Hybrid quantum-classical model of quantum measurements

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Two types of Hamiltonian hybrid quantum-classical theories are considered as potential models of the quantum measurement process. The two theories have the same Hamiltonian dynamics but differ in the association of states of the quantum system with states of the hybrid model. In the first type of association pure quantum states are modeled by pure states of the hybrid, while in the second type the pure quantum states are modeled by statistical mixtures of the hybrid pure states. It is shown that the second of the two theories describes correctly the quantum measurement while the first provides only an averaged description.

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The first phenomenological model of the measurement process in quantum mechanics (QM), which is mathematically explicit and rigorous, was given by von Neumann [1]. The model postulates mathematical form of the change undergone by the state vector of the system composed by the measurement apparatus (A) and the measured system (S), when an ideal measurement of a physical quantity, represented by a Hermitian operator, is performed. The type of change experienced by the system + apparatus (SA) state vector in the measurement process is called the collapse of the state vector and is qualitatively different from any other dynamical process occurring in an isolated quantum system. If the state vector is interpreted as a real property of the SA system, and with other standard axioms of the von Neumann type, the special status of the collapse process among all other physical processes demands an explanation. Many attempts have been made to provide at least an approximate description of the state vector collapse in terms of usual dynamical processes in a SA system [2,3]. One type of attempt to model quantum measurement, if not explain it, considers the SA system as a novel kind of so-called hybrid quantum-classical systems. Such models start with an isolated quantum system S and an isolated classical system A, which are then allowed to interact and form the hybrid SA system, with its own type of states and the corresponding hybrid dynamics. Hybrid systems are interesting independent of their application in modeling the measurement, and several hybrid theories have been proposed (for a recent review, see Ref. [4]). Some of the suggested hybrid theories are mathematically inconsistent, and "no-go" theorems have been formulated [5], suggesting that no consistent hybrid theory can be formulated. Nevertheless, mathematically consistent hybrid theories exist [4,6–8].

The goal of this communication is to compare descriptions of the quantum measurement in two mathematically consistent theories of hybrid quantum-classical systems. The dynamics in the two theories is the same and is described in terms of Hamiltonian dynamical systems, as for example in Refs. [4,9–11]. However, the association of states of the hybrid SA system with the states of the quantum SA system is different in the two hybrid theories, and due to this difference one of the hybrid theories gives a good model of the quantum measurement while the other theory provides only an averaged result. Let us stress right at the beginning that we treat the description provided by a hybrid theory as a convenient model of the quantum measurement, and we take no stance as to the ontological status of such hybrid model.

The hybrid theories to be discussed belong to the class of hybrid theories that are derived by considering part A of a bipartite quantum system SA as classical. These types of hybrid theories are formulated by giving two rules, one concerning the states of the hybrid and one concerning the hybrid dynamics. The state association rule explains how quantum states of the quantum SA system are modeled by states of the hybrid SA system. The hybrid dynamics rule fixes the dynamics of the hybrid SA system that models the Schrödinger dynamics of the quantum SA system. The hybrid theory is compatible with QM dynamics if the Schrödinger evolution of a quantum SA state followed by an application of the state association rule gives the same hybrid state as the state association applied on the initial quantum state followed by the hybrid dynamics. Pictorially, the two routes indicated in Fig. 1 end up in the same hybrid state. Let us point out that a hybrid theory might be compatible with QM only for certain types of dynamics or only over a certain type of states, and that here we are interested only in hybrid modeling of the quantum measurement process. The two hybrid theories considered are compatible with QM for the type of dynamics involved in the measurement process. We first present the hybrid dynamics, which is the same in both hybrid theories discussed, and then explain the state association rules of the two theories.

Hamiltonian evolution. Hamiltonian theory of the hybrid SA model is based on the mathematical possibility of describing the Hilbert space quantum mechanics as a Hamiltonian dynamical system. This Hamiltonian formulation of quantum mechanics has been described in many publications [12] and is repeated here. Necessary notation is be introduced and explained when needed. The hybrid theory is developed starting from the Hamiltonian formulation of a composite quantum SA system and imposing a constraint that one of the components, A, is behaving through the entire evolution as a classical system [10], in the sense that the total quantum fluctuation of the basic A observables $\hat{Q}$ and $\hat{P}$ in pure hybrid states is kept constant and minimal throughout the entire evolution. The evolution of pure states of the total SA hybrid system is described by Hamiltonian dynamical equation, and

