Cloning in nonlinear Hamiltonian quantum and hybrid mechanics

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The possibility of state cloning is analyzed in two types of generalizations of quantum mechanics with nonlinear evolution. It is first shown that nonlinear Hamiltonian quantum mechanics does not admit cloning without the cloning machine. It is then demonstrated that the addition of the cloning machine, treated as a quantum or as a classical system, makes cloning possible by nonlinear Hamiltonian evolution. However, a special type of quantum-classical theory, known as the mean-field Hamiltonian hybrid mechanics, does not admit cloning by natural evolution. The latter represents an example of a theory where it appears to be possible to communicate between two quantum systems at superluminal speed, but at the same time it is impossible to clone quantum pure states.

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I. INTRODUCTION

The impossibility of cloning unknown quantum states is a fundamental property of quantum systems [1,2]. It has been used as a basis for information theoretic axiomatization of quantum mechanics (QM) [3] and is crucial in several quantum-information-processing tasks [4]. Roughly speaking, state cloning is a process which involves at least two systems: an object system whose state is to be cloned and a target system whose state is transformed into the state which is equal to the state of the object system. Often, and in order to allow for the most general type of process, one includes also an ancilla system, which in the context of cloning is called the cloning machine. Standard simple proofs of no cloning involve properties of quantum processes, such as (a) linearity or (b) preservation of a nontrivial distance between quantum states, and also use (c) the direct product structure of composite quantum systems. The properties (a), (b), and (c) are not independent in QM, but each of them implies crucial differences between QM and classical mechanics (CM). Modifying any of the three properties leads to generalization of QM, which is also different from CM. Some of generalizations are mathematically inconsistent or in conflict with other fundamental physical theories like special relativity or thermodynamics [5,6]. Depending on the modification, cloning of states in the modified theory might, but need not, be possible. The possibility of cloning in a modified theory need not be related to superluminal signaling, like it is in standard QM. It is the purpose of this paper to discuss possibility of cloning in two types of modifications of QM. Both types of modified theories are formulated in the framework of Hamiltonian dynamical systems (HDSs). Standard QM can be formulated as a linear HDS on an appropriate phase space [7,8]. Mathematically consistent generalizations of QM can be obtained by modifying some of the standard QM properties but remaining within the framework of HDS. It is known that cloning is possible in classical mechanics with Hamiltonian dynamics [9]. Thus, it is interesting to investigate the possibility of cloning within different Hamiltonian generalizations of QM. The first class of modified theories that we study retains all the kinematical properties of QM in HDS formulation but allows evolution given by general nonlinear Hamiltonian equations. Weinberg [10] and Bialynicki–Birula and Mycielski [11] nonlinear Schrödinger equations are actually of this type. We abbreviate this type of theories as NHQM, which stands for nonlinear Hamiltonian QM. The second type of modified theory assumes that some of the degrees of freedom (DFs) of the HDS corresponding to a bipartite system are constrained to behave as classical DF [12,13]. We call this type Hamiltonian hybrid mechanics (HHM). The constraint implies nonlinear evolution of both classical-like DF (CDF) and of quantum DF (QDF) [12,14], but also changes the way in which the phase spaces of QDF and CDF are composed to form the phase space of the total hybrid system. Thus, in these types of theories the evolution is nonlinear and the tensor product rule is not valid for all DF. Our main results are (a) self-replication, i.e., a type of cloning in the restricted sense without the cloning machine, is impossible in NHQM; (b) inclusion of a *quantum* cloning machine makes the cloning in NHQM possible, and (c) cloning with the object and the target quantum systems and a *classical* cloning machine is also possible with nonlinear hybrid evolution. Thus, these two types of nonlinear generalizations of quantum mechanics, in which the evolution of the total system is Hamiltonian, allow the cloning of quantum states by natural evolution. However, cloning is impossible in a type of HHM with the Hamiltonian of a special mean-field form. These results are to be contrasted with the known result that the cloning is impossible within bipartite classical Hamiltonian systems (object and target), but becomes possible within three-partite systems [9] (object, target, and cloning machine). In the latter case the cloning can be achieved by a linear symplectic map [9]. Thus, it seems that if the object and the target are quantum (tensor product) and the evolution of the total system that includes the machine is Hamiltonian, then the cloning map is necessarily nonlinear, irrespective of the quantum or classical nature of the cloning machine. However, if all three systems are classical (Cartesian product), then the cloning is possible by linear transformations which are symplectic on the total phase space.

