^M = u^{S+M} v^{S-M}$, $M = -S, \dots, +S$ where $u = \cos(\theta/2)e^{i\phi/2}$, $v = \sin(\theta/2)e^{-i\phi/2}$ and the flux through the sphere is $2S$ in units of the flux quantum h/e . The stereographic projection from the sphere to a plane gives a one-to-one mapping of the wavefunctions in these

two geometries. When written on the sphere quantum Hall WF have a linear relation between flux and number of particles $2S = (1/\nu)N - \sigma$ when there is, in general, a nonvanishing offset σ called the “shift” in the FQHE literature wrt to the defining relation of the filling factor. On the sphere the finite number of orbitals leads to a finite set of occupation numbers hence the dominant partition is now given unambiguously by these numbers.

Finally when considering WFs for systems with more than one component (such as electrons with spin) it is convenient to define the Halperin wavefunctions²² with several Jastrow-type factors

$$\Psi_{mm'n} = \prod_{i,j \in A} (z_i - z_j)^m \prod_{k,l \in B} (z_k - z_l)^{m'} \prod_{a \in A, b \in B} (z_a - z_b)^n, \quad (9)$$

where there are two components and the respective indices belong to subsets A and B .

C. Gaffnian state

In the fermionic Laughlin wavefunction at filling factor $1/3$ any pair of particles have relative angular momentum at least three. If we consider the projector onto relative angular momentum one for each pair and sum these projectors then the Hamiltonian

$$\mathcal{H}_2^1 = \sum_{i < j} \mathcal{P}_2^1(ij) \quad (10)$$

has a densest ground state with zero energy which is exactly the Laughlin wavefunction. Similarly the bosonic Moore-Read Pfaffian is the densest zero-energy ground state of the Hamiltonian defined through $\mathcal{P}_3^2(ijk)$, excluding states where three particles have relative momentum two. One can ask now what is the densest zero-energy state when we consider excluding relative angular momentum three for three particles and the unique answer is the so-called bosonic Gaffnian WF introduced originally in Ref. 13 as a natural generalization of the Pfaffian state. Its coordinate first-quantized expression is

$$\Psi_G = \mathcal{S} \prod_{i < j \leq N/2} (z_i - z_j)^2 \prod_{N/2 < k < l} (z_k - z_l)^2 \prod_{m \leq N/2 < n} (z_m - z_n) \times \prod_{p \leq N/2} \frac{1}{z_p - z_{p+N/2}}, \quad (11)$$

where \mathcal{S} stands for symmetrization. It was recognized as the Jack polynomial $J_{\lambda_G}^{3/2}(\{z_i\})$ with dominant partition

$$\lambda_G = (2002002\dots). \quad (12)$$

While Jain wavefunctions are not in general Jack polynomials however they do satisfy restrictive rules on the partitions that appear when expanded in terms of monomials. Notably there is a dominant partition of Jain states and in the case of bosons at filling $\nu=2/3$ it is given by

$$\lambda_{2/3} = (2010110110102) \quad (13)$$

as found in Refs. 23 and 24. By counting the number of particles and fluxes, the relationship between the number of

particles and the number of flux quanta is the same as in the Gaffnian state: $N_\phi = \frac{3}{2}N_e - 3$. If we introduce one extra flux quantum in the Gaffnian state without changing the number of particles the new state may be described as an additional zero somewhere in the configuration of the Gaffnian (i.e., a Laughlin quasihole).²³ A state with a more uniform distribution of particles is obtained with a pair of half-flux non-Abelian quasiholes, where one quasihole is put on the north pole and the other on the south pole (in the sphere geometry). This is represented by the following partition:

$$\lambda_{1qh-1qh} = (11011011011011) \quad (14)$$

[compare with Eq. (12) the number of flux quanta and particles]. This is the bulk configuration of Jain state Eq. (13). Due to the same number of flux quanta and particles in Eqs. (12) and (13), i.e., Gaffnian and Jain state, this suggests that the Jain WF can be described as a Gaffnian WF with neutral quasiparticle-quasihole excitations on the boundaries of the system. This identification was first done in Ref. 23.

