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Parametric and geometric resonances of collective oscillation modes in Bose–Einstein condensates

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Abstract
We analytically and numerically study nonlinear dynamics in Bose–Einstein condensates (BECs) induced either by a harmonic modulation of the interaction or by the geometry of the trapping potential. To analytically describe BEC dynamics, we use a perturbative expansion based on the Poincaré–Lindstedt analysis of a Gaussian variational ansatz, whereas in the numerical approach we use numerical solutions of both a variational system of equations and the full time-dependent Gross–Pitaevskii equation. The harmonic modulation of the atomic s-wave scattering length of a BEC of ^{7}Li was achieved recently via Feshbach resonance, and such a modulation leads to a number of nonlinear effects, which we describe within our approach: mode coupling, higher harmonics generation and significant shifts in the frequencies of collective modes. In addition to the strength of atomic interactions, the geometry of the trapping potential is another key factor for the dynamics of the condensate, as well as for its collective modes. The asymmetry of the confining potential leads to important nonlinear effects, including resonances in the frequencies of collective modes of the condensate. We study in detail such geometric resonances and derive explicit analytic results for frequency shifts for the case of an axially symmetric condensate with two- and three-body interactions. Analytically obtained results are verified by extensive numerical simulations.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Collective oscillation modes of various physical systems provide important insights into their behavior and represent a valuable source of information about their properties. Collective modes are usually easily accessible experimentally, and a comparison of the measured values of their frequencies with corresponding analytical results obtained from a linear stability analysis provides an essential tool for quantitative assessment of the theoretical description of a given physical system. In Bose–Einstein condensate (BEC) systems [1, 2], collective oscillation modes were among the first properties to be measured [3, 4] and compared with theoretical results [5–9]. However, the prominent nonlinear features of BECs open up a rich variety of phenomena, such as solitons [10] or Faraday waves [11–15] that arise in different experimental setups. This is most dramatically seen through a resonant behavior, but a number of other phenomena are also observed. For instance, mode coupling due to nonlinearities is always present, and even when we excite only one given collective oscillation mode, in the realistic experimental setup other modes will also be excited eventually. Furthermore, resonances and pronounced nonlinear effects can also be caused purely by the geometry of the trapping potential, thus making the study of nonlinear effects quite important even for the design of BEC experiments. Therefore, a detailed theoretical and numerical description of nonlinear phenomena is of significant interest and analytic results, which we derive from the Poincaré–Lindstedt analysis, can contribute to a better understanding of them.

In this paper, following the approach introduced in [16], we study nonlinear effects in a BEC due to the harmonic modulation of the s-wave scattering length [17], as well as due to the geometry of the trap, where geometric resonances emerge. In section 2, we briefly introduce the mean-field description and the Gaussian variational approach for BEC with two- and three-body contact interactions. We present in section 3 our results on frequency shifts due to the modulation of two-body interactions and due to the geometry of the trap, whereas in section 4, we summarize our main results and give a brief outlook for future research on this topic.

2. Variational approach

If we take into account two- and three-body contact interactions, the dynamics of a BEC at zero temperature is described by the generalized Gross–Pitaevskii equation [18]

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t)=\left\{-\frac{\hbar^2}{2m}\Delta+V(\mathbf{r})+g_2\,|\Psi(\mathbf{r},t)|^2+g_3\,|\Psi(\mathbf{r},t)|^4\right\}\Psi(\mathbf{r},t), \tag{1}$$

where $\Psi(\mathbf{r},t)$ is a condensate wave function, the trapping potential is considered to be axially symmetric $V(\mathbf{r})=\frac{1}{2}m\omega_\rho^2\left(\rho^2+\lambda^2z^2\right)$ with anisotropy λ, while g_2 and g_3 are two- and three-body interaction strengths, respectively.