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$$\begin{tikzcd} |\Psi(t_0)\rangle \arrow[r, "\text{Schrödinger}", "\text{evolution}"'] \arrow[d, dashed, "\text{R.S.A.}"'] & |\Psi(t)\rangle \arrow[d, dashed, "\text{R.S.A.}"] \\ \rho_\Psi(t_0) \arrow[r, "\text{Hybrid}", "\text{evolution}"'] & \rho_\Psi(t) \end{tikzcd}$$

FIG. 1. Illustration of the compatibility of a hybrid model description with quantum SA dynamics. The sequence of operations consisting of the Schrödinger evolution followed by the application of the rule of state association (RSA) gives the same hybrid state as the one obtained through the state association followed by the hybrid evolution. The hybrid states $\rho_\psi(t_0)$ and $\rho_\psi(t)$ can be pure or statistical mixtures depending on the RSA and/or hybrid dynamics.

the subsystem A, in the appropriate macrolimit, behaves as a classical system.

The real symplectic phase space of the hybrid system $\mathcal{M}_{SA}$ is given by the Cartesian product $\mathcal{M}_{SA} = \mathcal{M}_S \times \mathcal{M}_A$ of the S subsystem phase space $\mathcal{M}_S$ and of the A subsystem phase space $\mathcal{M}_A$. The manifold $\mathcal{M}_A$ is isomorphic to the manifold of A coherent states $\{|q,p\rangle\}$, and $\mathcal{M}_S$ is the real symplectic phase space of the S system corresponding to the Hilbert space $\mathcal{H}_S$. Local canonical coordinates on the product are denoted $\{x,y,p,q\}$.

The Poisson bracket on $\mathcal{M}_{SA}$ of arbitrary functions of the local coordinates (x,y,q,p) is defined as

$$\{f_1,f_2\}_{\mathcal{M}_{SA}} = \sum_{i=1}^{n_A}\left(\frac{\partial f_1}{\partial q_i}\frac{\partial f_2}{\partial p_i} - \frac{\partial f_2}{\partial q_i}\frac{\partial f_1}{\partial p_i}\right) + \sum_{j}^{n_S}\left(\frac{\partial f_1}{\partial x_j}\frac{\partial f_2}{\partial y_j} - \frac{\partial f_2}{\partial x_j}\frac{\partial f_1}{\partial y_j}\right) \tag{1}$$

where n_A and n_S are numbers of A and S degrees of freedom respectively.

The Hamilton's function is in general comprised of three terms, $H_t(x,y,q,p) = H_A(q,p) + H_S(x,y) + V_{\text{int}}(x,y,q,p)$. Here H_A is the Hamilton's function of the A subsystem and $H_S(x,y) = \langle\hat{H}_S\rangle$ is the Hamilton's function of the S subsystem. The function $V_{\text{int}}(x,y,q,p) = \langle\psi_{x,y}|\hat{V}_{\text{int}}(q,p)|\psi_{x,y}\rangle$, where $\hat{V}_{\text{int}}(q,p)$ is an operator in the Hilbert space of the S subsystem which depends on the A coordinates (p,q), describes the interaction between the subsystems. For example, a common toy model of the SA premeasurement interaction, with $H_S = 0$ and $H_A = 0$ and the interaction given by $g(t)\hat{A}\hat{P}$ where $\hat{P}$ is the observable conjugated to the pointer observable $\hat{Q}$, is represented by the function $V_{\text{int}}(x,y,q,p) = g(t)\langle\hat{A}\rangle\langle p,q|\hat{P}|p,q\rangle = g(t)\langle\hat{A}\rangle p$.

The Hamiltonian dynamical equations for functions of pure states $f(x,y,p,q)$ of the hybrid are

$$\dot{f}(x,y,q,p) = \{f(x,y,q,p),H_t(x,y,q,p)\}_{\mathcal{M}_{SA}}, \tag{2}$$

and probability densities on $\mathcal{M}_{SA}$, representing mixed states of the hybrid, evolve according to the Liouville equation [11]

$$\dot{\rho}(x,y,q,p) = \{H_t(x,y,q,p),\rho(x,y,q,p)\}_{\mathcal{M}_{SA}}. \tag{3}$$

Correspondence between states of the quantum system and the hybrid model. The two hybrid theories that we discuss here have the same Hamiltonian dynamics of pure (2) and mixed states (3). However, the states of the hybrid SA system modeling a state of the quantum SA system are different in the two hybrid theories. In the first theory, a pure state of the quantum system is modeled by the corresponding pure state of the hybrid, and in the second theory a pure state of the quantum system is modeled by the corresponding mixed state of the hybrid. However, both state association rules have the property that the expectation values of the canonical observables $\hat{Q},\hat{P}$ of a system A in a state of the quantum SA system are the same as the expectations in the associated hybrid state of classically behaving variables q,p.

