Emergence of classical behavior from the quantum spin

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A classical Hamiltonian system of a point moving on a sphere of fixed radius is shown to emerge from the constrained evolution of quantum spin. The constrained quantum evolution corresponds to an appropriate coarse graining of the quantum states into equivalence classes, and forces the equivalence classes to evolve as single units representing the classical states. The coarse-grained quantum spin with the constrained evolution in the limit of the large spin becomes indistinguishable from the classical system.

I. INTRODUCTION

It is generally agreed that all systems in nature are described by appropriately formulated quantum theory. Therefore, the fact that some systems, usually of macroscopic size, can be described by a radically different theory, namely, the classical physics, requires an explanation. Clarifying the meaning in which the behavior described by the classical physics emerges from the quantum substrate is one of the main topics of the theory of quantum-to-classical relation (QCR).

Multifarious aspects of the problem of QCR have been analyzed since the early formulations of quantum mechanics. The relevant literature on the topic is vast and we shall single out as illustrative examples only a few reviews: The relevance of QCR to the notorious problem of quantum measurement is often discussed in the papers collected in Refs. [1] and [2], which contain references to more recent developments. Some of the more formal mathematical aspects of QCR are treated in Ref. [3]. Dynamical aspects of QCR have been intensively discussed within the framework of the correspondence principle and the semiclassical methods for classically chaotic systems [4]. Putative physical mechanisms and the appropriate ontological considerations underlying QCR are discussed from different points of view for example in Refs. [5–9]. During the last couple of decades, detailed experimental studies of the problems related to QCR have been performed (see, for example, [10–12]).

It has been realized many times that quantum and classical systems are related by some sort of coarse graining. The coarse graining enters differently in different theories of QCR, and is not always equally strongly emphasized. In the theories of decoherences [5,6], the emphasis is on the influence of the environment, but the description of the environment must be coarse grained to fulfill the desired decoherence effects. On the other hand, authors such as in Refs. [13] and [9,14], to mention just a few representatives of the approach from two different periods and backgrounds, emphasize the primary role of the coarse graining, associated with the limited precision of the devices used to observe the quantum system.

In this paper, we shall analyze the role of an appropriate coarse graining for the emergence of a class of classical Hamiltonian systems characterized by the spherical phase space. The points of the phase space are parameterized by the spherical

angles (θ,ϕ) or by the Cartesian coordinates (J_x, J_y, J_z) constrained by $J_z^2 + J_y^2 + J_z^2 = J^2$, and the classical Hamiltonian is a smooth function of $J_x(\theta,\phi), J_y(\theta,\phi), J_z(\theta,\phi)$. Our goal is to show how the classical systems on the sphere can be derived from the quantum spin of size J, i.e., quantum systems with su(2) dynamical algebra and $(2J + 1)$-dimensional Hilbert space of states. We shall see that the derivation of the classical system is done in two independent steps, both of which are necessary. The first step consists of an appropriate coarse graining, which introduces the classical phase space. A classical-like Hamiltonian system is then defined on this phase space by appropriately constraining the quantum Schrödinger evolution. The second step is the macroscopic limit applied on this coarse-grained and constrained system. As a result, the coarse-grained and constrained system becomes indistinguishable from a classical system on the sphere.

The same ideas have been recently utilized to study the appearance of a classical system of nonlinear oscillators from the corresponding quantum system [15]. The two examples suggest the formulation of a general procedure, which shall be briefly discussed.

The paper is organized as follows. In the next section, we recapitulate the geometric Hamiltonian formulation of quantum mechanics and of constrained quantum dynamics, with special emphasis on the system with su(2) dynamical algebra, i.e., the quantum spin. This representation is used in Sec. III to construct the classical model with the same dynamical algebra. Section IV discusses the appearance of the classical system from the macro limit of the coarse-grained and constrained quantum system. In Sec. V, we summarize our presentation.

II. HAMILTONIAN FORMULATION OF CONSTRAINED QUANTUM DYNAMICS

Investigations of the relations between classical and quantum mechanics are facilitated if both theories are expressed using similar mathematical language. The geometric Hamiltonian formulation [16–25] and the geometric theory of coherent states [26] are two such representations of quantum mechanics which are formulated in terms of mathematical objects that are typical for classical Hamiltonian mechanics. In this section, we shall briefly recapitulate the geometric Hamiltonian formulation for the case of a quantum system with finite-dimensional Hilbert space, since this will be the main tool of our analysis. In particular, we shall summarize

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the recently introduced description of constrained quantum systems [27–29] within this formalism. A group-theoretical and geometric treatment of the generalized coherent states, as it was introduced by Perelomov [26], shall be used when needed without prior recapitulation.