D. Case of Haldane-Rezayi state

The Haldane-Rezayi state was introduced in Ref. 10 as a FQHE state with some kind of pairing. It is a global spin-singlet that can be described as a collection of spin-singlet pairs with pairing function $g(z) \sim \frac{1}{z^2}$ at filling factor $\nu = \frac{1}{2}$ in the fermionic case.

$$\Psi_{\text{HR}} = \sum_{\sigma \in S_{N/2}} \text{sign } \sigma \frac{1}{(z_1^\uparrow - z_{\sigma(1)}^\downarrow)^2 \dots (z_{N/1}^\uparrow - z_{\sigma(N/2)}^\downarrow)^2} \times \prod_{i < j} (z_i - z_j)^q, \quad (15)$$

where $q=2$. Before realizing that Haldane-Rezayi (HR) is a critical (gapless) state¹¹ there were attempts^{25,26} to construct the edge theory for this system on the assumption that the HR system may represent a gapped phase even though it is related to a nonunitary CFT. One of these attempts²⁶ describes the edge of the HR system as the edge of a (331) Halperin two-component state, i.e., one of the well-known Halperin states²² that are certainly gapped.

We know show that, by inspection of the dominant partitions, there is evidence for a change in physics due to boundary insertions as we saw in the Gaffnian case. Explicitly the dominant partition of the Haldane-Rezayi state is

$$\lambda_{\text{HR}} = (\bar{2}000\bar{2}000\bar{2}0 \dots \bar{2}000\bar{2}), \quad (16)$$

where $\bar{2}$ means double occupancy of a *single* orbital with both spins $\uparrow\downarrow$. On the other hand the root configuration of the (331) Halperin state is

$$\lambda_{(331)} = (XX00XX00XX \dots XX00XX), \quad (17)$$

where XX stands for $\uparrow\downarrow + \downarrow\uparrow$, i.e., a symmetric superposition of the neighboring opposite spins.²⁷

It is important to note, by examining Eqs. (16) and (17), that for the same number of electrons there is one more orbital in the (331) case as can be expected by comparing flux and particle number relations in the Haldane-Rezayi case:

$N_\phi=2N_e-4$, and in the (331) case: $N_\phi=2N_e-3$. The extra flux quantum can be introduced in the Haldane-Rezayi state as an Abelian Laughlin quasihole and therefore as an extra zero²³ in Eq. (16) or as two non-Abelian quasiholes²⁸ symmetrically at the boundaries of the system as in Eq. (17). Therefore this suggests again that the (331) Halperin state can be derived by insertions of global non-Abelian excitations in a “parent” Haldane-Rezayi state. The case of neutral excitations is similar and is discussed in Appendix A.

E. Discussion

In this section we have shown that the special relationship via boundary excitations between Gaffnian and other $W_k(k+1, k+r)$ generalizations²⁹ at $\nu=\frac{k}{r}$ and Jain states of bosons at $\nu=\frac{k}{r}$ as demonstrated in Ref. 24 is not unique but extends to other non-Abelian gapless states such as HR, i.e., those states connected to nonunitary CFTs. These states are presumably at a phase boundary to a gapped FQHE state. Tweaking of interactions or imposing global change like with a magnetic field parallel to the two-dimensional electron-gas plane may lead the system from the critical point described by the non-Abelian gapless state into a stable Abelian gapped state and phase. In the next section we use the Coulomb-gas formulation to extend our argument beyond merely the dominant partition of the monomial expansion to the full WF.