The first type of state association, which we call type I, is given by the following rule:

$$|\psi\rangle_{SA} = \sum_i |\psi_i\rangle_S \otimes |\phi_i\rangle_A \longrightarrow \delta\big(x - x_{\psi_s}\big)\delta\big(y - y_{\psi_s}\big)\delta(q - q_a)\delta(p - p_a) \tag{4}$$

Pure quantum state of a general form on the left of Eq. (4) is modeled by the corresponding point $(x_{\psi_s},y_{\psi_s},q_a,p_a)$ from $\mathcal{M}_{SA}$. The point represents pure separable state $\langle q_a,p_a|\psi\rangle_{SA} \otimes |q_a,p_a\rangle \equiv |\psi_s\rangle \otimes |q_a,p_a\rangle$ with the property that the coordinates of the $|\psi\rangle_{SA}$-dependent coherent state $|q_a,p_a\rangle$ of the system A are such that the expectations of basic operators of the apparatus $\hat{Q}$ and $\hat{P}$ in the entangled state $|\psi\rangle_{SA}$ and in the separable state $\langle q_a,p_a|\psi\rangle \otimes |q_a,p_a\rangle$ are the same and equal to $\langle\hat{Q}\rangle = q_a, \langle\hat{P}\rangle = p_a$. The state $|\psi\rangle_S$ is explicitly given by $|\psi\rangle_S = \sum_i c_i\langle q_a,p_a|\psi_i\rangle_A|a_i\rangle$.

Such association between pure states of the quantum system and the hybrid model is consistent with the constraint that A behaves as a classical system and implies that there is no entanglement between S and A, but no restriction on the entanglement in S is imposed. This hybrid theory was studied, for example, in Refs. [4,10,11,13].

The association between states of the quantum and the hybrid systems in the second hybrid theory is based on the argument that predictions of quantum mechanics are statistical even if the quantum system is in a pure state. Therefore, predictions of the hybrid model should also be statistical. Hence, the state of the hybrid modeling a pure quantum state should be a proper statistical mixture of the pure states of the hybrid. We define such a rule in general, and call it type II, and then illustrate the rule using as examples the states relevant in the measurement process:

$$|\psi\rangle_{SA} = \sum_i |\psi_i\rangle_S \otimes |\phi_i\rangle_A \longrightarrow \sum_i |c_i|^2\delta(x - x_i)\delta(y - y_i)\delta(q - \langle q\rangle_{\phi_{iA}}) \times \delta(p - \langle p\rangle_{\phi_{iA}}). \tag{5}$$

By definition, the state of the hybrid SA modeling the quantum state $|\psi\rangle_{SA}$ is the statistical mixture of pure hybrid states $(x_i,y_i,\langle q\rangle_{\phi_{iA}},\langle p\rangle_{\phi_{iA}})$ with statistical weights $|c_i|^2$.

As the first example, consider a separable quantum SA state $|\psi\rangle = |\psi\rangle_S \otimes |\phi\rangle_A = (\sum_i |\langle a_i|\psi\rangle_S|a_i >_S) \otimes |\phi\rangle_A$, where $|a_i\rangle$ are the eigenstates of the measured observable. Notice that the state of the S subsystem is represented in the eigenbases of the S part of the Hamiltonian, which is in the case of the premeasurement just the measured observable $\hat{A}$.

The type II associated hybrid state is the following mixed state on $\mathcal{M}_{SA}$:

$$\rho_\psi(x,y,q,p) = \sum_i |\langle a_i|\psi\rangle_S|^2\delta(x-x_i) \times \delta(y-y_i)\delta(q-q_a)\delta(p-p_a), \quad (6)$$

where $q_a = \langle\phi_A|\hat{Q}|\phi_A\rangle$, $p_a = \langle\phi_A|\hat{P}|\phi_A\rangle$. In the case of a macroscopic A system already set in one of its coherent states $|q_0,p_0\rangle$, the hybrid mixture (6) is given by

$$\rho_\psi(x,y,q,p) = \sum_i |\langle a_i|\psi\rangle_S|^2\delta(x-x_i) \times \delta(y-y_i)\delta(q-q_0)\delta(p-p_0). \quad (7)$$

This is a statistical mixture of pure hybrid states $\delta(x-x_i)\delta(y-y_i)\delta(q-q_0)\delta(p-p_0)$ with probabilities $|\langle a_i|\psi\rangle_S|^2$.