The structure of the paper is as follows: The next section serves to recapitulate, very briefly, the Hamiltonian formulation of QM and of the HHM, and then to formulate the definitions of the the cloning and self-replication processes in NHQM and in HHM. In Sec. III we prove our main results concerning the cloning (and self-replication) in NHQM and in

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HHM. Section IV contains several remarks which provide a discussion of our results. A summary is given in Sec. V.

II. FORMULATION OF CLONING IN HAMILTONIAN QUANTUM AND HYBRID THEORIES

A. Hamiltonian formulation of quantum mechanics and of hybrid mechanics

1. Hamiltonian quantum mechanics and nonlinear generalizations

Quantum and classical mechanics can be formulated by using the same mathematical framework of Hamiltonian dynamical systems $(\mathcal{M},\omega,H)$, where $\mathcal{M}$ is a symplectic manifold, ω is the corresponding symplectic structure, and H is the Hamilton's function. Formulation of the classical mechanics of isolated conservative systems using $(\mathcal{M},\omega,H)$ is standard [15]. The formulation of quantum mechanics in terms of $(\mathcal{M},\omega,g,H)$, where g is an appropriate Riemann structure, is perhaps less well known but shall not be presented here in any detail since there exist excellent reviews [7,8]. Very briefly, the basic observation beyond the Hamiltonian formulation of quantum mechanics is that the evolution of a pure quantum state in a Hilbert space $\mathcal{H}^N$, given by the Schrödinger equation, can be equivalently described by a HDS on an Euclidean manifold $\mathcal{M}=\mathbb{R}^{2N}$. Here N is the complex dimension of the relevant Hilbert space. The manifold $\mathcal{M}$ is just the Hilbert space considered as a real manifold, with the symplectic and Riemann structures given by the real and the imaginary parts of the Hilbert scalar product. The manifold also possesses an almost complex structure $J^2=-I$ such that $g(x,y)=\omega(x,Jy)$. Normalization and global phase invariance of quantum states can be incorporated into the formulation of the phase space of quantum states which is the projective space $P\mathcal{H}^{N-1}\sim S^{2N-1}/S^1$, with the corresponding symplectic, Riemann, and almost complex structures. However, in our computation we shall use the Hamiltonian formulation based on $\mathbb{R}^{2N}$, so that, when treating the problem of cloning, we shall have to take care of the global phase invariance explicitly. Representing a normalized vector $|\psi\rangle\in\mathcal{H}^N$ in an arbitrary basis $\{|e_j\rangle\}_{j=1}^N$ as $|\psi\rangle=\sum_{j=1}^N c_j|e_j\rangle$, one can introduce the real canonical coordinates $x_j=(\bar{c}_j+c_j)/\sqrt{2\hbar}$, $y_j=i(\bar{c}_j-c_j)/\sqrt{2\hbar}$, $j=1,2,\dots,N$, where bar indicates complex conjugation. Change of the basis by a unitary map involves a linear symplectic transformation of the canonical coordinates. A generic point from $\mathcal{M}$ will also be denoted by X or X^a, where $a=1,2,\dots,2N$ is an abstract index, such that $X^a=x_a$, $a=1,2,\dots,N$ and $X^a=y_a$, $a=N+1,\dots,2N$. If we want to stress that the point X corresponds to the vector $|\psi\rangle\in\mathcal{H}^N$ we write X_ψ, and vice versa $|\psi_X\rangle$ for the vector corresponding to the point X. It should be stressed, perhaps, that the canonical coordinates (x_j,y_j) have nothing to do with the canonical coordinates of the classical system that after quantization gives the considered quantum system with the Hilbert space $\mathcal{H}^N$. The Hamilton's function $H(X)$ is given by the quantum expectation of the Hamiltonian $\hat{H}$ in the state $|\psi_X\rangle$: $H(X)=\langle\psi_X|\hat{H}|\psi_X\rangle$. The Schrödinger dynamical law is that of Hamiltonian mechanics:

$$\dot{X}^a=\omega^{ab}\nabla_b H, \tag{1}$$

where ω^{ab} is the standard unit symplectic matrix

$$\omega=\begin{pmatrix}\mathbf{0} & \mathbf{1}\\ -\mathbf{1} & \mathbf{0}\end{pmatrix}, \tag{2}$$

where $\mathbf{0}$ and $\mathbf{1}$ are zero and unit matrices of dimension N.