III. BOUNDARY INSERTIONS IN THE LANGUAGE OF COULOMB-GAS CORRELATORS

A. CFT formalism and FQH states

A bulk quantum Hall fluid is an incompressible liquid which is spatially featureless. When sitting on a sphere it will spread out to form a uniform film that is invariant by the rotation group acting upon the sphere. The corresponding quantum state should thus be annihilated by all the generators of the rotation group

$$L^+\Psi = L^-\Psi = L_z\Psi = 0. \tag{18}$$

The spherical geometry is of course a purely theoretical construct. We can translate these conditions on the realistic planar geometry by using the stereographic projection. The rotation operators are then differential operators acting upon the particle coordinates

$$L^+ = E_0, \quad L^- = N_\phi \sum_{i=1}^N z_i - E_2,$$

$$L_z = \frac{1}{2} NN_\phi - E_1,$$

where

$$E_n = \sum_{i=1}^N z_i^n \frac{\partial}{\partial z_i}. \tag{19}$$

If we suppose that the WF is given by a correlation function of some operators of a quantum field theory then we have the following conditions:

$$\sum_{i=1}^N \partial_i \langle 0 | \phi_1(z_1) \cdots \phi_N(z_N) | 0 \rangle = 0,$$

$$\sum_{i=1}^N (z_i \partial_i + h_i) \langle 0 | \phi_1(z_1) \cdots \phi_N(z_N) | 0 \rangle = 0$$

and

$$\sum_{i=1}^N (z_i^2 \partial_i + 2z_i h_i) \langle 0 | \phi_1(z_1) \cdots \phi_N(z_N) | 0 \rangle = 0. \tag{20}$$

These are the conditions for invariance under the global conformal group in two dimensions. It is thus clear that any CFT which is by definition invariant under the larger local conformal symmetry group will satisfy these conditions. In a given CFT the fields ϕ_i are the (quasi)primary fields and h_i are the corresponding conformal weights. Some quantum Hall WFs can be derived from correlators of operators of two-dimensional massless quantum field theories, the example of the Moore-Read Pfaffian is given in Appendix B.

B. “Gaffnian” state

The Gaffnian WF is built from the minimal^{13,30} model $\mathcal{M}_2(3,5)$ The central charge is this nonunitary CFT is

$$c = \frac{r(k-1)}{k+r} [1 - k(r-2)] = -\frac{3}{5}. \tag{21}$$

One way to construct this CFT and its correlators is to start from a free boson theory and introduce a background charge³¹⁻³⁵ by adding an extra term to the energy-momentum tensor

$$T(z) = -\frac{1}{2} : \partial x(z) \partial x(z) : + i\sqrt{2}\alpha_0 \partial^2 x(z), \tag{22}$$

where the free boson is field $x(z)$. This additional contribution leads to a central charge

$$c = 1 - 24\alpha_0^2. \tag{23}$$

One should then think³¹ of the background charge $-2\alpha_0$ as being “at infinity.” In the FQHE formulated on the sphere this means simply that the charge is located at the pole of the sphere which is sent to infinity by stereographic projection. The only nonvanishing correlators in the case of two-point function are

$$\langle V_\beta(z) V_{2\alpha_0-\beta}(w) \rangle = \frac{1}{(z-w)^{2\beta(\beta-2\alpha_0)}}, \tag{24}$$

where the vertex operators are given by

$$V_\beta(z) = : \exp[i\sqrt{2}\beta x(z)] :. \tag{25}$$

These two operators V_β and $V_{2\alpha_0-\beta}$ are adjoint to each other and their conformal weight is $h = \beta(\beta - 2\alpha_0)$. In our case we want $1 - 24\alpha_0^2 = -\frac{3}{5}$ so that $\alpha_0 = \frac{1}{\sqrt{15}}$.

We know that the non-Abelian quasihole derived from the Gaffnian state is described by a product of a field σ of the

minimal model $\mathcal{M}_2(3,5)$ (the neutral part) and a bosonic vertex operator (the charge part). The field σ has a conformal weight equal to $h_\sigma = -\frac{1}{20}$. The corresponding values of β 's in the bosonic representation are thus

$$\beta(\beta - 2\alpha_0) = -\frac{1}{20} \rightarrow, \quad (26)$$

$$\beta_{1,2} = \alpha_0 \pm \sqrt{\alpha_0^2 - \frac{1}{20}} = \frac{1}{\sqrt{15}} \left(1 \pm \frac{1}{2}\right). \quad (27)$$