A. Hamiltonian framework for quantum systems

The Schrödinger dynamical equation on a separable and complete Hilbert space $\mathcal{H}$ generates a Hamiltonian dynamical system on an appropriate symplectic manifold. The symplectic structure, which is needed for the Hamiltonian formulation of the Schrödinger dynamics, is provided by the imaginary part of the unitary scalar product on $\mathcal{H}$. In fact, the Hilbert space $\mathcal{H}$ is viewed as a real manifold $\mathcal{M}$ with a complex structure, given by a linear operator $\mathcal{J}$ such that $\mathcal{J}^2 = -1$. If $\mathcal{H}$ is finite n dimensional, then $\mathcal{M} \equiv \mathbb{R}^{2n}$, but, in general, $\mathcal{M}$ is an infinite-dimensional Euclid manifold. In what follows, we shall consider only the finite-dimensional Hilbert spaces, since the irreducible representations of the spin J, i.e., the su(2) algebra, are of finite dimension: $n \equiv \dim \mathcal{H} = 2J + 1$. Real coordinates (x_i, y_i) of a point $\psi \in \mathcal{H} \equiv \mathcal{M}$ are introduced using expansion coefficients c_i in some basis $\{|i\rangle, i = 1, 2, \ldots, n\}$ of $\mathcal{H}$ as follows:

$$|\psi\rangle = \sum_i c_i |i\rangle, \quad c_i = \frac{x_i + i y_i}{\sqrt{2}}, \tag{1a}$$

$$x_i = \sqrt{2}\, \mathrm{Re}(c_i), \quad y_i = \sqrt{2}\, \mathrm{Im}(c_i), \quad i = 1, 2, \ldots, n. \tag{1b}$$

The real manifold $\mathcal{M} = \mathbb{R}^{2n}$ has a Riemannian and symplectic structure. Since $\mathcal{M}$ is real, it is natural to decompose the unitary scalar product on $\mathcal{H}$ into its real and imaginary parts,

$$\langle \psi_1 | \psi_2 \rangle = \frac{1}{2\hbar} g_{\mathcal{M}}(\psi_1, \psi_2) + \frac{i}{2\hbar} \omega_{\mathcal{M}}(\psi_1, \psi_2). \tag{2}$$

It follows that $g_{\mathcal{M}}$ is Riemannian metric on $\mathcal{M}$ and that $\omega_{\mathcal{M}}$ is the symplectic form on $\mathcal{M}$. Thus, the manifold $\mathcal{M}$ associated with the Hilbert space $\mathcal{H}$ can be viewed as a phase space of a Hamiltonian dynamical system, additionally equipped with the Riemannian metric which reflects its quantum origin. A vector from $\mathcal{H}$, associated with a pure quantum state, is represented by the corresponding point in the phase space $\mathcal{M}$. We shall denote the point from $\mathcal{M}$ associated with the vector $|\psi\rangle$ by X_ψ.

In the coordinates (x_i, y_i), the Riemannian and the symplectic structures of $\mathcal{M}$ are given by

$$g_{\mathcal{M}} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \tag{3}$$

$$\omega_{\mathcal{M}} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}, \tag{4}$$

where $\mathbf{0}$ and $\mathbf{1}$ are zero and unit matrices of dimension equal to the dimension of the Hilbert space. Thus, coordinates (x_i, y_i) represent canonical coordinates of a Hamiltonian dynamical system. Consequently, the Poisson bracket between two functions F_1 and F_2 on $\mathcal{M}$ corresponding to the symplectic

form $\omega_{\mathcal{M}}$ is in the canonical coordinate (x_i, y_i) representation given by

$$\{F_1, F_2\}_{\mathcal{M}} = \sum_i \left(\frac{\partial F_1}{\partial x_i} \frac{\partial F_2}{\partial y_i} - \frac{\partial F_2}{\partial y_i} \frac{\partial F_1}{\partial x_i} \right). \tag{5}$$

A one-parameter family of unitary transformations on $\mathcal{H}$ generated by a self-adjoined operator $\hat{H}$ is represented on $\mathcal{M}$ by a flow generated by the Hamiltonian vector field $\omega_{\mathcal{M}}(-\mathcal{J}\hat{H}\psi, \cdot) = (dH)(\cdot)$, with the Hamilton's function given by

$$H(X_\psi) = \langle \psi | \hat{H} | \psi \rangle. \tag{6}$$

Thus, quantum observables $\hat{H}$ are represented by functions of the form $\langle \hat{H} \rangle_\psi$. Such and only such Hamiltonian flows, with the Hamilton's function of the form (6), also generate isometries of the Riemannian metric $g_{\mathcal{M}}$. More general Hamiltonian flows on $\mathcal{M}$, corresponding to the Hamilton's function which are not of the form (6), do not generate isometries and do not have the physical interpretation of quantum observables.