In the Hilbert space QM and in Hamiltonian CM the dynamical variables can be introduced formally as generators of the isomorphisms of the respective relevant structures. In QM these are self-adjoined operators generating unitary transformations that preserve the Hilbert scalar product. In the Hamiltonian formulation of QM the Hilbert scalar product generates both the symplectic and the metric Riemann structures. The symplectic structure is preserved by Hamiltonian vector fields of arbitrary smooth functions, but the metric is preserved only by the Killing vector fields, i.e., by the Hamiltonian vector fields generated by quadratic functions of the canonical variables. In particular, the unitarity of the QM evolution implies that the Hamilton equations (1) are linear. All observables are represented by quadratic functions $A(X)$ on $\mathcal{M}$ and are the quantum-mechanical expectations of the corresponding quantum observables $A(X)=\langle\psi_X|\hat{A}|\psi_X\rangle$. On the other hand, the canonical coordinates of the quantum phase space do not have physical interpretation. It is important to observe that the Poisson bracket between two quadratic functions is also a quadratic function and satisfies

$$\{A_1(X),A_2(X)\}=\frac{1}{i\hbar}\langle\psi_X|[\hat{A}_1,\hat{A}_2]|\psi_X\rangle. \tag{3}$$

In what follows we shall need to consider a bipartite quantum system composed of two systems with Hilbert spaces $\mathcal{H}_1^{N_1}$ and $\mathcal{H}_2^{N_2}$. The phase space of the total system is the manifold $\mathcal{M}_{12}=\mathbb{R}^{2N_1N_2}\sim\mathcal{H}_1^{N_1}\otimes\mathcal{H}_2^{N_2}$. Of course, the space $\mathcal{M}_{12}$ is much larger than the Cartesian product $\mathcal{M}_1\times\mathcal{M}_2$, which is relevant for the formation of classical compound systems. If $|e_j^1\rangle$ and $|e_k^2\rangle$ are basis vectors in $\mathcal{H}_1^{N_1}$ and $\mathcal{H}_2^{N_2}$ respectively, with the corresponding canonical coordinates (x_j^1,y_j^1) and (x_k^2,y_k^2), then the canonical coordinates (x_l^{12},y_l^{12}) corresponding to the basis $|e_j^1\rangle\otimes|e_k^2\rangle$ in $\mathcal{H}_1^{N_1}\otimes\mathcal{H}_2^{N_2}$ are given by rather complicated formulas in general. Fortunately, we shall need only the formulas in the most simple cases, further simplified by a special choice of the target system state before the cloning transformation. In what follows we denote the composition of phase spaces of two systems with phase spaces $\mathcal{M}_1$ and $\mathcal{M}_2$ by $\mathcal{M}_1\odot\mathcal{M}_2$, which means $\mathcal{M}_{12}$ in the quantum and $\mathcal{M}_1\times\mathcal{M}_2$ in the classical case.

The Hamiltonian formulation of QM suggests natural formal generalizations [7]. Several such generalizations could be seen as special cases of the theory called extended quantum mechanics which was introduced and extensively studied in Ref. [16]. The most obvious one is to consider a theory where the evolution can be generated by functions which are not quadratic [7,10,17] but to retain the assumption that only the quadratic functions correspond to physical observables, and to retain the composition rule for compound systems. This would correspond to a nonlinear Schrödinger evolution equation. Such a theory, which we abbreviate by NHQM, is still a HDS with the same set of states and observables as in QM, but the Hamiltonian evolution equations are nonlinear and the metric is not evolution invariant. Since the proofs of the no-cloning property in QM are based on linearity or unitarity of the QM evolution, it is interesting to investigate if the cloning is possible in NHQM.