In order to obtain analytic results on the low-lying collective modes of a BEC, we use the Gaussian variational ansatz for the ground state [7, 8]

$$\psi^{\mathrm{G}}(\rho,z,t)=\mathcal{N}(t)\exp\left[-\frac{1}{2}\frac{\rho^2}{u_\rho(t)^2}+i\rho^2\phi_\rho(t)\right]\times\exp\left[-\frac{1}{2}\frac{z^2}{u_z(t)^2}+iz^2\phi_z(t)\right], \tag{2}$$

with the normalization factor $\mathcal{N}(t)=(\pi^{\frac{3}{2}}u_\rho^2u_z)^{-1/2}$. Here $u_\rho(t)$, $u_z(t)$, $\phi_z(t)$ and $\phi_\rho(t)$ are variational parameters with a straightforward interpretation: $u_\rho(t)$ and $u_z(t)$ correspond to the radial and the axial condensate width, while $\phi_\rho(t)$ and $\phi_z(t)$ represent the corresponding phases. After minimization of the Lagrangian corresponding to the Gross–Pitaevskii equation, we arrive at the following nonlinear system of ordinary differential equations for condensate widths, which is our variational description of a BEC:

$$\ddot{u}_\rho(t)+u_\rho(t)-\frac{1}{u_\rho(t)^3}-\frac{p}{u_\rho(t)^3u_z(t)}-\frac{k}{u_\rho(t)^5u_z(t)^2}=0, \tag{3}$$

$$\ddot{u}_z(t)+\lambda^2u_z(t)-\frac{1}{u_z(t)^3}-\frac{p}{u_\rho(t)^2u_z(t)^2}-\frac{k}{u_\rho(t)^4u_z(t)^3}=0, \tag{4}$$

where

$$p=\frac{g_2N}{(2\pi)^{3/2}\hbar\omega_\rho\ell^3}\quad\text{and}\quad k=\frac{2g_3N^2}{9\sqrt{3}\pi^3\omega_\rho\hbar\ell^6}$$

are dimensionless two- and three-body interaction strengths and $\ell=\sqrt{\hbar/m\omega_\rho}$ denotes the harmonic oscillator length. Through extensive numeric simulations in [16], it was shown that the above Gaussian variational ansatz can be successfully used for describing the real-time dynamics of BEC systems with parameters similar to the experimental ones from [17].

3. Results

Nonlinear terms in the underlying Gross–Pitaevskii equation (1) and consequently in the variational system of equations (3) and (4) lead to a number of interesting effects in the properties of collective modes of a BEC. First we consider the important case of a harmonic modulation of the two-body interaction [17],

$$p(t)=p_0+q\cos\Omega t, \tag{5}$$

and neglect three-body effects. As we can see from figure 1(a) for a spherically symmetric trap, i.e. $\lambda=1$, when the two-body interaction strength is harmonically modulated with the external driving frequency Ω, collective modes exhibit a resonant behavior. The resonant frequencies correspond to collective modes calculated from the linear stability analysis and their higher harmonics. Close to resonances, frequencies of collective modes exhibit shifts from the corresponding linear stability results. By performing a Poincaré–Lindstedt perturbative expansion [19, 20] in the small modulation amplitude q, we can calculate this shift, which stems from secular terms in solving the hierarchy of equations obtained in the perturbation theory. It turns out that the first correction to the linear stability frequencies is quadratic in the modulation amplitude q and reads for the quadrupole mode

$$\omega_{\mathrm{Q}}=\omega_{\mathrm{Q0}}+q^2\frac{C_{\mathrm{Q}}}{2\omega_{\mathrm{Q0}}A_{\mathrm{Q}}}+\cdots. \tag{6}$$

Similar results can also be obtained for the breathing mode frequency. The coefficients A_{Q} and C_{Q} are calculated using the Mathematica code[5]. The structure of the coefficient A_{Q} shows that the quadrupole mode frequency contains poles at ω_{Q0}, $2\omega_{\mathrm{Q0}}$, $\omega_{\mathrm{B0}}-\omega_{\mathrm{Q0}}$, ω_{B0} and $\omega_{\mathrm{Q0}}+\omega_{\mathrm{B0}}$ to second order of perturbation theory. Higher-order calculations would lead to additional poles, which are, indeed, observed numerically [16]. Figure 1(b) compares the analytic result for the frequency shift (6) for an axially symmetric BEC at $\lambda=0.3$ with the numerical results obtained by solving the nonlinear variational equations (3)–(4) and performing their Fourier analysis. As we can see, even the first analytically calculated correction to the frequencies of collective oscillation modes is in excellent agreement with the full numerical results.