It can be seen easily that

$$\{H_1, H_2\}_{\mathcal{M}} = \frac{1}{i\hbar} \langle [\hat{H}_1, \hat{H}_2] \rangle. \tag{7}$$

In the remaining text, we will take $\hbar = 1$. The Schrödinger evolution equation

$$|\dot{\psi}\rangle = -i \hat{H} |\psi\rangle \tag{8}$$

is equivalent to the Hamilton's equations on $\mathcal{M}$,

$$\dot{X}_\psi^a = \omega^{ab} \nabla_b H(X_\psi). \tag{9}$$

In the canonical coordinates (x_i, y_i), the Schrödinger evolution is given by

$$\dot{x}_i = \frac{\partial H}{\partial y_i}, \quad \dot{y}_i = -\frac{\partial H}{\partial x_i}. \tag{10}$$

We have constructed the Hamiltonian dynamical system corresponding to the Schrödinger evolution equation on $\mathcal{H}$. In fact, phase invariance and arbitrary normalization of the quantum states imply that the proper space of pure quantum states is not the Hilbert space used to formulate the Schrödinger equation, but the projective Hilbert space. This is also a Kähler manifold and can be used as a phase space of a geometrical Hamiltonian formulation of quantum mechanics. Nevertheless, we shall continue to use the formulation in which points of the quantum phase space are identified with the vectors from $\mathcal{H}$, since it is sufficient for our main purpose.

B. Constrained quantum dynamics

The Hamiltonian framework for quantum dynamics enables one to describe the evolution of a dynamical system generated by the Schrödinger equation with quite general additional constraints [27–29]. Suppose that the evolution given by the Hamiltonian H is further constrained onto a submanifold Γ of $\mathcal{M}$ given by a set of k independent functional equations,

$$f_l(X) = 0, \quad l = 1, 2, \ldots, k. \tag{11}$$

The equations of motion of the constrained system are in general obtained using the method of Lagrange multipliers.

In the Hamiltonian form, developed by Dirac [30–32], the method assumes that the dynamics on Γ is determined by the following set of differential equations:

$$\dot{X} = \omega_{\mathcal{M}}(\nabla X, \nabla H_{\text{tot}}), \quad H_{\text{tot}} = H + \sum_{l=1}^{k} \lambda_l f_l, \quad (12)$$

which should be solved together with the equations of the constraints (11). For notational convenience, we do not indicate in the gradient ∇ that it is defined on $\mathcal{M}$. The Lagrange multipliers λ_l are functions on $\mathcal{M}$ that are to be determined from the following so-called compatibility conditions:

$$
\begin{aligned}
0 = \dot{f}_l &= \omega_{\mathcal{M}}(\nabla f_l, \nabla H_{\text{tot}}) \\
&= \omega_{\mathcal{M}}(\nabla f_l, \nabla H) + \sum_{m=1}^{k} \lambda_m \omega_{\mathcal{M}}(\nabla f_l, \nabla f_m) \quad (13)
\end{aligned}
$$

on the constrained manifold Γ. We shall not go into the details of the standard Dirac's procedure that emphasizes the distinction between the first and the second class constraints. In order to apply the standard procedure, the constraints have to be regular. A set of constraints is irregular if there is at least one such that the derivative of the constraint with respect to at least one of the coordinates is zero in at least one point on the constrained manifold. Otherwise, the constraints are regular. In our case of finite-dimensional $\mathcal{M}$, the constraints are regular if for all l,

$$\frac{\partial f_l}{\partial x_i} \neq 0, \quad \frac{\partial f_l}{\partial y_i} \neq 0, \quad (14)$$

for all $i, j = 1, 2, \ldots, n$ and everywhere on the constrained manifold Γ. If this is not satisfied, then the Dirac's classification into the first and the second class is blurred and the straightforward application of Dirac's recipe is not possible. It will turn out that the case of interest here involves precisely irregular constraints that must be described in the most convenient way. However, if the constrained manifold is symplectic, then the Dirac's procedure of constructing the constrained system and reducing it on the constrained manifold can be bypassed. In fact, the result of the procedure is known to be a Hamiltonian system defined on the constrained manifold. The Hamilton's function of the reduced constrained system is just the original Hamiltonian evaluated at the constrained manifold [32].

