Decoherence of qubits interacting with a nonlinear dissipative classical or quantum system

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Quantum trajectories given by the quantum state diffusion theory are used to study the dynamics of a pair of qubits interacting with the quantized Duffing oscillator. The latter is a nonlinear dissipative and driven quantum system that in the semiclassical limit displays typical bifurcations and chaotic behavior of the classical model. It is shown that the oscillator part of the qubits-oscillator system in the semiclassical limit is well described by the classical model despite the oscillator-qubit interaction. On the other hand, exact quantum dynamics of the qubits pair is completely different from the classical model as long as there is non-negligible qubit-oscillator interaction. Dynamics of the interqubits entanglement and the qubits pair von Neumann entropy is studied in the deeply quantum and various degrees of semiclassical regimes and for different values of the oscillator's bifurcation parameter. It is concluded that the decoherence of the qubits pair by the dissipative nonlinear oscillator is more effective when the oscillator is in the more classical regime, and if the semiclassical oscillator dynamics is chaotic rather than regular.

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I. INTRODUCTION

The theory of quantum measurement requires a division of the analyzed system into two parts. One of them, the small system denoted by (q,p), is described exactly and with all details, using quantum mechanical formalism, while the other part, the large system, is described only in a coarse-grained way by using a small number of relevant degrees of freedom, denoted by (Q,P). Coarse graining leads to decoherence. The coarse graining of the large system is commonly performed by averaging over its irrelevant degrees of freedom. Such coarse graining renders the dynamics of the relevant degrees of freedom (Q,P) of the large system irreversible with dissipative and stochastic contributions. Depending on the properties of the particular large system, such dynamics can be predominantly quantum, with characteristic phase space delocalization, or predominantly classical with the well-defined approximate phase space orbits. Furthermore the deterministic part of the (Q,P) evolution can be nonlinear and complicated. The dynamics of the small quantum part, with (q,p) degrees of freedom is influenced by the dynamics of the (Q,P) variables. The primary effect of such influence is the entanglement between (q,p) and (Q,P) degrees, which via the averaging over the neglected degrees of the large system leads to the decoherence of the (q,p) system [1]. The mechanism of decoherence is universal if the coarse-grained system consists of a large number of noninteracting harmonic oscillators [2]. In fact, common experimentally studied systems of entangled qubits are of the form of few qubits with interaction mediated via a third system of one or few degrees of freedom, which can display dissipation and decoherence. Examples include atoms in a leaky cavity in interaction with the specific electromagnetic field that can itself be coupled to the electromagnetic environment [3,4] and trapped ions in a chain interacting via exchanged phonons [5]. It is interesting to analyze the dynamics of the coupled small-large systems when the large system displays complex dynamical properties.

In this paper we shall analyze the decoherence dynamics of a pair of qubits [playing the role of the small (q,p) system] in interaction with the quantum dissipative Duffing oscillator [large (Q,P) system]. The later is characterized by two important parameters: the parameter β related to the classicality of the system and the bifurcation parameter g determining the qualitative properties of the dynamics of the system in the classical limit. In fact, the classical driven Duffing oscillator goes through a sequence of bifurcations from the stable fixed point up to the chaotic attractor as the parameter g is increased [6]. The corresponding quantized open dissipative system in the semiclassical limit of small β displays similar qualitative dependence on g in the sense that the expectation values $\langle Q\rangle$, $\langle P\rangle$ evolve as trajectories of the classical system and then the long term dynamics of $\langle Q\rangle$, $\langle P\rangle$ corresponds to qualitatively different attractors including the chaotic attractor depending on the value of the parameter g [7,8]. The large quantum system with such behavior enables us to study the decoherence dynamics in the typical quantum system of the two qubits under the influence of qualitatively different dynamics of the large system.

Understanding of the entanglement becomes particularly important since it was discovered that it enables quantum systems to perform useful tasks that cannot be performed by systems obeying classical physics (for a recent review of the literature please see [9]). One can consider different forms of entanglement and various quantities have been introduced in order to quantitatively characterize the degree of entanglement between quantum systems. One such quantity is the entanglement of formation for a pair of qubits in a pure or in a mixed state. In order to characterize the dynamics of decoherence we shall use the dynamics of the entanglement of formation between the two qubits and the von Neumann entropy of the density matrix representing the state of the pair of qubits. This can be considered as measuring the entanglement between the pair of qubits and the rest represented by the dissipative oscillator.

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Our analyses are related to the recent works on the decoherence by chaotic environments, modeled typically by kicked tops [10], kicked rotators [11,12], spin chains [13], or abstract models [14–16]. Chaoticity of the quantum systems that are used in these references to play the role of the environment is identified with certain properties of their spectral distributions or with the chaoticity of the corresponding classical model. Neither of these is equivalent to the existence of chaotic approximate phase space trajectories, i.e., existence of the phase space localized wave packet whose centroid approximately follows the classically chaotic evolution, which is the property of the dissipative Duffing oscillator used here. Related to the decoherence by complex or chaotic environment is the problem of entanglement dynamics in closed or open quantum systems that can be considered chaotic in some sense [17–30]. In this case the complicated dynamics is produced by the quantum system internal dynamics, and in the case studied here the interest is in the consequences of the interaction of a simple quantum system with a macroscopic system with complicated dynamics. We would also like to mention the recent paper [31] where the dynamics of a single qubit in interaction with a driven dissipative linear oscillator was studied using quantum trajectories similar to our approach.

The structure of the paper is as follows. In the next section we introduce the model of two qubits interacting with the dissipative quantum Duffing oscillator. In Sec. III, we use the quantum trajectories of the quantum state diffusion theory [32] to compare the exact dynamics of the large and small subsystems of the dissipative oscillator-qubits system with the corresponding dynamics of a classical approximation. Presented are all typical dynamical situations that could occur in the qubits-oscillator system depending on the values of the three parameters that characterize (a) the classicality of the oscillator, (b) the qualitative properties of its dynamics, and (c) the strength of the qubits-oscillator interaction. In Sec. IV we study the decoherence of the qubit pair by discussing the dynamics of entanglement between the two qubits and between the qubits and the dissipative oscillator in the different dynamical regimes. We consider the decoherence of the initially maximally entangled qubits states with the oscillator in the quantum or classical regime, and the entanglement creation (dynamics) from initially separable qubits states due to interaction with the oscillator in the quantum regime. Summary and discussion of our results are presented in Sec. V.

II. THE MODEL

Chaotic property of a classical dynamical system is commonly defined in terms of instability of its phase space trajectories. Lack of such trajectories for isolated quantum systems represent a problem if the definition of the chaotic dynamics with its consequences is to be extended from classical to quantum systems [33,34]. Therefore, there are several inequivalent definitions of what should be considered as a chaotic quantum system [33–38,29]. However, isolated systems represent an idealization, and in the case of more realistic open quantum systems the interaction with the environment can lead to dynamics which preserves wave packet localization in phase space [32,39,1]. At the same time the quantum dynamics of the centroid of such well localized wave packet could be very similar with the dynamics following from classical equations, and in particular the centroid could evolve chaotically leading to the chaotic evolution of the quantum expectation values of the dynamical variables [7,8,40].