The vertex operator “at infinity” is

$$V_{\beta_0=-2\alpha_0} =: \exp\left[-i2\sqrt{2}\frac{1}{\sqrt{15}}x(z=\infty)\right]:. \quad (28)$$

It appears as an additional insertion in correlation functions

$$\langle V_{\beta_0}(z=\infty) \cdots \rangle. \quad (29)$$

We can recover the ordinary bosonic theory if we insert vertex operators with $\beta = -\frac{\beta_0}{2} = \alpha_0$ at two ends— $z=\infty$ and $z=0$, i.e., the two poles of the sphere in the following way:

$$\langle V_{\beta_0/2}(z=\infty) \cdots V_{-\beta_0/2}(z=0) \rangle \quad (30)$$

or

$$\langle V_{\beta_0}(z=\infty) \sigma \sigma^\dagger(z=\infty) \cdots \sigma \sigma^\dagger(z=0) \rangle, \quad (31)$$

where we have defined

$$\sigma =: \exp\left[i\sqrt{2}\frac{3}{2}\frac{1}{\sqrt{15}}x(z)\right]:$$

and

$$\sigma^\dagger =: \exp\left[-i\sqrt{2}\frac{1}{2}\frac{1}{\sqrt{15}}x(z)\right]:. \quad (32)$$

These two operators are related to the non-Abelian quasiparticle.³⁶ In Eq. (31) we introduce a quasihole excitation σ through the vertex operator Eq. (25) with exponent $\beta_1 = \frac{1}{\sqrt{15}}(1 + \frac{1}{2})$. The second vertex operator that we use for the quasihole has exponent $\beta_2 = \frac{1}{\sqrt{15}}(1 - \frac{1}{2}) > 0$ and the same conformal dimension. We construct the quasiparticle excitation σ^\dagger through vertex operator with exponent $-\beta_2$.

The most important implication of the boundary insertions in CFT correlators is that by additional neutralizing background charges we recover a standard bosonic description without background charges usually associated with Abelian FQHE states. Indeed we have

$$\begin{aligned} & \langle \exp[i\sqrt{2}\beta x(z)] \exp[-i\sqrt{2}\beta x(w)] \rangle_{\text{with neutralizing insertions}} \\ & \sim \frac{1}{(z-w)^{2\beta^2}} \end{aligned} \quad (33)$$

as in the usual Coulomb gas formulation. In the Gaffnian case, though we can not reproduce the full wavefunction of the Jain state, we note that the neutral part of the state we obtain can be considered as a spin-singlet state of “spinons,” i.e., excitations³⁷ created by vertex operators with $\beta = \pm \frac{1}{2}$.

Thus the usual correlator of the neutral Coulomb gas can reproduce a Halperin (221) state of bosons that is closely related to the Jain state at $\nu = \frac{2}{3}$ (they share the same low-energy description³⁸). We find that the dominant partition of this (221) state is

$$\lambda_{(221)} = (XX0XX0XX0XX0XX) \quad (34)$$

in the notation of Sec. II, to be compared with the bulk pattern of Jain state in Eq. (13).

Finally we mention that this construction with background charges can be generalized to other $\nu = \frac{k}{r}$ cases deduced from CFTs associated with $W_k(r+1, r+k)$ algebras using their multicomponent Coulomb gas representations.

C. Haldane-Rezayi state

In the case of Haldane-Rezayi state¹⁰ the CFT has central charge $c = -2$, it is a nonunitary “scalar fermion” theory.²⁵ We now use the Coulomb gas mapping established for this non-unitary ghost system in Ref. 39. For the background charge $q = -2\alpha_0$ we should have

$$1 - 24\alpha_0^2 = -2, \quad (35)$$

hence we have

$$\alpha_0 = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}}. \quad (36)$$