III. COARSE-GRAINED DESCRIPTION OF THE SPIN

A. Equivalence of states

The phase space of the classical system is the sphere denoted Γ of some radius J. Pure states of the classical system are the points of the sphere, parameterized by the spherical angles (θ, ϕ), or, equivalently, by the corresponding vector $\mathbf{J}(\theta, \phi) = (J_x(\theta, \phi), J_y(\theta, \phi), J_z(\theta, \phi))$ of fixed length $|\mathbf{J}(\theta, \phi)| = J$, or by the points of the complex plane $z = -\tan(\theta/2)\exp(-i\phi)$. The symplectic structure of the classical phase space is expressed, for example, in terms of z by

$$\omega_\Gamma = 2iJ \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}. \quad (15)$$

The real coordinates (q, p) given by

$$q + ip = \frac{\sqrt{4J}z}{\sqrt{1 + |z|^2}} \quad (16)$$

are the canonical coordinates with respect to ω_Γ. The basic variables $J_x(\theta, \phi)$, $J_y(\theta, \phi)$, $J_z(\theta, \phi)$ form the su(2) algebra $\{J_x, J_y\}_\Gamma = J_z$ (and cyclic permutations) with respect to the Poisson bracket induced by (15). Thus, the dynamical algebra of the classical system is the su(2) algebra. The Hamilton function of the classical system is not necessarily an element of the su(2) algebra, but is assumed to be expressible as a simple function of $J_x(\theta, \phi)$, $J_y(\theta, \phi)$, $J_z(\theta, \phi)$.

The quantum system with the same dynamical algebra is the quantum spin. Cartesian coordinates $\hat{J}_x$, $\hat{J}_y$, $\hat{J}_z$ of the spin operator $\hat{\mathbf{J}}$ satisfy the su(2) commutation relations: $[\hat{J}_x, \hat{J}_y] = i\hat{J}_z$ (and cyclic permutations). The Hilbert space of the spin of size J is the space of the $(2J + 1)$-dimensional irreducible su(2) representation.

The su(2) coherent states $|\Omega\rangle$ can be defined as the quantum states that minimize the total quantum fluctuation of $\hat{\mathbf{J}}$ [33]: $\Delta_\psi^2 \hat{J}_x + \Delta_\psi^2 \hat{J}_y + \Delta_\psi^2 \hat{J}_z$. Thus, Ω is the coherent state if

$$\Delta_\Omega^2 \hat{J}_x + \Delta_\Omega^2 \hat{J}_y + \Delta_\Omega^2 \hat{J}_z = J. \quad (17)$$

This is the main property of the coherent states for our purposes. Alternatively, the coherent states are defined as the eigenstates corresponding to the maximal eigenvalue of the operator

$$
\begin{aligned}
(\hat{J}_x \sin\theta\cos\phi &+ \hat{J}_y \sin\theta\sin\phi + \hat{J}_z\cos\theta)|\Omega(\theta,\phi)\rangle \\
&= J|\Omega(\theta,\phi)\rangle. \quad (18)
\end{aligned}
$$

The set of coherent states is parameterized by the points of the two-dimensional spherical submanifold Γ of the $2(2J + 1)$-dimensional quantum phase space $\mathcal{M}$. Furthermore, each coherent state satisfies $J_x^2(\Omega) + J_y^2(\Omega) + J_z^2(\Omega) = J^2$, where $J_i(\Omega) \equiv \langle\Omega|\hat{J}_i|\Omega\rangle$. Thus, the coherent states are in a one-to-one relation with the points of the phase space of the classical system.

A classical pure state (θ, ϕ) does not make a distinction between pure quantum states such that the average of the vector operator $\hat{\mathbf{J}}$ is a vector collinear with $\mathbf{J}(\theta, \phi)$. Thus, we define an equivalence relation on $\mathcal{M}$ (i.e., on $\mathcal{H}$) as follows:

$$X_1 \sim X_2, \text{ if } \mathbf{J}(X_1) = \kappa \mathbf{J}(X_2), \quad (19)$$

where κ is a positive scalar. The two quantum states are equivalent if they give collinear expectation vectors $\langle\hat{\mathbf{J}}\rangle_X = \mathbf{J}(X) \equiv (J_x(X), J_y(X), J_z(X))$.

Each equivalence class of quantum pure states $[X]$ contains one and only one coherent state, i.e., an $\Omega_X \sim X$ such that $J_x^2(\Omega_X) + J_y^2(\Omega_X) + J_z^2(\Omega_X) = J^2$. The partition of $\mathcal{M}$ by the equivalence relation ($\sim$) represents the coarse-grained description of the space of quantum pure states. The coarse-grained quantum states, i.e., the coherent states, are parameterized by the classical pure states.