The complete description of a state of an open quantum system is given by its density matrix ρ, and the evolution is commonly described by the corresponding master equation for $\rho(t)$ [39]. This description corresponds to an ensemble of quantum systems, and is not suitable if we want to characterize the chaoticity of the quantum dynamics in terms of orbits of the expectation values of dynamical variables. However, the evolution of the state vector of a single open quantum system can often be described by the Schrödinger equation with additional terms due to dissipation and stochastic fluctuations [41–43,39]. Such stochastic Schrödinger equations (SSE) are obtained by unraveling the master equation for $\rho(t)$. They are all consistent with the requirement that the solutions of the master equation and of the SSE satisfy

$$\hat{\rho}(t) = E[|\psi(t)\rangle\langle\psi(t)|], \tag{1}$$

where $E[|\psi(t)\rangle\langle\psi(t)|]$ is the expectation with respect to the distribution of the stochastic process $|\psi(t)\rangle$.

In the case of continuous Markov evolution the unique SSE for the stochastic state vector $|\psi(t)\rangle$, which has the same transformation properties as the Markov master equation in the Lindblad form for $\rho(t)$ [44,39], is given by the theory of quantum state diffusion (QSD) [32]. The resulting SSE is of the form of a diffusion process on the Hilbert space of pure states,

$$|d\psi\rangle = -i\hat{H}|\psi\rangle dt + \left[\sum_k 2\langle\hat{L}_k^\dagger\rangle\hat{L}_k - \hat{L}_k^\dagger\hat{L}_k - \langle\hat{L}_k^\dagger\rangle\langle\hat{L}_k\rangle\right]|\psi(t)\rangle dt + \sum_k (\hat{L}_k - \langle L_k\rangle)|\psi(t)\rangle dW_k, \tag{2}$$

where $\langle\ \rangle$ denotes the quantum expectation in the state $|\psi(t)\rangle$ and dW_k are independent increments of complex Wiener c-number processes $W_k(t)$ satisfying

$$E[dW_k] = E[dW_k dW_{k'}] = 0, \quad E[dW_k d\bar{W}_{k'}] = \delta_{k,k'}dt, \quad k = 1,2,\ldots,m. \tag{3}$$

Here $E[\cdot]$ denotes the expectation with respect to the probability distribution given by the multidimensional process W, and $\bar{W}_k$ is the complex conjugate of W_k. The first term $-iH|\psi\rangle\, dt$ describes the unitary part, the rest proportional to dt represents the drift, and the last term proportional to dW represents the diffusion. The Lindblad operators L_k are interpreted and inferred from different types of influence that the environment exerts on the system and are the same as in the Lindblad master equation. The correspondence between the QSD equations and the Lindblad master equations is unique, and is not related to a particular measurement scheme, or the form of the Markov environment.

Besides the practical advantage in numerical computations of using pure states N dimensional bases rather than the bases of $N\times N$ matrices, the description of the evolution by the QSD equation has the advantage that often the environmental influence leads to permanent localization of the wave packets $|\psi(t)\rangle$ onto a small areas in phase space. This is important in the study of the chaotic semiclassical behavior, and therefore is the primary reason for us to use the QSD approach to modeling quantum chaotic dissipative systems.

We chose the quantization of the Duffing oscillator as an example of the chaotic dissipative system which shall play the role of the large quantum system [7,45]. The Hamiltonian H_l and the single Lindblad operator L are

$$H_l = P^2/2 + \beta^2 Q^4/4 - Q^2/2 + g\cos(t)Q/\beta + \gamma(QP+PQ)/2, \tag{4}$$

$$L = \sqrt{2\gamma}a = \sqrt{2\gamma}(Q-iP)/\sqrt{2}, \tag{5}$$

when substituted in the QSD equation, and in an appropriate classical limit, represented by $\beta\to 0$, reproduce the dynamics of the classical Duffing oscillator, given by the second order nonautonomous equation

$$\frac{d^2Q}{dt^2} + 2\gamma\frac{dQ}{dt} + Q^3 - Q = g\cos(t). \tag{6}$$

Depending on the parameter g the classical system (6) can have simple regular attractors such as fixed points or periodic orbits, or a complicated chaotic attractor. The dissipative quantum evolution (2) with Eqs. (4) and (5), in an appropriate limit, reproduces the classical dynamics. The parameter β characterizes the classicality of the system in the sense that the classical limit is realized by rescaling $\beta\to 0$, which leads to the large ratio of the phase space covered by the system's motion and the area of the Planks cell. Also appropriate values of γ imply good localization in the sense that the dispersion of the dynamical variables is negligible with respect to their variations during the motion. If β is sufficiently small and for appropriate γ the QSD equation with Eqs. (4) and (5) reproduces the qualitative and quantitative properties of the classical Duffing oscillator. For example, for $g=0.3$ and $\gamma=0.125$ the averages $\langle Q(t)\rangle$, $\langle P(t)\rangle$ of the open quantum system reproduce the chaotic trajectories of the classical Duffing oscillator [7].

We chose a pair of qubits in an external fixed field to represent the typical small quantum system. The Hamiltonian reads

$$H_s = \omega\sigma_x^1 + \omega\sigma_x^2, \tag{7}$$

where $\sigma_{x,y,z}^{1,2}$ denote x, y, and z components of the Pauli operators for the first or the second qubit. We shall also use σ^i, $i=1,2$ to denote the vector of Pauli operators for the first or second qubit.

The qubits do not interact directly but are both linearly coupled to the Q variable of the large system, i.e., to the Duffing oscillator

$$H_{ls} = \mu Q\sigma_z^1 + \mu Q\sigma_z^2. \tag{8}$$

The total Hamiltonian is given by

$$H = H_l + H_s + H_{ls}. \tag{9}$$

Notice that the qubits self-Hamiltonian H_s and the interaction Hamiltonian H_{ls} do not commute.

Equation (2) with the Hamiltonian (9) and the Lindblad operator (5) represent our model of the small quantum system, the two qubits, interacting with the large nonlinear dissipative system, the dumped Duffing oscillator, which can be quantum, when $\beta=1$, or approximately classical when $\beta=0.01$. We are interested in the decoherence of the qubits pair corresponding to different dynamical regimes of the large system. This problem will be studied using numerically obtained trajectories of the system (2), (9), and (5).

III. QUALITATIVE PROPERTIES OF DYNAMICS IN THE SEMICLASSICAL REGIME

In this section we concentrate on possible dynamical properties of single orbits of the evolution equation (2) with Eqs. (9) and (5) for different values of the relevant parameters. In particular, we shall be interested to see if a classical model can reproduce the quantum dynamics of the oscillator and the qubits in some domain of the parameters. Qualitative properties of the evolution of the average values $\langle Q\rangle$, $\langle P\rangle$, $\langle\sigma^1\rangle$, $\langle\sigma^2\rangle$ along an orbit of the QSD equation (2) with Eqs. (9) and (5) are determined in different ways by the values of the parameters β, γ, g, and μ. If β is sufficiently small and for sufficient dissipation γ the dispersions ΔQ and ΔP are simultaneously small during the evolution. Then, the parameters g and μ determine the qualitative properties of the dynamics. The parameter g determines if the dynamics of $\langle Q\rangle$, $\langle P\rangle$ is regular or chaotic. The effect of the different values of the parameter μ on the dynamics of averages $\langle\sigma^1\rangle$, $\langle\sigma^2\rangle$ and dispersions $\Delta\sigma^1$, $\Delta\sigma^2$ is tricky and depends on whether $\langle Q\rangle$, $\langle P\rangle$ dynamics is regular or chaotic.