2. *Hamiltonian hybrid theory*

There is no unique generally accepted theory of interaction between micro and macro degrees of freedom, where the former are described by quantum and the latter by classical theory. The reason is primarily because each of the suggested theories has some unexpected or controversial features (see Ref. [13] for an informative review). Partial selection of hybrid theories can be found in Refs. [18–23]. Some of the suggested hybrid theories are mathematically inconsistent, and "no-go" type theorems have been formulated [24], suggesting that no consistent hybrid theory can be formulated. Nevertheless, mathematically consistent but inequivalent hybrid theories exist [13,22,23]. The Hamiltonian hybrid theory, as formulated and discussed, for example, in Refs. [12,13], has many of the properties commonly expected of a good hybrid theory. In fact, the dynamical formulas of the Hamiltonian theory are equivalent to the well-known mean-field approximation, the main novelty being that the theory is formulated entirely in the framework of the theory of Hamiltonian dynamical systems, which enables useful insights and methods of analysis [25–27]. Analysis of cloning in the Hamiltonian hybrid system is one such application. In fact, we analyze the possibility of cloning in general HHM where the Hamiltonian is not necessarily of mean-field form and contrast the results with the HHM of the restricted type where the Hamiltonian is of mean-field form.

The phase space in the Hamiltonian theory of a hybrid classical-quantum system, denoted by $\mathcal{M}$, is considered as a Cartesian product $\mathcal{M} = \mathcal{M}_c \times \mathcal{M}_q$ of the classical subsystem phase space $\mathcal{M}_c$ with $\dim \mathcal{M}_c = 2N_c$ and of the quantum subsystem phase space $\mathcal{M}_q$ with $\dim \mathcal{M}_q = 2N_q$. Local coordinates on the product are denoted (q,p,x,y), where $(q,p) \in \mathcal{M}_c$ are called the classical degrees of freedom (CDF) and $(x,y) \in \mathcal{M}_q$ are called the quantum degrees of freedom (QDF). Notice that the classical and the quantum parts are composed as if both were classical, i.e., there is no possibility of entanglement between CDF and QDF. Generalized Hamiltonian hybrid theory is given by a Hamiltonian dynamical system on the phase space $\mathcal{M} = \mathcal{M}_c \times \mathcal{M}_q$. In the general case, nothing is supposed about the total Hamiltonian, and it is only the structure of the phase space that justifies the terminology of hybrid quantum-classical systems. The Poisson bracket on $\mathcal{M}$ of arbitrary functions of the local coordinates (q,p,x,y) is defined as

$$\{f_1,f_2\}_{\mathcal{M}} = \sum_{i=1}^{N_c}\left(\frac{\partial f_1}{\partial q_i}\frac{\partial f_2}{\partial p_i} - \frac{\partial f_2}{\partial q_i}\frac{\partial f_1}{\partial p_i}\right) + \frac{1}{\hbar}\sum_{j=1}^{N_q}\left(\frac{\partial f_1}{\partial x_j}\frac{\partial f_2}{\partial y_j} - \frac{\partial f_2}{\partial x_j}\frac{\partial f_1}{\partial y_j}\right). \quad (4)$$

Thus, the Hamiltonian form of the hybrid dynamics on $\mathcal{M}$ as the phase space reads

$$\dot{q} = \{q,H\}_{\mathcal{M}}, \quad \dot{p} = \{p,H\}_{\mathcal{M}}, \quad \dot{x} = \{x,H\}_{\mathcal{M}}, \quad \dot{y} = \{y,H\}_{\mathcal{M}}, \quad (5)$$

where H is an arbitrary smooth function on the total phase space $\mathcal{M}$.

A particular case of HHM, treated, for example, in Refs. [12,13] and equivalent to the mean-field approach, is obtained by further assumptions about the form of the Hamiltonian. The evolution equations of the hybrid system are in this type of HHM given by the Hamiltonian of the following form:

$$H_t(q,p,x,y) = \langle\psi_{x,y}|\hat{H}_q + \hat{V}_{\text{int}}(q,p)|\psi_{x,y}\rangle + H_c(q,p) = H_c(q,p) + H_q(x,y) + V_{\text{int}}(q,p,x,y), \quad (6)$$

where H_c is the Hamilton's function of the classical subsystem, $H_q(x,y) = \langle\psi_{x,y}|\hat{H}_q|\psi_{x,y}\rangle$ is the Hamilton's function of the quantum subsystem, and $V_{\text{int}}(q,p,x,y) = \langle\psi_{x,y}|\hat{V}_{\text{int}}(q,p)|\psi_{x,y}\rangle$, where $\hat{V}_{\text{int}}(q,p)$ is a Hermitian operator in the Hilbert space of the quantum subsystem which depends on the classical coordinates (q,p) and describes the interaction between the subsystems. Despite the fact that the Hamiltonian is a quadratic function of QDF (and arbitrary function of CDF) the evolution of the QDF is nonlinear because of the coupling between QDF and CDF.