Next, we consider the interplay between the geometry of the trap, which is represented by the anisotropy λ, and

[5] SCL BEC MATHEMATICA codes, http://www.scl.rs/speedup

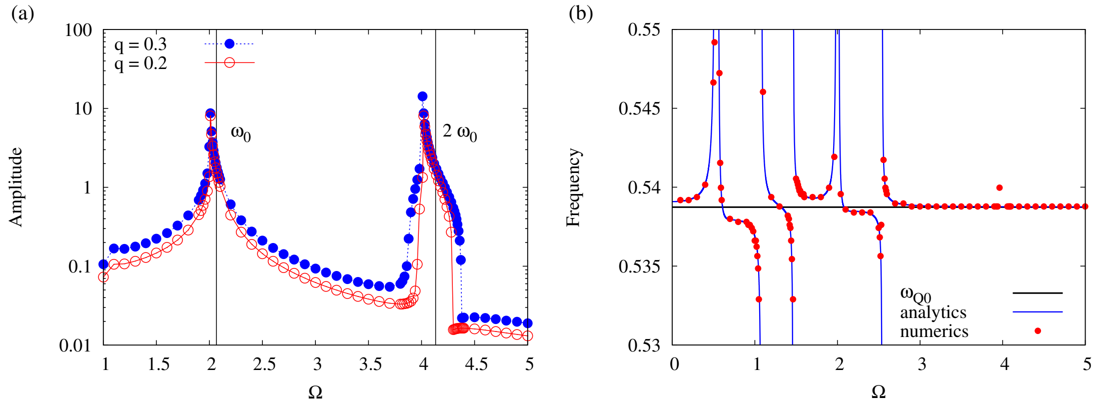

Figure 1. (a) Oscillation amplitude $(u_{\max} - u_{\min})/2$ versus driving frequency Ω for $p_0 = 0.4$, $k = 0$ for a spherically symmetric BEC. The shape and value of a resonance occur at a driving frequency Ω, which differs from the linear stability frequency ω_0, and depends on the modulation amplitude q. (b) Frequency of the quadrupole mode ω_{Q} versus driving frequency Ω for $p_0 = 1$, $q = 0.2$ and $k = 0$ for an axially symmetric BEC with anisotropy $\lambda = 0.3$.

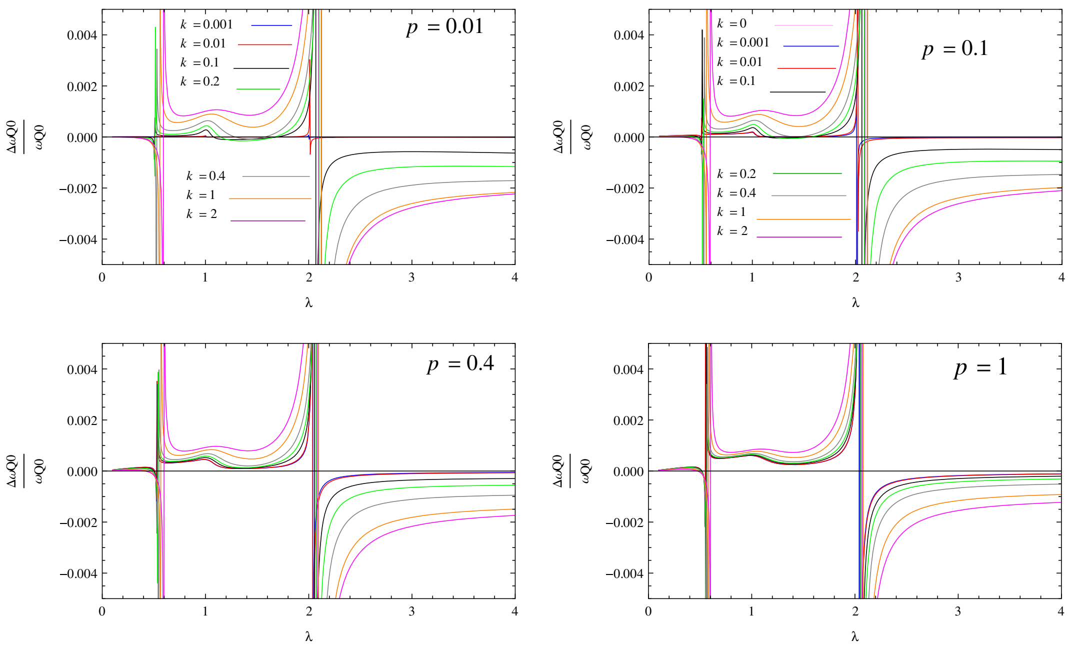

Figure 2. Relative frequency shift of the quadrupole oscillation mode versus the trap aspect ratio λ for values of the dimensionless two-body interaction $p = 0.01$, 0.1, 0.4 and 1 and for several values of the dimensionless three-body interaction k.