A. QSD evolution of the averages versus classical approximations

The classical analog of the quantum system (2) with Eqs. (9) and (5) is given by the corresponding differential equations on the six-dimensional classical phase space. The classical model is obtained by approximating the evolution of the averages $\langle Q\rangle$, $\langle P\rangle$, $\langle\sigma_{x,y,z}^{1,2}\rangle$ exactly described by the QSD equation (2). There are two crucial approximations in the construction of the classical model: (a) the average values of operators given by nonlinear expressions of operators Q, P, and $\sigma_{x,y,z}^{1,2}$ can be replaced by the same nonlinear expressions of the average values $\langle Q\rangle$, $\langle P\rangle$, $\langle\sigma_{x,y,z}^{1,2}\rangle$; and (b) the construction of the phase space of the composite system is according to the rules of the classical mechanics obtained as the product of the plane $\mathbb{R}^2$ for the Duffing part and the two Bloch spheres $S^2\times S^2$ for the two qubits. The plane $\mathbb{R}^2$ is the set of harmonic oscillator coherent states and is invariant on the dynamics generated by the quadratic harmonic Hamiltonian. As we shall see, the two assumptions are justified for the case of the Duffing oscillator with small β, but are not justified for the qubit part of the model. The equations of the classical model are written in terms of the classical coordinates $(Q,P)\in\mathbb{R}^2$ for the oscillator and (q_2,p_2,q_3,p_3) which are the symplectic coordinates on the product $S^2 \times S^2$, and are related to the single qubit coherent state averages of the qubits components $\langle\sigma^{1,2}_{x,y,z}\rangle$ by the following formulas [33,46,47]:

$$\sigma^i_x = \frac{q_i}{2}\sqrt{2 - q_i^2 - p_i^2}, \tag{10}$$

$$\sigma^i_y = \frac{p_i}{2}\sqrt{2 - q_i^2 - p_i^2}, \tag{11}$$

$$\sigma^i_z = (q_i^2 + p_i^2 - 1), \quad i = 1,2. \tag{12}$$

The equations of the classical model read

$$\frac{dQ}{dt} = P, \tag{13}$$

$$\frac{dq_1}{dt} = \mu p_1 Q - \frac{p_1 q_1}{2\sqrt{(2 - p_1^2 - q_1^2)}},$$

$$\frac{dq_2}{dt} = \mu p_2 Q - \frac{p_2 q_2}{2\sqrt{(2 - p_2^2 - q_2^2)}}, \tag{14}$$

$$\frac{dP}{dt} = - \gamma P + Q - \beta Q^3 - \mu(p_1^2 + p_2^2 + q_1^2 + q_2^2)/2 - g \sin(t)/\beta, \tag{15}$$

$$\frac{dp_1}{dt} = \mu q_1 Q - \frac{2 - p_1^2 - 2q_1^2}{2\sqrt{(2 - p_1^2 - q_1^2)}},$$

$$\frac{dp_2}{dt} = \mu q_2 Q - \frac{2 - p_2^2 - 2q_2^2}{2\sqrt{(2 - p_2^2 - q_2^2)}}. \tag{16}$$

Notice that if there is no qubit-oscillator interaction, i.e., $\mu = 0$, the qubit part of the classical model (14) and (16) represent the exact quantum evolution of the average values $\langle\sigma^{1,2}(t)_{x,y,z}\rangle$, where the state $|\,\rangle$ is a product of the qubit's coherent states, in the form of the Hamilton's dynamical equations $\dot{q}_i = \partial\langle H\rangle/\partial p_i$, $\dot{p}_i = -\partial\langle H_s\rangle/\partial p_i$. This is a consequence of the fact that the Hamiltonian H_s is linear in the generators $\sigma^{1,2}(t)_{x,y,z}$ and there is no interaction between qubits. Thus, the purely qubit part of the quantum model is represented exactly by the classical model, and it is only the Hamiltonian form of the equations that is classical. It is the interaction between qubits and the quantum dissipative oscillator and the nonlinear parts of the oscillator potential that are not exactly described by the classical model.

When the parameter β is small, and for sufficient localization due to dissipation γ, the exact dynamics of averages $\langle Q\rangle$, $\langle P\rangle$, given by Eq. (2) with Eqs. (9) and (5), reproduce the dynamics of the dissipative Duffing oscillator part of the classical model (13) and (15), including the qualitative dependence on the chaoticity parameter g. This is true even for large coupling $\mu \approx 1$ between the oscillators and the qubits. On the other hand, provided that the coupling strength μ is not negligibly small, the part of the classical system that corresponds to the two qubits, that is the dynamics of $\langle\sigma^{1,2}_{x,y,z}\rangle$ as given by Eqs. (14), (16), and (10)–(12), completely fails to describe the exact qubit dynamics of the quantum system given by Eq. (2) with Eqs. (9) and (5), irrespective of the values of the parameter g. In fact, the interaction between a qubit and the oscillator introduces entanglement between the two systems which is not captured by the product construction of the classical phase space. On the other hand, the effect of the qubit-oscillator entanglement on the oscillator is destroyed by the action of the Lindblad operator (5) on the oscillator state and dynamics. The effect of the Lindblad operator is to localize the state onto small regions of the $\mathbb{R}^2$ phase space which turns the dynamics of the centroid of such localized wave function into the dynamics of the classical model (13) and (15). Similar impossibility to capture the exact dynamics of the qubit part of Eq. (2) with Eqs. (9) (5) is shown also by the modification of the classical model in which one would use the classical equations (13) and (15) to describe the oscillator dynamics and replace the qubit part (14) and (16) by the Schrödinger differential equations on the two qubit Hilbert space C^4 [48]. These qubit equations will depend on the values of $\langle Q(t)\rangle$ as a time-dependent parameter. Such mixed classical-quantum description of the oscillator-qubit system describes well the oscillator part (in the $\beta \to 0$ limit) but also fails to describe the qualitative properties of the exact qubit dynamics. The qualitative properties of the qubits must be described by the full system (9) using the QSD equation (2) and the Lindblad (5).

In the rest of this section we shall describe the typical dynamical regimes of the qubit and the oscillator dynamics. The dynamics of entanglement between the qubits and between the two qubits and the dissipative oscillator is described in the next section. In all our computations the initial state is a product of the oscillator and the qubit pair states, and for the oscillator state we use some of the harmonic oscillator coherent states. All illustrations of our results use the dimensionless time $\tau = \omega t$.