It is important to mention the evolution of statistical ensembles of hybrid systems in this type of HHM. Such an ensemble is described by a probability distribution $\rho(q,p,x,y)$, which evolves by the Liouville equation with the Hamiltonian (6). The following expression:

$$\hat{\rho}(t) = \int_{\mathcal{M}} \rho(q,p,x,y;t)\hat{\Pi}(x,y)dxdydqdp = \int_{\mathcal{M}} \rho_q(x,y;t)\hat{\Pi}(x,y)dxdy = \int_{\mathcal{M}} \hat{\rho}_{cl}(q,p;t)dqdp, \quad (7)$$

where $\hat{\Pi}(x,y)$ is a normalized projector onto the vector $|\psi_{x,y}\rangle$, is a well-defined density matrix representing a state of the QDF at each t. There are many $\rho_q(x,y;t)$ giving the same density matrix $\hat{\rho}(t)$. From the evolution equation satisfied by Eq. (7), or from Eq. (5), it is seen that a pure state $|\psi(t)\rangle\langle\psi(t)|$ obtained from an initial ensemble $\rho(q,p,x,y) = \delta(q-q_0)\delta(p-p_0)\delta(x-x_0)\delta(y-y_0)$ with CDF (and QDF) in pure states is always a pure state of QDF. The evolution equation satisfied by this pure state is in the form of a (linear) Schrödinger equation with the Hamiltonian which is a Hermitian operator that depends explicitly on $(q(t),p(t))$. On the other hand, if the CDF are initially in a mixed state, a pure state of the QDF will evolve into a mixed state. Furthermore, it was shown in Ref. [26] that the evolution of a general $\rho(t)$ will involve explicitly the convex expansion (7), and not only $\rho(t)$. Therefore, it seems that this type of HHM can be used for superluminal communication between distant subparts of the quantum DF.

Discussion of cloning within the restricted type of HHM with the classical part playing the role of the cloning machine requires special treatment as compared with the general HHM.

B. Definitions of cloning and self-replication

Cloning is a process involving three systems: the object system S_o with the state space $\mathcal{S}_o$, a target system S_t with the state space $\mathcal{S}_t$ the same as that of S_o, and an auxiliary system, the cloning machine S_m, with the state space of dimension M that is not specified in advance. It is said that cloning of some arbitrary object state $X_o \in \mathcal{S}_o$ is possible if there is a state of the target $X_{t,in} \in \mathcal{S}_t$ and a state of the machine $X_{m,in} \in \mathcal{S}_m$ such that

$$X_o \odot X_{t,in} \odot X_{m,in} \to X_o \odot \{X_t = X_o\} \odot X_m(X_o). \quad (8)$$

The arbitrary state of the object system is conserved by cloning; one fixed state of the target and another fixed state of the machine are chosen as initial, independently of the object state. The fixed initial target state is mapped into the initial state of the object. The final state of the machine might depend on the object state X_o. It is not assumed that the final machine state is uniquely related to X_o. Observe that the possibility of cloning does not imply that the cloning is achieved with any initial target and machine states, but only with a specific choice of these states. The domain and the range of the cloning map (8) are proper subsets of the sets of possible states of the object + target + machine system.

The system $S_o \bigcup S_t \bigcup S_m$ is characterized by its natural evolution, and the question is if the cloning map belongs to that class. In our case the natural evolution is given by a Hamiltonian flow on $\mathcal{S}_o \odot \mathcal{S}_t \odot \mathcal{S}_m$ and thus preserves the symplectic structure on $\mathcal{S}_o \odot \mathcal{S}_t \odot \mathcal{S}_m$. In NHQM all three systems are quantum and, as was stated in the previous section, $\odot$ is the tensor product. In HHM we shall consider the case when the object system and the target are quantum and the machine is classical. Thus, in this case, $\odot$ between the machine and object + target is the Cartesian product. Alternatively, which we shall not do, one could analyze cloning with all three systems of the hybrid nature. The only fixed property of the cloning problems within the Hamiltonian framework is the canonical Hamiltonian evolution and the fact that pure states are represented by points in the corresponding phase spaces. If X_o and X_t represent phase space points in the Hamiltonian formulation corresponding to the vectors $|\psi_o\rangle$ and $|\psi_t\rangle$, respectively, then it is natural to assume that the cloning is successful if at the output $|\psi_t\rangle \exp(i\theta) = |\psi_o\rangle$, i.e.,