nonlinearities due to interactions. In the case of two-body interactions, this was studied within the hydrodynamic approach of [21] and more recently in other formalisms [22, 23]. Here we also consider three-body contact interactions, and describe the BEC system by the variational set of equations (3) and (4). Linear stability analysis yields the frequencies of collective modes:

$$\omega_{\mathrm{B0,Q0}} = \left(\frac{m_1 + m_3 \pm \sqrt{(m_1 - m_3)^2 + 8m_2^2}}{2} \right)^{1/2}, \tag{7}$$

$$m_1 = 1 + \frac{3}{u_{\rho 0}^4} + \frac{3p}{u_{\rho 0}^4 u_{z0}} + \frac{5k}{u_{\rho 0}^6 u_{z0}^2}, \tag{8}$$

$$m_2 = \frac{p}{u_{\rho 0}^3 u_{z0}^2} + \frac{2k}{u_{\rho 0}^5 u_{z0}^3}, \tag{9}$$

$$m_3 = \lambda^2 + \frac{3}{u_{z0}^4} + \frac{2p}{u_{\rho 0}^2 u_{z0}^3} + \frac{3k}{u_{\rho 0}^4 u_{z0}^4}, \tag{10}$$

and $u_{\rho 0}$ and u_{z0} are equilibrium widths, obtained as stationary solutions.

To study nonlinear effects in real-time dynamics, we consider a BEC in the initial state corresponding to the stationary ground state with a small perturbation proportional to the eigenvector of the quadrupole mode. This perturbation, proportional to the small parameter ϵ, leads to quadrupole mode oscillations. However, due to nonlinear effects in a BEC, the breathing mode is also excited eventually, as well as other, higher harmonics, which include linear combinations of both modes. The frequency of collective modes depends on the anisotropy λ and, as was shown in [21], exhibit resonances for specific values of λ. For trap geometries with anisotropies close to resonant values, frequencies of collective oscillation modes are significantly shifted from their linear stability analysis values. If we take into account three-body interactions and apply a Poincaré–Lindstedt perturbative expansion [19, 20] in the small parameter ϵ, we obtain the frequency shift of the quadrupole mode in the form

$$\omega_{\rm Q} = \omega_{\rm Q0} + \varepsilon^2 \frac{f(\omega_{\rm Q0}, \omega_{\rm B0}, u_{\rho 0}, u_{z0}, p, k, \lambda)}{2\omega_{\rm Q0}^2(\omega_{\rm B0} - 2\omega_{\rm Q0})(\omega_{\rm B0} + 2\omega_{\rm Q0})}. \quad (11)$$

From this, we can immediately read off poles for the values of λ determined by the condition $\omega_{\rm B0} = 2\omega_{\rm Q0}$. Similar results are obtained for the breathing mode. The frequency shifts for the quadrupole mode are illustrated in figure 2 for various values of dimensionless two- and three-body interaction strengths. As we can see from the graphs, for small values of the two-body interaction, the three-body interaction can have a significant effect on the frequency of collective modes. Furthermore, we see that the trap anisotropy can be fine-tuned in such a way that the frequency shift is completely removed. However, as two-body interactions increase, three-body interaction effects become less important and eventually just represent a small correction to the leading two-body behavior.

4. Conclusions

In this paper, we have studied prominent nonlinear effects that arise in BECs due to two- and three-body contact interactions. We have used a Gaussian variational approach which was shown to well describe BEC systems in the range of parameters that are relevant for current experimental setups [16]. Using the Poincaré–Lindstedt perturbation theory, we have calculated frequency shifts due to a harmonic modulation of the s-wave scattering length, motivated by a recent experiment [17]. We have also studied in detail the delicate interplay between nonlinear effects due to two- and three-body interactions and the trap geometry. Within the variational approach and the Poincaré–Lindstedt method, we have calculated frequency shifts and identified the geometric resonances of collective modes of axially symmetric BEC systems. We have also shown that the observed geometric resonances can be eliminated if two- and three-body interactions can be appropriately fine-tuned.

We plan to extend this research and further study the interplay of two- and three-body interactions by considering the case of attractive three-body interaction, when competing effects between repulsive two-body and attractive three-body interactions may give rise to interesting phenomena. We also plan to study mode coupling and energy transfer between quadrupole and breathing oscillation mode due to nonlinear effects.

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