B. Dynamics of the oscillator subsystem

Figure 1 illustrates qualitatively different dynamics of $\langle Q\rangle$, $\langle P\rangle$ for the whole system (9), obtained for three typical values of g and for the fixed values of the parameters $\beta = 0.01$ and $\gamma = 0.125$. These values of β and γ lead to the very similar dynamics of $\langle Q\rangle$, $\langle P\rangle$ generated by the classical equations on the phase space on one hand and by the QSD equation (2) with Eqs. (9) and (5) and the corresponding coherent initial state on the other. The fixed point [Fig. 1(a)], the periodic [Fig. 1(b)] or the chaotic orbit [Fig. 1(c)] are generated by the QSD equation with Eqs. (9) and (5), and are very similar to the Q,P part of the corresponding solutions of the phase space classical approximation. This good approximate coincidence of the quantum $\langle Q\rangle$, $\langle P\rangle$ and the classical Q,P is the consequence of strong localization of the wave function on the points in the Q,P part of the phase space due to the action of the Lindblad operator (5) proportional to the oscillators annihilation operator. Indeed the dispersions of both Q and P are negligible compared to $\langle Q\rangle$, $\langle P\rangle$, as is illustrated only for the coordinate in Figs. 1(d)–1(f). On the other hand,

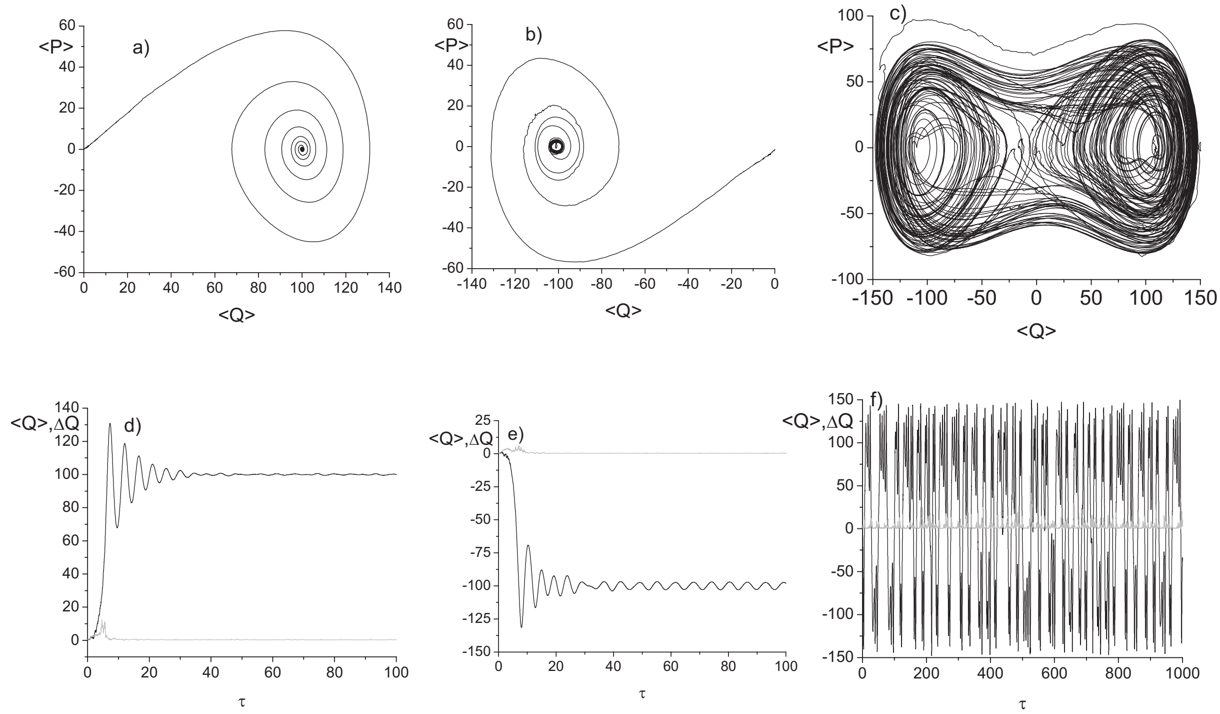

FIG. 1. Qualitatively different dynamics for different values of the bifurcation parameter g in the semiclassical regime of the oscillator part of the full system (2), (9), and (5). (a), (b), and (c) illustrate $\langle P\rangle$ (dimensionless) vs $\langle Q\rangle$ (dimensionless) and (d), (e), and (f) $\langle Q\rangle$ and ΔQ and where $|\,\rangle$ is a solution of the QSD equation. In (a) and (d) $g=0.003$, (b) and (e) $g=0.03$, and (c) and (f) $g=0.3$. Other parameters are $\beta=0.01$, $\mu=0.1$, $\gamma=0.125$.

in the quantum regime, i.e., for $\beta=1$, the correspondence between $\langle Q\rangle$, $\langle P\rangle$ and Q,P is completely lost. Qualitatively the same quantum dynamics of $\langle Q\rangle$, $\langle P\rangle$ is obtained for different g corresponding to qualitatively different dynamics of Q,P. In Fig. 2 we illustrate dynamics of $\langle Q\rangle$, $\langle P\rangle$ [Figs. 2(a) and 2(b)] and $\langle\sigma_z^1\rangle$ and $\Delta\sigma_z^1$ [Figs. 2(c) and 2(d)] for $g=0.3$ [Figs. 2(a) and 2(c)] corresponding to the chaotic classical dynamics and for $g=0.003$ [Figs. 2(b) and 2(d)] corresponding to the regular case. The coupling strength in Figs. 2(a)–2(d) is $\mu=1$, corresponding to strong interaction, but

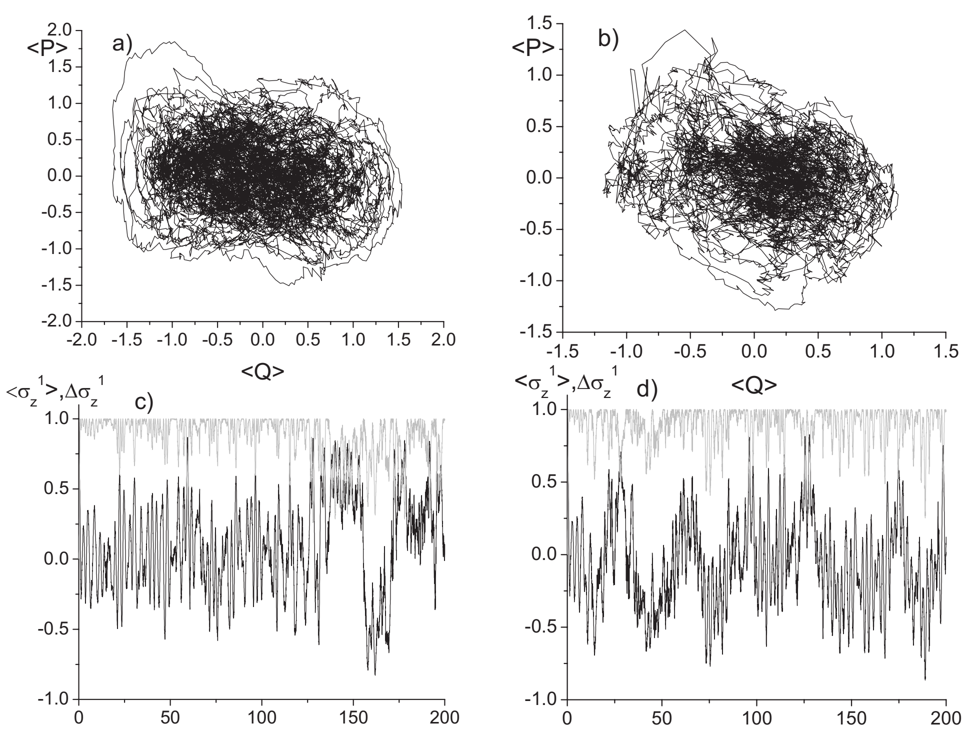

FIG. 2. Typical dynamics in the deeply quantum regime $\beta=1$. (a), (b) illustrate $\langle P\rangle$ (dimensionless) vs. $\langle Q\rangle$ (dimensionless) and (c) and (d) $\langle\sigma_z^1\rangle$ (dimensionless) and $\Delta\sigma_z^1$ (dimensionless) and $|\,\rangle$ is a solution of the QSD equation. The coupling is $\mu=1$ and in (a) and (c) $g=0.3$, and in (b) and (d) $q=0.003$.