$$x_t^i \cos\theta - y_t^i \sin\theta = x_o^i,$$
$$y_t^i \cos\theta + x_t^i \sin\theta = y_o^i,$$
$$i = 1,2,\ldots,N. \quad (9)$$

The role of the machine DF can be justified from two different points of view. One is the operational point of view, where the appearance of the cloning machine is natural. The other role of the cloning machine is to actually enable the object + target subsystem to evolve in a non-Hamiltonian way. Quite analogously to the role of the machine in the standard QM formulation of cloning, here the presence of the cloning machine enables the total object + target + machine system to evolve canonically while enabling a more general type of evolution of the subsystem object + target. In this respect a related more restrictive problem with no cloning machine is sometimes considered. Such a process has been termed self-replication and consists of mapping a fixed state of the target system into an arbitrary state of the object system, the latter remaining unchanged, but without any influence of the third system. In the self-replication process the object + target system is considered as isolated. Together with the problem of proper cloning within NHQM (with the cloning machine) we shall also analyze the possibility of self-replication in such theories.

III. MAIN RESULTS

Our strategy to analyze the possibility of self-replication and cloning will be the same in NHQM and HHM. Let us denote by $\mathcal{M}_{otm}$ the total phase space of the object + target + machine system. By $\mathcal{M}_{in} \subset \mathcal{M}_{otm}$ we denote the submanifold of the total phase space of the form $\mathcal{M}_o \odot X_{t,in} \odot X_{m,in}$, where $X_{t,in}$ and $X_{m,in}$ are specific initial vectors representing states of the target and the machine, respectively. We shall all the time deal with vectors of unit norm. Similarly, we denote by $\mathcal{M}_f \subset \mathcal{M}_{otm}$ the submanifold which is the image of $\mathcal{M}_{in}$ by the cloning map. Points in $\mathcal{M}_f$ are of the form $X_o \odot X_o \odot X_{m,f}(X_o)$, $X_o \in \mathcal{M}_o$, and thus $\dim \mathcal{M}_f = \dim \mathcal{M}_{in} = \dim \mathcal{M}_o$. We then choose an arbitrary point $X \in \mathcal{M}_{in}$ and two arbitrary normalized tangent vectors $g_X, h_X \in T_X(\mathcal{M}_{in}) \subset T_X(\mathcal{M}_{otm})$. The value of the symplectic area $\omega_X(g_X,h_X)$ is then computed. Cloning (or self-replication) is represented by the mapping $\phi : \mathcal{M}_{in} \to \mathcal{M}_f$ with the tangent map $\phi_\star : T_X(\mathcal{M}_{in}) \to T_{\phi(X)}(\mathcal{M}_f)$. Symplectic area between the images of the two vectors $\omega_{\phi(X)}(\phi_\star g_X, \phi_\star h_X)$ is then computed. If ϕ is a symplectic map, i.e., can be generated by a piecewise smooth Hamiltonian flow, then

$$\omega_{\phi(X)}(\phi_\star g_X, \phi_\star h_X) = \omega_X(g_X, h_X). \quad (10)$$

If Eq. (10) is not satisfied, for any choice of $X_{t,in}, X_{m,in}$, and $X_{m,f}$, then the cloning (self-replication) map ϕ cannot be realized by a Hamiltonian flow. To apply the procedure, we shall write explicitly the cloning map ϕ and its tangent map $\phi_\star$, corresponding to the phase spaces $\mathcal{M}_{in}$ and $\mathcal{M}_f$ with a specific choice of the initial target and machine states in the NHQM and HHM. The only difference will be in the way the machine phase space $\mathcal{M}_m$ is added to the phase space of the object + target.