The σ —field needed for the neutral part description of the non-Abelian quasihole, has conformal weight

$$h_\sigma = -\frac{1}{8}. \quad (37)$$

Therefore we have

$$\beta_{1,2} = \alpha_0 \pm \sqrt{\alpha_0^2 - \frac{1}{8}} = \alpha_0 = \frac{1}{2\sqrt{2}}. \quad (38)$$

The background charge is given by the insertion of the following vertex operator “at infinity:”

$$V_{\beta_0=-2\alpha_0}(z) = \exp[-ix(z)]. \quad (39)$$

We implement the σ -field as

$$\sigma = \exp\left(i\sqrt{2}\frac{1}{2\sqrt{2}}x(z)\right) = \exp\left(i\frac{1}{2}x(z)\right). \quad (40)$$

Therefore to recover a usual bosonic theory we can insert σ operators at two ends, $z=\infty$ and $z=0$, in the following manner:

$$\langle V_{\beta_0}(z=\infty) \sigma(\infty) \cdots \sigma(0) \rangle. \quad (41)$$

This parallels the boundary-insertion relationship we found in the previous section that led to the (221) state that is naturally described in the Coulomb gas formalism. If we use the neutral fermion field instead of the σ -field operator we have $\beta_{1,2} = \alpha_0 \pm \sqrt{\alpha_0^2 + 1} = \frac{1}{\sqrt{8}}(1 \pm 3)$ and again by “trivial insertions” of a single field on both ends (i.e., trivial because $\beta_1 + \beta_2 = 2\alpha_0$ is always satisfied) we obtain again an insertion

ansatz in the CFT formalism that leads to an Abelian state described by ordinary Coulomb-gas formalism. This state should be closely related to the hierarchy/Jain’s spin-singlet state at filling factor 1/2 although we have not yet been able to find the precise relationship.

Related to this is a comment we want to make that according to (a) what we found about the root configuration of HR state, i.e., how complex its definition is, and (b) that the neutral part of the HR state can be decomposed into a product of Cauchy determinant and permanent, the CFT associated with the HR state may be more general than a single scalar fermion theory. This would imply more than one Coulomb gas necessary to describe the neutral sector of the state and its excitations, which is quite expected given that the Coulomb-gas description of the neutral part of the (boundary-insertions-related) hierarchy and Jain spin-singlet state at $\frac{1}{2}$ requires two Coulomb gases. (The K matrix of these states is a 3×3 matrix.⁴⁰) Nevertheless a single scalar fermion theory is, as we already seen, able to capture the basic mechanism of neutralization that is at work in this case.

D. Permanent state

The physics of the so-called permanent state was first described in Ref. 8. This spin-singlet state is defined in the case of electrons at filling factor one. The state contains one power of the Laughlin-Jastrow factor (which is the Vandermonde determinant) and has also a BCS-like pairing part with a pairing function is $\frac{1}{z}$. It can be written as

$$\Psi_{per} = \sum_{\sigma \in S_{N/2}} \frac{1}{(z_1^\uparrow - z_{\sigma(1)}^\downarrow) \dots (z_{N/1}^\uparrow - z_{\sigma(N/2)}^\downarrow)} \prod_{i < j} (z_i - z_j)^q, \tag{42}$$

where $q=1$ This is the densest zero-energy state of the projector that penalizes the closest possible approach of three spin-1/2 particles for total spin 1/2. We find by direct expansion of Eq. (42) and examination of the terms that the dominant partition of the permanent state is

$$\lambda_{per} = (\bar{2}\bar{0}\bar{2}\bar{0}\bar{2}\bar{0}\bar{2}\bar{0}\dots) \tag{43}$$

in the notation of Sec. II. The CFT that corresponds to this permanent state is the so-called β, γ (nonunitary) commuting spinor ghost system. It is explained in Ref. 39 that the ghost system allows a representation by *two* Coulomb gases. Only one of them needs a background charge and represents a pair of scalar fermions as in the CFT formalism for the HR state. The boundary-condition changing field σ (or the spin field) can be represented by a vertex operator of a Bose field that does not need a background charge. Therefore the insertions of this field σ at the ends of a general correlator do *not* lead to a complete neutralization of the background. Thus, since the σ field in the case of permanent CFT generates a non-Abelian excitation, its insertions on the boundaries of the permanent system cannot lead to an Abelian gapped state contrary of the HR state. Indeed this can be guessed already from the partition analysis: the insertions will transform Eq. (43) into a dominant partition of the Halperin (111) state

$$(1111111111), \tag{44}$$

i.e., a dominant partition of a state that is known⁴¹ to be gapless.