Consider now an entangled state of the quantum SA system of the following form: $|\psi\rangle_{SA} = \sum_i c_i|a_i\rangle_S \otimes |q_i,p_0\rangle$. Such states appear as the result of the premeasurement interaction between S and A where $|a_i\rangle$ are the eigenstates of the measured observable and $|q_i,p_0\rangle$ are the relative coherent states of the apparatus. The hybrid state, in the macrolimit applied on the A system, is by definition given by the following density on $\mathcal{M}_{SA}$:

$$\rho_\psi(x,y,q,p) = \sum_i |c_i|^2\delta(x-x_i) \times \delta(y-y_i)\delta(q-q_i)\delta(p-p_0). \quad (8)$$

The state of the hybrid SA modeling this quantum state is the statistical mixture of pure hybrid states (x_i,y_i,q_i,p_0) with statistical weights $|c_i|^2$.

Dispersions of $\hat{Q}$ and $\hat{P}$ in the hybrid mixed state can be large, but this is a consequence of the classical statistical indeterminacy of the apparatus state. This is guaranteed to be the case for all times because of the constrained Hamiltonian evolution. Dispersions of the observables of the S system which do not commute with the Hamiltonian are of the classical statistical but also of the quantum nature. However, the hybrid system is always in some state with a sharp value of the S part of the Hamiltonian or the measured observable. Which such state the hybrid is in can only be specified with some probability, and the mixed state is understood as representing a statistical ensemble. Like in the type I rule, there is no entanglement between S and A systems. The possibility of entanglement between S degrees of freedom depends on the type of interaction between S and A systems.

The hybrid dynamics (2) and (3) and the state associations of type I (4) and type II (5) are introduced here independently of each other. However, as was pointed out, if the hybrid theory is considered as a model of some bipartite quantum system with one part behaving as classical, then the state association and dynamics must be compatible in the sense of Fig. 1. Compatibility of type I and II state association rules with the dynamics (2) and (3) is demonstrated for the special type of dynamics modeling the premeasurement process.

Hybrid models of a quantum measurement. We now discuss descriptions of the quantum measurement provided by the two hybrid models. We consider the S system to be a qubit, the measured observable to be $\hat{\sigma}_z$, and the apparatus to be a free one-dimensional (1D) particle. The premeasurement Hamiltonian is $\hat{H}_{\rm int} = \mu\hat{\sigma}_z\hat{P}$. The SA system is initially set in the state $|\psi\rangle_0 = \sum_i c_i|\sigma_z\rangle_i \otimes |q_0,p_0\rangle_A$, where $i = +,-$ and $|\sigma_z\rangle_\pm$ are the eigenstates of $\hat{\sigma}_z$. The quantum premeasurement process results in the entangled state: $\sum_i c_i|\sigma_z\rangle_i \otimes |q_0 - \mu t\langle\hat{\sigma}_z\rangle_i,p_0\rangle_A$. The interaction is such that the manifold of A coherent states is invariant. Of course, the duration of the premeasurement is considered finite and short.

The corresponding hybrid Hamiltonian is $H_t = \mu p(y_1^2 - y_2^2 + x_1^2 - x_2^2)/2$ where x_1,x_2,y_1,y_2 are the canonical coordinates in qubit phase space R^4 associated with the Hilbert space C^2. Hybrid equations of motion in the Hamiltonian form are

$$\begin{aligned} &\dot{x}_1 = p\mu y_1, \quad \dot{y}_1 = -p\mu x_1 \quad \dot{x}_2 = -p\mu y_2, \quad \dot{y}_2 = p\mu x_2,\\ &\dot{q} = \mu\big(y_1^2 - y_2^2 + x_1^2 - x_2^2\big)/2 \quad \dot{p} = 0. \end{aligned} \quad (9)$$

The solutions of these equations are such that $\langle\hat{\sigma}_z\rangle = (y_1^2 - y_2^2 + x_1^2 - x_2^2)/2 = \text{const} = \langle\hat{\sigma}_z\rangle_0$, $p = p_0$ and $q(t) = q_0 + \mu t\,\langle\hat{\sigma}_z\rangle_0$. In what follows we conveniently denote the solution $q(t)$ by $q(t;\langle\hat{\sigma}_z\rangle_0)$ since it depends on $\langle\hat{\sigma}_z\rangle_0$.