In our discussion, we consider the simplest possible systems as object, target, and machine. The object and the target are each taken to be a single qubit. An arbitrary state of the object qubit is a normalized $\mathbb{C}^2$ vector with complex coefficients (α,β) corresponding to some basis of $\mathcal{S}_o$. Furthermore, the initial state of the target qubit will be represented by vector (1,0) in a basis of $\mathcal{S}_t$ chosen in the same way as the basis in $\mathcal{S}_o$. This does not seem to be a restriction with crucial consequences, but grossly simplifies explicit formulas for the self-replication (and later cloning) map.

In the case of NHQM the machine is also a quantum system and the coupling of it with the object + target is via tensor product. In order to demonstrate that, in NHQM, cloning by a symplectic (nonlinear) transformation is possible, it is enough to assume that the cloning machine is also a qubit, set initially in the state $(\alpha_m,\beta_m) = (1,0)$, represented in some basis of $\mathcal{S}_m$. Cloning is also possible by a symplectic map in the case of general HHM, when the machine is a classical system with two degrees of freedom and is coupled to the object + target via the Cartesian product. However, an additional argument is used to show that in the specific HHM with the Hamiltonian of the form (6), i.e., quadratic in the QDF, cloning of the quantum state is impossible by symplectic transformation generated by the Hamilton functions of the stated form.

A. Impossibility of self-replication in NHQM

Let us first illustrate the computations for the case of self-replication in NHQM. The real dimension of $\mathcal{M}_{in}$ with normalized object states is three. In the complex notation the initial point in $\mathcal{M}_{in}$ representing the state of object + target before self-replication is

$$X_{in} = (\alpha,0,\beta,0), \quad |\alpha|^2 + |\beta|^2 = 1. \tag{11}$$

Two normalized tangent vectors g and h in $T(\mathcal{M}_{in})$ at X_{in} are given as

$$g_{re} = (-g_1\alpha_{im} + g_3\beta_{re},0, - g_3\alpha_{re} - g_2\beta_{im},0), \tag{12a}$$

$$g_{im} = (g_1\alpha_{re} + g_3\beta_{im},0, - g_3\alpha_{im} + g_2\beta_{re},0), \tag{12b}$$

with arbitrary real numbers g_1, g_2, and g_3 chosen to respect the unity norm. Analogous formulas apply to h_{re} and h_{im}. Subscripts re and im stand for real and imaginary parts. The skew product of the two tangent vectors is

$$\omega(g,h) = [g_3(h_1 - h_2) + (g_2 - g_1)h_3](\alpha_{re}\beta_{re} + \alpha_{im}\beta_{im}). \tag{13}$$

In formulas (12) and (13) we have, for the sake of brevity, skipped the subscript indicating the related point X_{in}.

Image by the self-replication map ϕ of X_{in}, again in the complex coordinates, is given by

$$X_f = (\alpha^2,\alpha\beta,\beta\alpha,\beta^2)\exp[i\theta(\alpha,\beta)]. \tag{14}$$

Notice the arbitrary phase factor added to the result of the self-replication operation. Images of g and h by the tangent map $\phi_\star$ are given by rather long formulas which we do not reproduce here. However, the skew product of $\phi_\star g$ and $\phi_\star h$ at the point X_f is given by

$$\omega(\phi_\star g,\phi_\star h) = 2[g_3(h_1 - h_2) + (g_2 - g_1)h_3] \times (\alpha_{re}\beta_{re} + \alpha_{im}\beta_{im})(|\alpha|^2 + |\beta|^2). \tag{15}$$

Notice that the previous result is independent of arbitrary phase factor. The ratio of the symplectic areas after and before the application of the self-replication map is

$$\frac{\omega(\phi_\star g,\phi_\star h)}{\omega(g,h)} = 2(|\alpha|^2 + |\beta|^2) = 2. \tag{16}$$

Thus, the self-replication map does not preserve the skew product and therefore cannot be realized by any symplectic map between $\mathcal{M}_{in}$ and $\mathcal{M}_f$.

B. Possibility of cloning in NHQM

Consider now the proper cloning map in NHQM with the quantum machine included. Since we shall see that the cloning map is symplectic with the cloning machine given by a qubit, it is enough to assume this simplest realization of the machine. The final state of the machine (α_{mf},β_{mf}) is free to chose, and the choice can be done such that the factor of two appearing in the result of self-replication (16) can be canceled.