IV. BOUNDARY INSERTIONS AND THE UNITARY PFAFFIAN CASE

A. Introduction

The examples we have given for quantum Hall states connected to nonunitary theories are known in the literature as critical states—see Ref. 13 for the case of Gaffnian and Ref. 11 for the Haldane-Rezayi case. They are recognized to be at the phase boundary to the Abelian states that we described here via boundary insertions. Therefore our construction has the following physical interpretation—it tells whether and in what manner a quantum Hall state connected to a nonunitary CFT and therefore gapless can be transmuted, via some global change in parameters described by boundary insertions, into unitary theory with Abelian braiding properties of excitations. Then the natural question to ask is what happens if we apply boundary insertions to unitary non-Abelian states; whether they will be transmuted, if the neutralization of the background charges is complete while using CFT constructions, into Abelian unitary states. If they are “immune” that would give an insight into a stability of a particular state and a stability of its non-Abelian property. In the following we will discuss the effect of the boundary insertions on the Moore-Read Pfaffian state.

B. Case of Pfaffian

The Pfaffian can be built from the $\mathcal{M}_2(3,4)$ minimal model or Ising CFT. The central charge is $c=1/2$ and this is the simplest unitary theory which as a minimal model can be represented in the Coulomb-gas formalism. Then it is not hard to repeat the algebra as in the Gaffnian case in Sec. III B to find $\alpha_0 = \frac{1}{2\sqrt{12}}$ and corresponding β ’s for the non-Abelian quasihole field σ are $\beta_1 = 3\alpha_0$ and $\beta_2 = -\alpha_0$. Therefore in this case it is impossible to introduce a quasiparticle vertex operator reversing the sign of β_1 or β_2 and achieve the neutralization of the background charge by two quasiparticle-quasihole pairs like in the Gaffnian case.

Therefore we established that the Pfaffian state is stable wrt global insertions of quasiparticle-quasihole pairs; insertions will not lead to an Abelian state. Nevertheless we should also examine trivial insertions [see below Eq. (41)], i.e., those that are made by placing a single field on both ends of the system. By doing this we may be still just in an excited sector of the non-Abelian theory but as we saw in the Haldane-Rezayi case we may as well enter or make a space for an Abelian theory [(331) in the Haldane-Rezayi case]. The CFT construction can not give us an answer for that and we have to resort to examining root configuration that correspond to this kind of trivial insertions, to see if the outcome may be an Abelian state. The basic root configuration of the fermionic Pfaffian at $\nu=1/2$ is

$$\lambda_{pf} = (11001100110011). \quad (45)$$

Insertions of a neutral fermion on both ends would lead to the following $L=0$ state:

$$\lambda_{nf} = (10110011001101). \quad (46)$$

The bulk configuration did not change and we do not have a reason to believe that this structure can be connected to an Abelian state. Even if we start with two-component picture of the structure that is ensuing after the neutralization [we end up with two Coulomb gases—compare the discussion in Sec. III B and the relationship between Gaffnian, (221), and Jain's state] this will not take us out of Pfaffian. Namely, the root in Eq. (46) can be related to the root configuration of two component (331) state but its (anti)symmetrization leads back to Pfaffian.