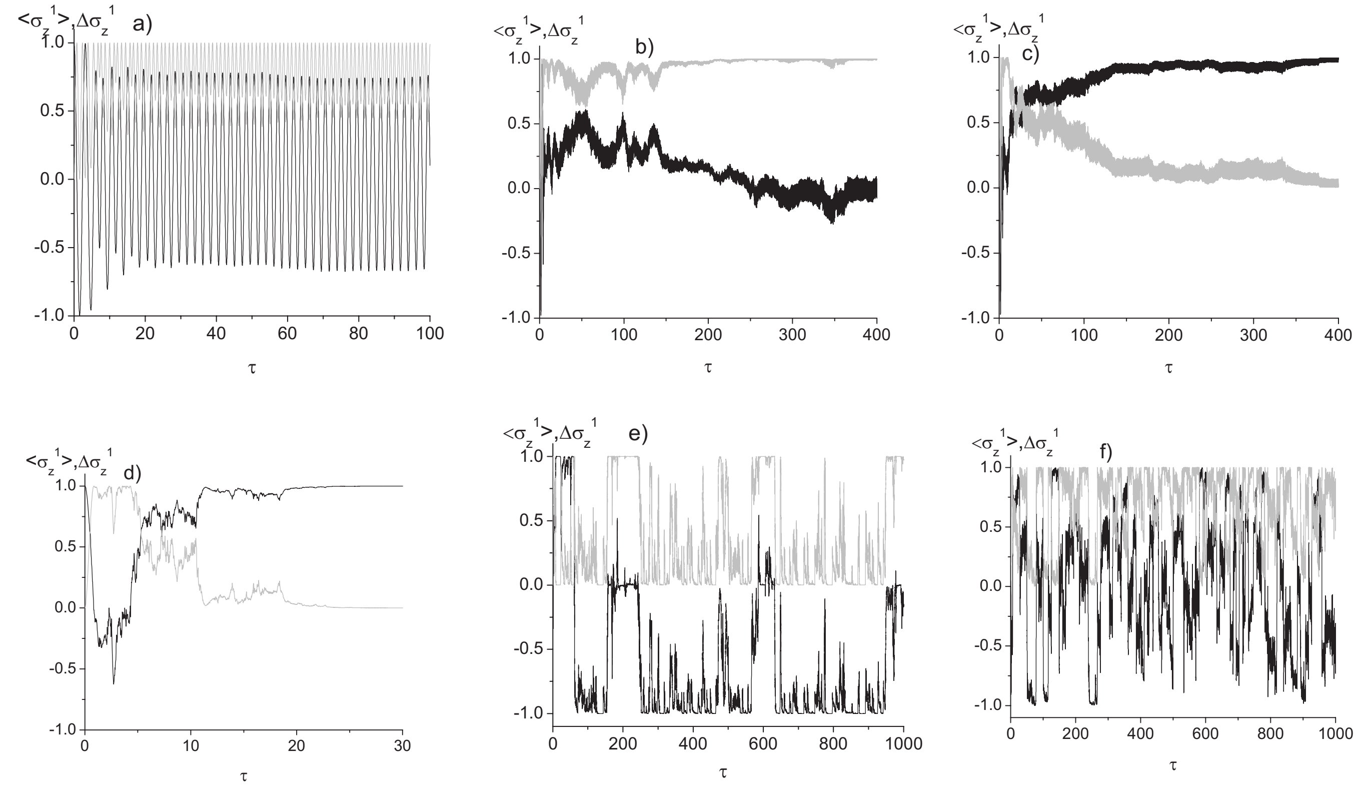

FIG. 3. Typical dynamics in the semiclassical regime of the qubit part of the full system (2), (9), and (5) starting from qubits in the separable initial state. All figures illustrate time series $\langle \sigma_z^1 \rangle$ (black) (dimensionless) and $\Delta\sigma_z^1$ (gray) (dimensionless) where $|\,\rangle$ is a solution of the QSD equation and fixed parameters are β=0.01, γ=0.125. In (a) μ=0.01, g=0.003, (b) μ=0.1, g=0.03, (c) μ=1, g=0.003, (d) μ=1, g=0.03, (e) μ=1, g=0.3, (f) μ=0.1, g=0.3.

the similar dynamics of $\langle Q \rangle$, $\langle P \rangle$ is obtained for other values of μ. Of course, the dynamics of $\langle \sigma_z^1 \rangle$ strongly depends of μ. The dispersions of oscillator observables Q and P are of the same magnitude as the averages and are not shown. The values of the parameters correspond to the stable fixed point or to the chaotic dynamics in the semiclassical regime. We see that in the deeply quantum regime of the dissipative oscillator β=1 there is no qualitative dependence of the dynamics of either the oscillator or the qubits subsystems on the values of the classical bifurcation parameter g.

C. Dynamics of the qubits subsystem

The dynamics of the qubits degrees of freedom for small β is illustrated in Figs. 3 and 4, which correspond to the typical dynamics from a separable initial state $|\uparrow\rangle\,|\uparrow\rangle$ (Fig. 3 and the maximally entangled state $(|\uparrow\rangle|\downarrow\rangle+|\downarrow\rangle|\uparrow\rangle)/\sqrt{2}$ (Fig. 4). The dynamics of the qubits is sensitive to the coupling strength μ and to the parameter g. The dynamics is illustrated over a time interval which is sufficiently long to infer the qualitative properties of the asymptotic states and dynamics. The time intervals strongly depend on the parameter g.

In general, weak coupling $\mu \ll 1$ and small g [Fig. 3(a)] or large g [Fig. 3(f) and Fig. 4(d)] lead to regular or chaotic oscillations of the $\sigma_z^{1,2}$ components, respectively, together with large dispersions $\Delta\sigma_z^{1,2}$. Thus, a large dispersion is typical for weak coupling μ. The behavior for small g and weak coupling depends also on the properties of the initial state. If the initial state is maximally entangled then the dispersion of $\sigma^{1,2}$ remains maximal all the time [Fig. 4(a)]. Strong coupling $\mu \approx 1$ implies localization of the σ_z component, i.e., small dispersion in the case of g values that correspond to the fixed point [Fig. 3(c)] or to the periodic $\langle Q \rangle$, $\langle P \rangle$ orbit [Fig. 3(d) and Fig. 4(b)]. In these two cases of regular $\langle Q \rangle$, $\langle P \rangle$ dynamics and strong coupling the component σ_z localizes onto its eigenstate. However, the values of g that correspond to chaotic $\langle Q \rangle$, $\langle P \rangle$ dynamics imply chaotic dynamics of $\langle \sigma^{1,2} \rangle$ and comparably large dispersion $\Delta\sigma_z^{1,2}$ irrespective of the value of the coupling strength [Figs. 3(d), 3(f), and 4(c)]. Let us stress again that the qubit part of the classical phase space model [Eqs. (14) and (16)] completely fails to predict the qualitative behavior of the averages $\langle \sigma^{1,2} \rangle$.