B. Classical dynamics: Constraining the quantum dynamics

The Schrödinger evolution equation for $\psi(t)$, or its Hamiltonian form for $X_\psi(t)$, does not preserve the equivalence

classes of quantum states (19), and the manifold of coherent states is not invariant. On the other hand, the system with the same Hamiltonian and additional constraints introduced in such a way that the manifold of coherent states is invariant also preserves the equivalence classes of quantum states. This constrained Hamiltonian system, when restricted on the manifold of coherent states, generates by definition the dynamics of the coarse-grained reduced quantum system. The constrained evolution of the quantum system with the corresponding total Hamiltonian always preserves the small total quantum fluctuation of $\hat{\mathbf{J}}$, which is the crucial property in the analysis of its macro limit. It is our goal in this section to construct the constrained Hamiltonian evolution such that the manifold of coherent states is preserved. Due to the unique representation of the equivalence classes by coherent states, this condition on the evolution also implies that the coarse-grained states, i.e., the equivalence classes, evolve as a single unit.

The manifold of coherent states Γ is uniquely determined by the total quantum fluctuation of the basic operators, $\hat{J}_x, \hat{J}_y, \hat{J}_z$, which is minimal if and only if the state is a coherent state. Thus, the constraint

$$\Phi(X) = \Delta_X^2 \hat{J}_x + \Delta_X^2 \hat{J}_y + \Delta_X^2 \hat{J}_z - J = 0 \qquad (20)$$

determines the sphere of coherent states. The evolution of the coarse-grained system is defined to be the constrained Hamiltonian evolution with the given Hamiltonian $H(X)$ and the constraint (20).

The master constraint that we want to fulfill is given by the function (20), but that constraint is not regular because it is equivalent with $J_x^2(X) + J_y^2(X) + J_z^2(X) - J^2 = 0$. Similarly to the case of oscillators treated in Ref. [15], the application of Dirac's procedure with this constraint as the initial primary one is not straightforward, and would imply an additional number of secondary constraints. However, since the constrained manifold Γ is known to be symplectic, the constrained system reduced on Γ is Hamiltonian with the Hamilton's function given simply by $H(\Omega) = \langle \Omega | \hat{H} | \Omega \rangle$.

Alternatively, one could obtain the constrained evolution equations on the full space $\mathcal{M}$ using an appropriate form of the primary constraints and then reduce the constrained system on the constrained manifold. Following the idea presented in Ref. [15], one should replace the irregular primary constraint (20) by a more effective equivalent primary constraint, formulated using the equivalence of states. The primary constraint that should be imposed would require that the average of the Hamiltonian $H(X)$ is equal to its average in the equivalent coherent state Ω_X,

$$\Phi(X) = H(X) - H(\Omega_X) = 0. \qquad (21)$$

The Lagrange multiplier in the total Hamiltonian $H_{\text{tot}}(X) = H(X) - \lambda \Phi(X)$ is simply $\lambda = 1$, and thus the total Hamiltonian is

$$H_{\text{tot}}(X) = H(\Omega_X). \qquad (22)$$

As already stated, the restriction of the evolution of the resulting constrained system onto the constrained manifold of coherent states is guided simply by the Hamiltonian $H(\Omega)$.

In summary, the Hamilton's function of the coarse-grained and reduced system is just the $\langle \Omega | \hat{H} | \Omega \rangle$. The states of the coarse-grained system are equivalence classes represented by the coherent states. The quantum constrained dynamics preserves the equivalence classes and the total quantum fluctuation remains minimal (but nonzero) throughout the evolution. Of course, the coarse-grained system is not classical. The coherent states have nonzero overlap and the quantum fluctuations are, although minimal, nonzero. For example, if the quantum Hamiltonian is a nonlinear expression in terms of the basic operators $\hat{J}_x, \hat{J}_y, \hat{J}_z$, then its expectation $\langle \Omega | \hat{H}(\hat{J}_x, \hat{J}_y, \hat{J}_z) | \Omega \rangle$ is different from $H(\langle \Omega | \hat{J}_x | \Omega \rangle, \langle \Omega | \hat{J}_y | \Omega \rangle, \langle \Omega | \hat{J}_z | \Omega \rangle)$ where $H(\cdot)$ is of the same form as $\hat{H}(\cdot)$. Due to the constraint dynamics of the coarse-grained model, the total quantum fluctuations are minimal and the difference between those two expressions is always bounded by terms of the leading order $1/J$. The difference becomes arbitrarily small as J becomes sufficiently large. The macro limit of the coarse-grained system is discussed in the next section.