Next we can consider putting non-Abelian quasiholes on two ends of the system. The corresponding root configuration in the fermionic Pfaffian case is

$$\lambda_{qh} = (101010101010101). \quad (47)$$

If we again invoke the two-component interpretation that the CFT allows, the bulk configuration of the root in Eq. (47) can be related to the bulk configuration of the root of the Jain state, $\chi_{1,1}\chi_2\chi_1$ in the usual Jain notation, at $\nu=1/2$ as described by Eqs. (A5) and (A6). The state can be rewritten as

$$\frac{\chi_{1,1}\chi_1\chi_2\chi_1}{\chi_1}. \quad (48)$$

$\chi_{1,1}\chi_1$ is nothing but a (221) state which under appropriate inclusion of derivatives and a symmetrization procedure can be transformed into the Jain bosonic state, $\chi_2\chi_1$. Therefore this case, with non-Abelian quasihole insertions, is nontrivial in the sense that it might lead to a non-Abelian composite Jain state:⁴²

$$\frac{(\chi_2\chi_1)^2}{\chi_1} \quad (49)$$

but again non-Abelian which shows how the Pfaffian physics at $\nu=1/2$ is immune to abelianization but can be transmuted by changing parameters of the system into another non-Abelian state.

It is interesting to note that the (bosonic) Read-Rezayi states at $\nu=\frac{k}{r}$; $r=2, k=3,4$ allow the abelianization by quasiparticle-quasihole pairs as we described in the Gaffnian case (Sec. III B). This is not surprising given that these constructions can be considered at the same time with some hierarchy (Abelian) constructions as viable candidates for corresponding filling factors.

V. CONCLUSION

We have shown how to construct an Abelian gapped FQHE state starting from a FQHE state deriving from a non-unitary CFT. This construction is done in the Coulomb-gas language by the introduction of some background charges. Since we expect that states constructed from a nonunitary CFT are gapless it means that we have a way to construct a

gapped Abelian state whose boundary in some parameter space presumably contains the gapless state. It is interesting to note that the Abelian/non-Abelian character of the states is not preserved: in the two examples discussed in this paper the gapped state is Abelian while it is the critical theory which is non-Abelian. Of course the non-Abelian character of a gapless theory is a bit formal since it is not possible to perform an adiabatic exchange of excitations to obtain their braiding properties: there is no adiabatic limit since there is no gap to protect the excitations.

While we have treated in some detail the case of the Gaffnian and the HR states the Coulomb-gas construction shows that it is more general. However it cannot be completely general. Indeed we have an example, the permanent state, for which this construction is impossible with non-Abelian quasihole insertions. It would be interesting to have a clearer understanding of this special case.

Finally we applied the boundary insertion ansatz to the unitary Pfaffian case. The Pfaffian character and non-Abelian behavior remain preserved under boundary insertions pointing out to the stability of this state in the context of the FQHE of polarized electrons at $\nu=1/2$.

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APPENDIX A: NEUTRAL BOUNDARY INSERTIONS AND HALDANE-REZAYI STATE

We now ask whether boundary insertions can be done in the HR state while keeping flux and particle relation fixed, i.e., in a neutral way, to transform the HR state into a gapped state. The basic neutral excitations of the HR system are neutral fermions and they carry only a spin degree of freedom. After an inspection of which partitions with boundary insertions are still uniform ($L=0$) states we conclude that the following dominant partition:

$$\lambda_N = (\bar{2}00\uparrow 0\downarrow 0\uparrow 0\downarrow 0\uparrow 0\downarrow 00\bar{2}) \quad (A1)$$

together with the configuration that we find by exchanging ups and downs

$$\lambda_{N'} = (\bar{2}00\downarrow 0\uparrow 0\downarrow 0\uparrow 0\downarrow 0\uparrow 00\bar{2}), \quad (A2)$$

describes the neutral fermion insertions.⁴³

On the other hand from (a) the study of paired fermion states¹¹ and (b) the work on spin-singlet hierarchy⁴⁰ and possible spin-singlet candidates⁴⁴ at fillings $\frac{1}{q}$, q even, we know that there is an Abelian incompressible phase closely connected with the HR state. In the hierarchy picture this is a spin-singlet state that can be constructed by condensing spinless quasielectrons on the top of the Halperin (332) state at $\nu=\frac{2}{5}$. We will use the expression of the state in the Jain picture.⁴⁴