IV. DYNAMICS OF ENTANGLEMENT

The superposition principle is the hallmark of quantum behavior. For composite quantum systems the superposition of more than one product state, i.e., superposition of states in the form of the product of states of the system's constituents, lead to the entanglement, that is, typically quantum correlation, between the considered parts of the whole system. Interaction of a composite quantum system with the macroscopic environment often results in dynamical instability of the entangled states. The entanglement among parts of the system is transferred by interaction to the entanglement be-

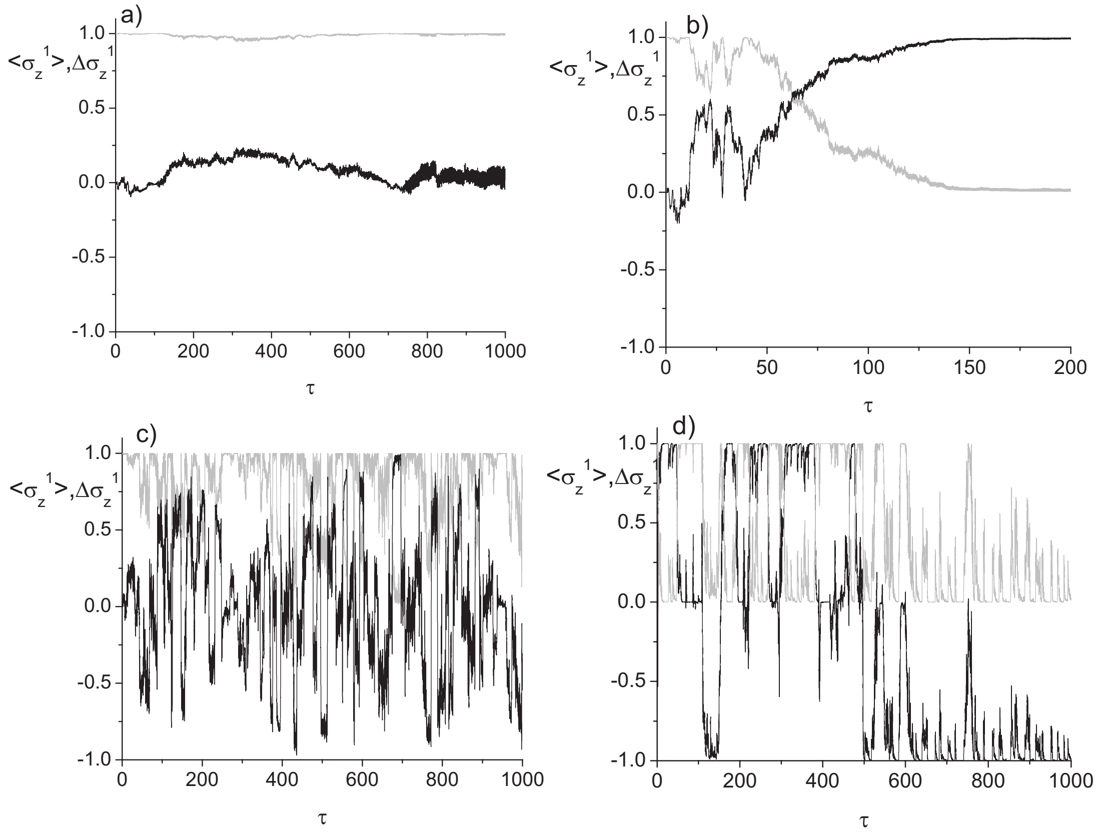

FIG. 4. Typical dynamics in the semiclassical regime of the qubit part of the full system (2), (9), and (5) starting from qubits in the maximally entangled initial state. All figures illustrate time series $\langle\sigma_z^1\rangle$ (black) (dimensionless) and $\Delta\sigma_z^1$ (gray) (dimensionless) where $|\,\rangle$ is a solution of the QSD equation and fixed parameters are $\beta=0.01$, $\gamma=0.125$. In (a) $\mu=0.01$, $g=0.003$, (b) $\mu=0.1$, $g=0.03$, (c) $\mu=0.1$, $g=0.3$, (d) $\mu=1$, $g=0.3$.

tween the system and environmental degrees of freedom and is subsequently almost completely lost because of the decoherence in the macroscopic system. On the other hand, it is known that in some cases macroscopic systems can mediate interaction that leads to entanglement between different parts of a composite quantum system [49,50]. In this section we present the results of our analyses of the dynamics of entanglement between (a) the two qubits in Eq. (9) and (b) the pair of qubits and the Q,P degree of freedom of the large dissipative system. We shall be especially interested in the dependence of the entanglement dynamics on the classicality parameter β and on the bifurcation parameter g.

The study of entanglement always involves correlations and thus an ensemble of systems. If the system can be considered as isolated than it can in principle be represented all the time during its evolution by an ensemble of equal pure states. Otherwise, if the considered system is a part of a larger system then the ensemble representing the state of the subsystem, and completely ignoring the rest of the total system, must be taken to represent a convex combination of different pure states of the subsystem. In our case, the pair of qubits is always interacting with the dissipative oscillator and thus its state needs to be represented by a mixture ρ_s of pure states. Furthermore, the entire system of the qubit pair and the dissipative oscillator includes dissipation and thus cannot be considered as isolated. It is also described by a mixture ρ. The qubit state ρ_s is in principle related to the total state ρ by the partial trace $\rho_s=\mathrm{Tr}_l[\rho]$. On the other hand, entries of the matrix ρ_s can be expressed in terms of correlations between qubit's components: $\mathrm{Tr}_s[\rho_s\sigma_i^1\sigma_j^2]$, $i,\ j=x,y,z$. The dynamics of these correlations can be computed by averaging the quantum expectations $\langle\psi(t)|\sigma_i^1\sigma_j^2|\psi(t)\rangle$ over many realizations $|\psi(t)\rangle$ of the stochastic process give by the QSD equation, according to the unraveling formula (1). This double averaging, first over the pure sample state $\langle\psi(t)|\sigma_i^1\sigma_j^2|\psi(t)\rangle$ and then over many such samples $E[\langle\psi(t)|\sigma_i^1\sigma_j^2|\psi(t)\rangle]$, gives $\mathrm{Tr}_s[\rho_s\sigma_i^1\sigma_j^2]$, $i,\ j=x,y,z$. These correlations are all that we need in order to calculate the measure of entanglement between the qubits, known as the entanglement of formation, using the procedure discovered by Wootters [51,52]. Entanglement of formation between two qubits in a mixed state ρ_s is calculated as follows. First, a concurrence is calculated by the following formula:

$$C(\rho_s)=\max\{0,\sqrt{\lambda_1}-\sqrt{\lambda_2}-\sqrt{\lambda_3}-\sqrt{\lambda_4}\},\qquad(17)$$

where $\lambda_1>\ldots\lambda_4$ are the eigenvalues of the matrix $\rho_s(\sigma_y^1\otimes\sigma_y^2)\bar{\rho}_s(\sigma_y^1\otimes\sigma_y^2)$, where $\bar{\rho}_s$ is the complex conjugate of ρ_s calculated in the standard bases. The entanglement of formation is then given via the function

$$h(x)=-x\log_2 x-(1-x)\log_2(1-x)$$

by the following formula:

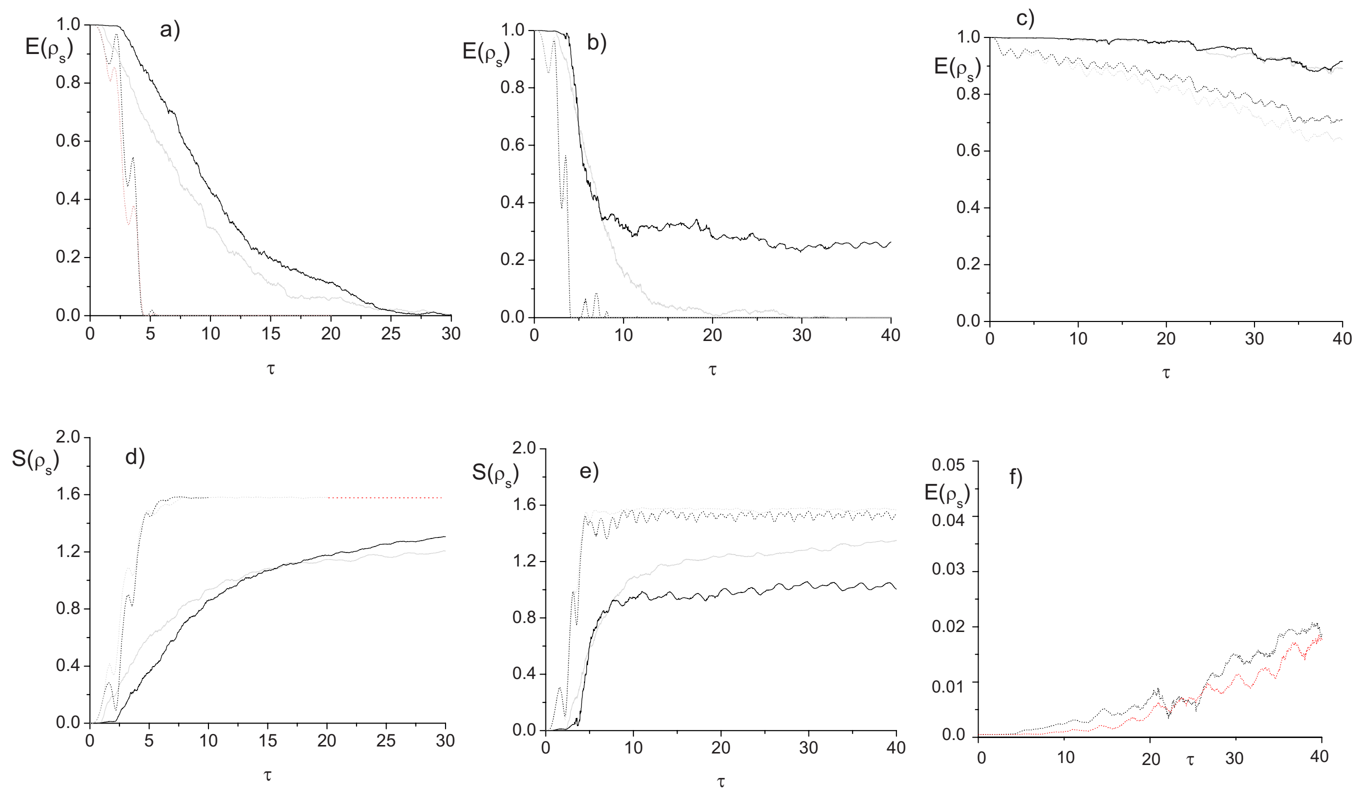

FIG. 5. (Color online) Dynamics of entanglement between qubits $E(\rho_s)$ (dimensionless) (a), (b), (c), (f) and von Neumann entropy (dimensionless) (d) and (e) of the qubit pair $S(\rho_s)$ for $\beta=0.01$ (a) (d); $\beta=0.1$ (b), (e) and $\beta=1$ (c), (f). Parameter values on various curves are $\mu=0.1$, $g=0.3$ (dotted gray) (strong coupled, chaotic); $\mu=0.1$, $g=0.03$ (dotted black) (strong coupled, regular); $\mu=0.01$, $g=0.3$ (full gray) (weak coupled, chaotic); $\mu=0.01$, $g=0.03$ (full black) (weak coupled, regular). Entanglement creation from separable $|\uparrow\uparrow\rangle$ initial state is illustrated in (f).

$$E(\rho_s) = h\left(\frac{1+\sqrt{1-C(\rho_s)^2}}{2}\right). \tag{18}$$

Besides the dynamics of entanglement between the qubits we shall also be interested in the entanglement of the qubit pair as one quantum subsystem and the dissipative oscillator as the other subsystem. An estimate of the maximum possible entanglement between the qubits and the dissipative oscillator is given by the von Neumann entropy of the reduced density matrix of the qubits pair $S(\rho_s) = \mathrm{Tr}[\rho_s \log_2(\rho_s)]$. Let us stress that the total system consisting of qubits and the dissipative oscillator as described by the QSD formalism is always in a pure state during its evolution. Nevertheless, all consistent experimental predictions are obtained from the mixed state that is obtained by averaging (1). However, the entanglement of formation is not given by stochastic averaging over many trajectories satisfying Eq. (2) of the entanglement calculated as if the system is in the pure state. The definition of entanglement of formation requires minimization over all equivalent representations of the mixed state in terms of convex combinations of pure states [9,53]. This is why we first computed, using the QSD approach, the reduce density matrix ρ_s, which represents the description of the qubits pair state, and then used it for the computations of the concurrence $E(\rho_s)$ and the entropy $S(\rho_s)$.

A. Numerical results

The results of our computations are presented in Fig. 5. We have illustrated the dynamics of the qubit entanglement $E(\rho_s)$ and $S(\rho_s)$ for the oscillator system with the parameter $\beta=0.01$ [Figs. 5(a) and 5(d)], 0.1 [figs. 5(b) and 5(e)], and $\beta=1$ [figs. 5(c) and 5(f)]. As the initial state we use the product of the coherent state $|\alpha\rangle$ with $\alpha=0.25+i0.2$ for the oscillator and for the qubit pair we used either the maximally entangled state $(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle)/\sqrt{2}$ [in Figs. 5(a)–5(e)], or a product of pure states $|\uparrow\uparrow\rangle$ [in Fig. 5(f)]. The entanglement dynamics depends on all three parameters β, μ, and g. Notice that the characteristic time intervals of the entanglement dynamics in Figs. 5(a)–5(e) are an order of magnitude shorter then the time intervals that are needed to infer the long-term dynamics of the qubits in Figs. 3 and 4.