Notice that the reduced constrained Hamilton's function $H(\Omega) = \langle \hat{H} \rangle_\Omega$ is a valid Hamiltonian for any Hermitian operator $\hat{H}$, while the Hamilton's function of the classical model must be a function of the expectations of the basic operators $\langle \hat{J}_x \rangle_\Omega, \langle \hat{J}_y \rangle_\Omega, \langle \hat{J}_z \rangle_\Omega$.

Sometimes the Hamiltonian of a quantum system is a sum of the terms linear in the dynamical algebra generators and a small perturbation containing some nonlinear terms. In such cases, the manifold of coherent states of the dynamical algebra is approximately invariant over some finite time. This fact has been used (see, for example, [34], or, more recently, [35,36]) to propose and study an approximation of the exact quantum dynamics, from some coherent state or a mixture of such, by the equations of the classical Hamiltonian form with the Hamilton's function given by $\langle \hat{H} \rangle_\Omega$. In our approach, the reduced constrained Hamiltonian equations of the coarse-grained system appear as the evolution equations of the equivalence classes of the quantum states. The equations are the evolution equations of the coarse-grained quantum system, whatever the form of the Hamilton's operator.

IV. MACRO LIMIT OF THE REDUCED CONSTRAINED SYSTEM AND THE CLASSICAL MODEL

In this section, we want to demonstrate that for sufficiently large J, the states of the coarse-grained system and their evolution display properties of a classical system. In particular, we shall show that for sufficiently large J, (a) the coarse-grained system appears to be in one and only one classical state, (b) the state of the coarse-grained system can be determined without measurable disturbance, (c) the evolution of the coarse-grained system is such that the property (a) is valid for all times, or, in other words, ratios $\Delta \hat{J}_i / \langle \hat{J}_i \rangle (i = x, y, z)$ remain arbitrarily small during the evolution, and (d) the evolutions of the coarse-grained system and of the classical model become indistinguishable. As a consequence of these properties, the reduced constrained system with sufficiently large J is the same as the classical model for all observational devices with arbitrary but finite accuracy.

The properties (a), (b), and (c) are based on the fact that the overlap between the coherent states, $|\langle \Omega | \Omega' \rangle| = \cos^{2J}(\alpha/2)$,

becomes arbitrarily small for sufficiently large J, where α is the angle between directions (θ,ϕ) and (θ',ϕ') corresponding to Ω and Ω', respectively. Thus, for an observation with arbitrary but finite accuracy, different coherent states appear as orthogonal.

The property (a) is, in fact, the same as the near orthogonality of the coherent states for large J. To demonstrate the property (b), consider the measurement in the overcomplete basis given by the coherent states. Upon such measurement, the representative of the coarse-grained state $|\Omega\rangle$ is transformed into $|\Omega'\rangle = \int d\Omega''|\Omega''\rangle\langle\Omega''|\Omega\rangle$, which is approximately equal to $|\Omega\rangle$ due to the near orthogonality of the coherent states for large J. Thus, disturbance of the states of the reduced system with sufficiently large J by the measurement of classical properties is negligibly small.

The evolution of the reduced system is defined precisely such that the dispersions $\Delta\hat{J}_i$ ($i = x,y,z$) remain small, and thus the ratios $\Delta\hat{J}_i/\langle\hat{J}_i\rangle$ are arbitrarily small for sufficiently large J. This implies, in particular, that $\langle\Omega|f(\hat{J}_x,\hat{J}_y,\hat{J}_z)|\Omega\rangle = f(\langle\Omega|\hat{J}_x|\Omega\rangle,\langle\Omega|\hat{J}_y|\Omega\rangle,\langle\Omega|\hat{J}_z|\Omega\rangle) + O(1/J)$, where f is an arbitrary polynomial expression. Thus, the evolution of the coarse-grained and reduced system with large J and the Hamiltonian $\langle\Omega|\hat{H}(\hat{J}_x,\hat{J}_y,\hat{J}_z)|\Omega\rangle$ is indistinguishable from the classical evolution generated by the Hamilton's function $H(\langle\Omega|\hat{J}_x|\Omega\rangle,\langle\Omega|\hat{J}_y|\Omega\rangle,\langle\Omega|\hat{J}_z|\Omega\rangle)$. We can conclude that the reduced constrained system for large J displays all of the typical properties of a classical Hamiltonian system in all physically possible observations.