$$\Psi = \chi_{1,1}\chi_2\chi_1, \quad (\text{A3})$$

where we used the usual Jain notation for χ_1 , the Jastrow-Laughlin factor for the filled LLL (Vandermonde determinant), χ_2 as the wave function for two filled LLs of all particles and $\chi_{1,1}$ as the filled LLL of both spins, i.e., (110) state in the Halperin notation. The spinless part of the wave function ($\chi_2\chi_1$) is the Jain state at $\nu=\frac{2}{3}$ for bosons which the dominant partition is in Eq. (13), i.e.,

$$\lambda_{2/3} = (2010110110102). \quad (\text{A4})$$

Inserting the fluxes that carry spin by $\chi_{1,1}$, after a little inspection we find

$$\lambda_{1/2} = (\bar{2}00\uparrow 0\downarrow 0\uparrow 0\downarrow 0\uparrow 0\downarrow 00\bar{2}) \quad (\text{A5})$$

and

$$\lambda'_{1/2} = (\bar{2}00\downarrow 0\uparrow 0\downarrow 0\uparrow 0\downarrow 0\uparrow 00\bar{2}) \quad (\text{A6})$$

as the basic configurations that describe the Jain state at $\frac{1}{2}$. By comparing what we found out about neutral fermion constructions in the HR state [Eqs. (A1) and (A2)] we conclude that this Jain state at $\frac{1}{2}$ can be realized by implementing boundary insertions of neutral fermions in the HR state, at least when considering dominant partitions.

APPENDIX B: MOORE-READ STATE AS A CFT CORRELATOR

The Moore-Read state is given by

$$\Psi_{MR} = \prod_{i<j} (z_i - z_j)^m Pf\left(\frac{1}{z_i - z_j}\right), \quad (\text{B1})$$

where

$$Pf\left(\frac{1}{(z_i - z_j)}\right) = \sum_{\sigma \in S_N} \text{sgn } \sigma \frac{1}{(z_{\sigma(1)} - z_{\sigma(2)})} \dots \frac{1}{(z_{\sigma(N-1)} - z_{\sigma(N)})} \quad (\text{B2})$$

and we have a pairing part (Pfaffian) or neutral part that corresponds to a correlator of N Majorana fermion fields. The Laughlin part or charge part is a correlator of special bosonic vertex operators with a background charge.⁵ Explicitly for the Pfaffian part

$$\Psi_{Pf} = Pf\left[\frac{1}{(z_i - z_j)}\right], \quad (\text{B3})$$

we have $h_i = h = \frac{1}{2}$, $\forall i$ that is (in the previous notation) $(E_2 + Z)\Psi_{Pf} = 0$ and $(E_1 + N\frac{1}{2})\Psi_{Pf} = 0$ so that $M = -N\frac{1}{2}$, i.e., $N_\phi = -1$, and for the Laughlin part

$$\Psi_L^m = \prod_{i<j} (z_i - z_j)^m \quad (\text{B4})$$

the correlator is given by

$$\langle \exp[-iN\sqrt{m}\Phi(\infty)] \exp[i\sqrt{m}\Phi(z_1)] \dots \exp[i\sqrt{m}\Phi(z_N)] \rangle \quad (\text{B5})$$

for a boson field Φ and with the background charge at $z = \infty$, which (as we will explain more later) shifts the value of the conformal weight of $\exp[i\sqrt{m}\Phi(z)]$ from $\frac{m}{2}$ to $\frac{m}{2} - \frac{m}{2}N$ so that: $[E_2 + Z(m - mN)]\Psi_L^m = 0$ and $[E_1 + \frac{N}{2}(m - mN)]\Psi_L^m = 0$, i.e., $M = \frac{mN(N-1)}{2}$ and $N_\phi = m(N-1)$. Together, Ψ_{Pf} and Ψ_L^m lead to the Moore-Read WF with $N_\phi = m(N-1) - 1$ as expected.

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