Type I association is compatible with the hybrid and quantum descriptions of the premeasurement dynamics. A pure hybrid state $(x_1,y_1,x_2,y_2,q,p)_0$ is associated with the initial quantum state $|\psi\rangle_0 = |\psi_0\rangle_S \otimes |q_0,p_0\rangle_A$ by the type I rule. The final hybrid state $(x_1,y_1,x_2,y_2,q,p)_t$, given by the solutions of Eq. (9), is associated by the type I rule with the quantum result of the premeasurement, which is the entangled pure state $|\psi(t)\rangle = \langle\sigma_{z+}|\phi_0\rangle|\sigma_{z+}\rangle \otimes |q_0 - \mu t,p_0\rangle + \langle\sigma_{z-}|\phi_0\rangle|\sigma_{z-}\rangle \otimes |q_0 + \mu t,p_0\rangle$. From Eq. (9) we see that the type I rule and the hybrid dynamics establish correlation between the value $q(t;\langle\hat{\sigma}_z\rangle_0)$ of the pointer observable and the expectation of the measured observable in the initial state $\langle\hat{\sigma}_z\rangle_0$. Thus, type I theory, as it stands, does not reproduce the results of a single measurement for arbitrary initial states (the same result but with different basic motivation was obtained in Ref. [13]).

Consider the hybrid theory with the type II association. The quantum initial state and the final state are the same as before, and the hybrid dynamics is also determined by Eq. (9). The initial hybrid state is the mixed state represented by

$$\begin{aligned}\rho(x,y,q,p;t_0) = {}& p_+\delta(\langle\hat{\sigma}_z\rangle - \langle\hat{\sigma}_{z+}\rangle)\delta(q-q_0)\delta(p-p_0)\\ &+ p_-\delta(\langle\hat{\sigma}_z\rangle - \langle\hat{\sigma}_{z-}\rangle)\delta(q-q_0)\delta(p-p_0),\end{aligned} \quad (10)$$

where the notation is somewhat imprecise but is quite obvious and suggestive. This state, evolved by the Hamiltonian $H_t = \mu p\langle\hat{\sigma}_z\rangle$, is transformed into the mixed state

$$\begin{aligned}&\rho(x,y,q,p;t)\\ &\quad = p_+\delta(\langle\hat{\sigma}_z\rangle - \langle\hat{\sigma}_{z+}\rangle)\delta(q(t;\langle\hat{\sigma}_z\rangle) - q_0)\delta(p_0-p_0)\\ &\qquad + p_-\delta(\langle\hat{\sigma}_z\rangle - \langle\hat{\sigma}_{z-}\rangle)\delta(q(t;\langle\hat{\sigma}_z\rangle) - q_0)\delta(p_0-p_0), \end{aligned} \quad (11)$$

where the solutions of Eq. (9) are used. This is equivalent to

$$\begin{aligned}&\rho(x,y,q,p;t)\\ &\quad = p_+\delta(\langle\hat{\sigma}_z\rangle - \langle\hat{\sigma}_{z+}\rangle)\delta(q(t;\langle\hat{\sigma}_{z+}\rangle) - q_0)\delta(p_0-p_0)\\ &\qquad + p_-\delta(\langle\hat{\sigma}_z\rangle - \langle\hat{\sigma}_{z-}\rangle)\delta(q(t;\langle\hat{\sigma}_{z-}\rangle) - q_0)\delta(p_0-p_0). \end{aligned} \quad (12)$$

Equation (12) is just the hybrid state associated by the type II rule with the entangled quantum state, which appears as the result of the quantum premeasurement evolution. This demonstrates compatibility of the type II rule with the quantum and hybrid descriptions of the premeasurement dynamics. The final state of the hybrid SA system is a classical statistical mixture of pure SA states. In each of these pure states the pointer variable of the apparatus corresponds to an eigenvalue, either $\langle \hat{\sigma}_{z+} \rangle$ or $\langle \hat{\sigma}_{z-} \rangle$. This is just the result of an ideal $\hat{\sigma}_z$ measurement. Statistical mixture is here a consequence of the initial incomplete knowledge represented by the initial mixture of hybrid pure states. Thus, Hamiltonian model of the hybrid dynamics with the type II rule for the state modeling successfully describes the measurement process.

In summary, we have studied description of the quantum measurement process in two hybrid theories. The hybrid theories have the same dynamics but differ in the rules that associate a state of the hybrid system with the state of the quantum system. The type I rule associates pure hybrid states with a pure quantum state. The type II rule associates statistical mixtures of hybrid pure state with pure quantum states. It is motivated by the fact that the predictions of a quantum system are statistical even when it is in a pure state and therefore the corresponding state of the hybrid system must also give statistical predictions. It is shown that the hybrid theory with the type II association rule correctly models the measurement process while the hybrid theory with the same dynamics but with the type I association models only averaged results of measurements. Nevertheless, it might be possible, as it seems to be indicated in Ref. [13], to supplement the type I hybrid theory with an additional decoherence process, caused perhaps by macroscopic features of the apparatus and occurring before the premeasurement, which would then imply the postmeasurement hybrid state similar to the result of the type II hybrid theory.

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