B. Entanglement dynamics from initially maximally entangled state

When $\beta=0.01$ the dissipative oscillator behaves similar to the classical system. The dependence on the interaction strength dominates, but for a fixed value of μ the dependence of $E(\rho_s)$ and $S(\rho_s)$ on g is also clearly manifested. The chaotic oscillator induces more efficient decrease of the entanglement $E(\rho_s)$ between the qubits than the regular one. It is interesting to observe [Figs. 5(b) and 5(e)] that variation of

g for fixed weak coupling μ and $\beta=0.1$ can change the asymptotic dynamics of the entanglement from $E(\rho_s)\to 0$ when $g=0.3$ (chaotic) to $E(\rho_s)>0$ when $g=0.03$ (regular periodic). Qualitatively similar dependence on the value of μ and g is observed also in the deeply quantum regime. Even for $\beta=1$, larger values of μ, and for fixed μ larger values of g, lead to a faster decrease of the entanglement between the qubits [Fig. 5(c)]. Thus it is not clear if the observed dependence of the entanglement dynamics on the parameter g is due to the qualitatively different dynamics of the oscillator or just due to the dependence on the value of the parameter g. In fact the latter conclusion is supported by the fact that in the strongly quantum regime no qualitatively different dynamics of the oscillator is displayed for different values of g, but nevertheless the entanglement dynamics shows similar dependence on g as in the semiclassical regime.

C. Generation of entanglement by the common environment

It is known that the entanglement between the (q,p) degrees of freedom can be enhanced by their interaction with the common "environment," described by the (Q,P) system [49,50]. In our case it can be observed that the evolution of $E(\rho_s)$ for strong coupling μ is not monotonic even when $\beta=0.01$. When $\beta=0.1$ one can clearly see [Fig. 5(b)] that $E(\rho_s)$ can be created at times after its been completely annulated, i.e., from separable states. It is known that standard environments consisting of a sufficiently large number of harmonic oscillators can mediate interaction and induce entanglement in a pair of qubits initially in a separable state. However, notice that the analyzes presented in [49] does not apply here since the qubits self-Hamiltonian H_s (7) cannot be neglected, and does not commute with Eq. (8). Nevertheless we have observed such environment-induced creation of entanglement $E(\rho_s)$ between the qubits starting from a separable initial state [Fig. 5(f)]. The reappearance of entanglement that can be seen for strong coupling $\mu=0.1$ and $\beta=0.1$ is such that the entanglement is nonzero for a brief period of time and eventually is annulated permanently. On the other hand, in the deeply quantum regime when $\beta=1$ the entanglement can be created from separable initial states and is preserved for a long period of time. In this regime the nonlinear dissipative oscillator appears as a very unsuccessful decoherer for the qubit system, as is clearly seen from Figs. 5(c)–5(f). corresponding to $\beta=1$ and to some extent to $\beta=0.1$ [Fig. 5(b)]. The creation from separable initial states of small but long lasting nonzero entanglement when $\beta=1$ is illustrated in Fig. 5(f). The value of the coupling constant in this figure is $\mu=0.1$ and corresponds to cases illustrating the faster decrease of entanglement for the stronger coupling in Figs. 5(a)–5(c). The entanglement created from the initially separable state decreases with the weaker coupling, and for $\mu=0.01$ and any g, is negligibly small, i.e., numerically it is zero, during the studied time interval.

V. SUMMARY AND DISCUSSION

We have studied decoherence dynamics of a pair of qubits due to their interaction with the dissipative nonlinear quantum system, exemplified by the quantized driven Duffing oscillator. Due to the dissipation in the Duffing oscillator part the system must be treated as an open quantum system. Dynamics of an individual quantum system in interaction with the environment is described by stochastic unraveling of the density matrix dynamical master equation. We have used a particularly convenient form of such an unraveling provided by the theory of quantum state diffusion. Description of individual quantum systems in terms of their trajectories in the Hilbert space, and the localization, due to the dissipation and fluctuations, of these trajectories onto trajectories in properly defined phase space enables one to treat the qualitative properties of the quantum system dynamics in a way that is as close as possible to the classical theory of dynamical systems. Variations of two parameters crucially alter the dynamics of the quantum Duffing oscillator. In the limit when the parameter $\beta\to 0$, which is analogous to $\hbar\to 0$, the quantum Duffing oscillator behaves as the corresponding classical system, and then the parameter g is the bifurcation parameter that determines the qualitative properties of the oscillators dynamics. Small g imply regular stationary or periodic dynamics and large g entail the chaotic dynamics.

Our main results concern two related types of problems. First, we have compared the dynamics of the single quantum system with the corresponding classical model, and second, we have studied the dynamics of entanglement between qubits with the nonlinear environment in different regimes.

Concerning the comparison of the classical approximations with the quantum system in different regimes the following picture is suggested by our numerical computations. Due to the dissipation by the environments interaction with the oscillator part, the state of the total system is localized on small areas of the oscillator's phase space and consequently the oscillator's dynamics in the limit of small β is well approximated by the classical model, despite the interaction with the qubits. This is true for all qualitatively different dynamical regimes of the oscillator. On the other hand, the exact quantum dynamics of the qubits is not reproduced by the classical model, even when the oscillator is behaving classically. This observation is similar to the conclusions about quantum-classical transition in coupled linear oscillator-single spin system reported in Ref. [40]. The reason for such failure of the classical model in the qubits domain is that the entanglement between each of the qubits with the oscillators cannot be described by the classical way of treating the qubit-oscillator interaction.

We have also studied the evolution of entanglement between the qubits and of the qubits von Neumann entropy, as indicators of the decoherence of the qubits by the interaction with the dissipative oscillator. Behavior of the entanglement and the entropy from initially maximally entangled and from separable qubits state was investigated. It is found that decoherence is much more effective when the dissipative oscillator is in the semiclassical rather then in the fully quantum regime, for the same values of the parameters characterizing the nonunitary effects in the oscillator dynamics. Furthermore, larger values of the bifurcation parameter g, corresponding to the chaotic dynamics of the oscillator in the semiclassical regime, imply more rapid decrease of the qubits pair entanglement and faster increase of the qubits von

Neumann entropy than the smaller values of g corresponding to regular motion. In the semiclassical regime, i.e., for β sufficiently small any value of g and qubit-oscillator coupling strength μ leads to annulation of the interqubit entanglement after finite time. Contrary to the case of environment consisting of a large number of harmonic oscillators in the thermal equilibrium, the decrease of the interqubit entanglement is not monotonic. On the other hand, in the fully quantum regime the entanglement between the qubits can be created from initially separable state and appears to converge at long times to finite nonzero values. Te variations in the typical behavior of the entanglement as the parameter β is gradually decreased from $\beta=1$ to $\beta \ll 1$ seems to be continuous.

Properties of decoherence of a quantum system by another coarse-grained quantum system considered as environment are related to the correlation function of the system that plays the role of the environment. In our case, the open quantum system could be the pair of qubits, in which case the oscillator is treated as a part of the nonlinear environment that can show classical or quantum features depending on β, or the open system could be the pair of qubits in interaction with conservative quantum nonlinear oscillator. In the later case the open system is immersed in an environment that interacts only with the oscillator and produces dissipation, localization, and renders the classical behavior of the large oscillator, i.e., in the classical regime $\beta \to 0$. The dynamical properties of the systems playing the role of the environment in the two possible arrangements, including their correlation functions, are different and yet the decoherence of the qubits systems in the two cases is by the construction the same. This illustrates the importance of the type of coupling between the quantum system and the environment on the properties of decoherence of the quantum system.

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