V. AN EXAMPLE

If the Hamiltonian of the quantum system is a nonlinear expression of the basic operators of the dynamical algebra, then the Schrödinger evolution with the Hamiltonian $\hat{H}(\hat{J}_x,\hat{J}_y,\hat{J}_z)$, the evolution of the reduced constrained system by the Hamiltonian H_{tot} (22), and the evolution of the classical model with the Hamiltonian $H(\langle\hat{J}_x\rangle,\langle\hat{J}_y\rangle,\langle\hat{J}_z\rangle)$, starting from the same coherent state, are all different. However, when the spin size J increases, the difference between the evolution of the coarse-grained and reduced system and of the classical model decreases. Thus, for sufficiently large J, the differences become negligible over arbitrarily large periods of time. These facts are illustrated in this section using as an example the following Hamiltonian:

$$\hat{H} = \epsilon\hat{J}_z - \lambda\hat{J}_x + \mu\hat{J}_z^2, \qquad (23)$$

where ϵ,λ, and μ are parameters. The Hamiltonian (23) appears as the most convenient form of the two-mode Bose-Hubbard model [36].

The constrained system reduced on Γ is a Hamiltonian system on Γ with the Hamilton's function given by $\langle\hat{H}\rangle_\Omega$. To express it as a function of the canonical coordinates (p,q) of Γ, we need the appropriate expectations of the linear and quadratic operators in terms of (p,q). The relevant formulas
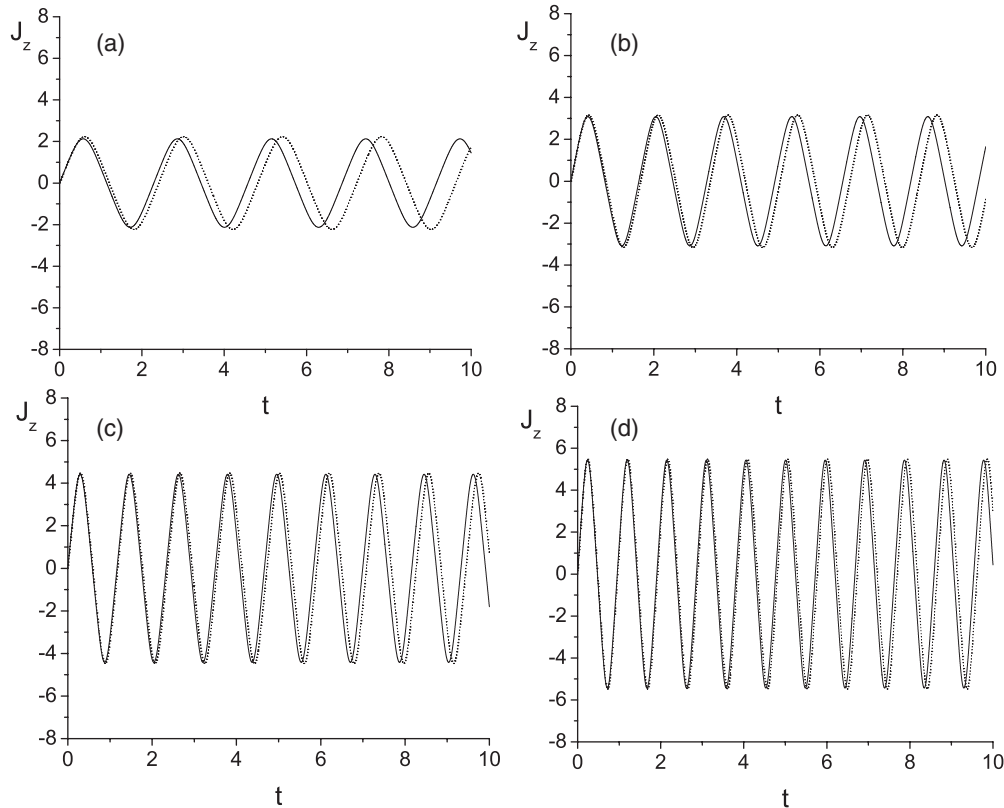

FIG. 1. Illustration of the constrained (dotted line) and classical (solid line) evolution for $J = $ (a) 5, (b) 10, (c) 20, and (d) 30. Time series $J_z(t) = \langle\Omega(t)|\hat{J}_z|\Omega(t)\rangle$ are shown. All quantities are dimensionless. Parameters are $\lambda = \mu = 1, \epsilon = 0$ and the initial state is the coherent state $|\Omega_0\rangle = (\theta = \pi/2, \phi = 0)$.

are given by

$$\langle \hat{J}_x \rangle (p,q) = \frac{q}{2}(4J - q^2 - p^2)^{1/2},$$

$$\langle \hat{J}_y \rangle (p,q) = -\frac{p}{2}(4J - q^2 - p^2)^{1/2}, \qquad (24)$$

$$\langle \hat{J}_z \rangle (p,q) = \frac{1}{2}(q^2 + p^2 - 2J),$$

and

$$\left\langle \hat{J}_z^2 \right\rangle = \langle \hat{J}_z \rangle^2 (p,q) + \frac{1}{8J}(p^2 + q^2)(4J - p^2 - q^2). \quad (25)$$

The last term in Eq. (25), proportional to $1/J$, represents the quantum correction to the expectation of the nonlinear operator $\hat{J}_z^2$.

The Hamilton's function $H(p,q) \equiv H(\Omega(p,q))$ of the constrained system reduced on Γ is given by

$$H(p,q) = \frac{\epsilon}{2}(q^2 + p^2 - 2J) - \lambda \frac{q}{2}(2J - p^2 - q^2)^{1/2}$$

$$+ \mu \left[\frac{1}{4}(q^2 + p^2 - 2J)^2 + \frac{1}{8J}(p^2 + q^2) \right.$$

$$\left. \times (4J - p^2 - q^2) \right]. \qquad (26)$$

The Hamilton's function of the classical model is obtained by assuming that the last term in square brackets in Eq. (26) is, in fact, equal to zero, i.e., $\langle \Omega | \hat{J}_z^2 | \Omega \rangle = \langle \Omega | \hat{J}_z^2 | \Omega \rangle^2$. The classical Hamiltonian reads

$$H_{cl}(p,q) = \frac{\epsilon}{2}(q^2 + p^2 - 2J) - \lambda \frac{q}{2}(2J - p^2 - q^2)^{1/2}$$

$$+ \frac{\mu}{4}(q^2 + p^2 - 2J)^2. \qquad (27)$$

In Fig. 1, we illustrate the evolution of the average $\langle \hat{J}_z \rangle$ generated by the constrained system (26) and by the classical model (27). The initial state for the constrained system is the coherent state ($\theta = \pi/2, \phi = 0$) implying ($p_0 = \sqrt{2J}, q_0 = 0$). The main conclusion of Fig. 1 is that the constrained and the classical evolutions become indistinguishable as J becomes sufficiently large.

VI. DISCUSSION AND SUMMARY

We would now like to compare the coarse graining, whose fundamental role is analyzed in this paper, with two different types of coarse graining commonly used in the studies of micro-macro transition. We consider the coarse graining (a) in the mean-field-type approach to the appearance of macro properties and (b) motivated by the finite precision of the measuring instruments. Let us first discuss the coarse-graining of the type (a). A typical example is the treatment of macroscopic magnetization defined as the average of the spin components over an ensemble of spins: $\hat{J}_i = \sum_k \hat{\sigma}_i^k$ ($i = x, y, z$). All states

of the ensemble which give the same average of the large spin $\hat{\mathbf{J}}$ components are considered equivalent. However, the large spin is a quantum system which can be in states corresponding to superpositions of macroscopically distinct eigenvalues of its observables. No classical behavior is implied by the coarse graining that replaced the ensemble of spins by the single large spin [14,37]. Furthermore, the coarse graining that treats as indistinguishable the eigenstates of an observable with nearby eigenvalues also does not introduce the classical behavior. Thus the coarse graining of type (a) or of type (b), although justified, is not crucial for the explanation of the emergence of classical systems.

To summarize, we have analyzed the conditions for the emergence of a classical Hamiltonian dynamical system on the sphere from the quantum spin. The main condition behind the classical appearance is that only a limited set of averages of quantum observables is distinguished in the classical system. This naturally leads to the corresponding equivalence relation among the quantum states. The equivalence relation represents a type of coarse graining, such that in each equivalence class, there is one and only one state with minimal total quantum fluctuation, i.e., the corresponding coherent state. If the equivalence classes or their representatives are to be identified with states of the appropriate classical system, then they must evolve as single units. This leads to the constrained Hamiltonian dynamics, which preserves the manifold of coherent states. For an observer with sufficiently limited observational accuracy, the constrained system reduced on the manifold of coherent states displays all of the characteristic properties of a classical system. The upper limit on the observational accuracy with which the system appears as classical can be increased as the size of the spin J is increased, but is ultimately limited by the physical nature of possible observational devices.

In Ref. [15], we have applied the same ideas and methods to explain the emergence of the classical system of oscillators from the corresponding quantum system. The important difference, of ultimately geometrical origin, between the system of oscillators and the spin is in the form of the equivalence relation, i.e., the form of the coarse graining. Nevertheless, in both cases, the role of the coarse graining is the same. The two examples of the oscillators and the spin suggest a general explanation of the emergence of classical models from quantum systems with the same dynamical Lie algebra.

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