Formulas for the initial point and its image by the cloning map for the indicated choice of initial states of the target and the machine in the complex notation are given by:

$$X_{in} = (\alpha,0,0,0,\beta,0,0,0), \tag{17}$$

$$X_f = (\alpha^2\alpha_{mf},\alpha^2\beta_{mf},\alpha\beta\alpha_{mf},\alpha\beta\beta_{mf}, \times \alpha\beta\alpha_{mf},\alpha\beta\beta_{mf},\beta^2\alpha_{mf},\beta^2\beta_{mf}), \tag{18}$$

where (α_{mf},β_{mf}) denote the final state of the machine. The tangent vector g is given by

$$g_{re} = (-g_1\alpha_{im} + g_3\beta_{re},0,0,0, -g_3\alpha_{re} - g_2\beta_{im},0,0,0), \tag{19a}$$

$$g_{im} = (g_1\alpha_{re} + g_3\beta_{im},0,0,0, -g_3\alpha_{im} + g_2\beta_{re},0,0,0), \tag{19b}$$

and analogously for h. The skew product between g and h is

$$\omega(g,h) = [g_3(h_1 - h_2) + h_3(g_2 - g_1)](\alpha_{re}\beta_{re} + \alpha_{im}\beta_{im}). \tag{20}$$

The images of g and h by the tangent map, their skew product, and the ratio $\omega(\phi_\star g,\phi_\star h)/\omega(g,h)$ are given by rather long formulas, which depend on the final machine state. However, we have found that the choice of final machine state as $(\alpha_{mf},\beta_{mf}) = (\bar{\alpha},\bar{\beta})$, where the bar indicates complex conjugation, renders the ratio equal to unity for the normalized state (α,β) of the object. Therefore, the cloning map can be realized by a symplectic transformation. From the standard QM it follows that the symplectic cloning transformation in NHQM must be nonlinear.

C. Possibility of cloning in general HHM

We chose the object and the target to be the same systems and to be in the same states as in the case of NHQM. The machine is chosen to be a convenient classical system with two DF and coordinates $(q_{1m},q_{2m},p_{1m},p_{2m})$ or in complex notation $(q_{1m} + ip_{1m},q_{2m} + ip_{2m}) = (\alpha_m,\beta_m)$. Formulas for the initial point for the indicated special choice of initial target and machine states are given in the complex coordinates by

$$X_{in} = (\alpha,0,\beta,0,1,0)\,. \tag{21}$$

The machine final state is free to choose. With the choice $(\alpha_{mf} = \alpha_{im} + i\alpha_{re},\beta_{mf} = \beta_{im} + i\beta_{re})$ the state after cloning operation is

$$X_f = (\alpha^2,\alpha\beta,\beta\alpha,\beta^2,\alpha_{im} + i\alpha_{re},\beta_{im} + i\beta_{re}). \tag{22}$$

Tangent normalized vector g is given by

$$g_{re} = (-g_1\alpha_{im} + g_3\beta_{re},0, -g_3\alpha_{re} - g_2\beta_{im},0,0,0), \tag{23a}$$

$$g_{im} = (g_1\alpha_{re} + g_3\beta_{im},0, -g_3\alpha_{im} + g_2\beta_{re},0,0,0), \tag{23b}$$

and similarly for tangent vector h. The skew product between g and h is given by

$$\omega(g,h) = [g_3(h_1 - h_2) + h_3(g_2 - g_1)](\alpha_{re}\beta_{re} + \alpha_{im}\beta_{im}). \tag{24}$$

The images of the normalized tangent vectors and their skew product are again given by rather long formulas. However, the above choice of the machine final state renders the ratio

$$\frac{\omega(\phi_\star g,\phi_\star h)}{\omega(g,h)} = 1, \tag{25}$$

for normalized initial object states. Again, the cloning map can be realized by a symplectic transformation.

D. Impossibility of cloning in the HHM with the specific form of the Hamiltonian

Special form of the hybrid Hamiltonian (6) implies special status of the cloning operation in this type of HHM, as compared with the general case. In fact, due to the properties of the evolution of pure hybrid states, summarized in Sec. II, pure states of QDF remain pure if the initial state of CDF is also pure. Furthermore, the scalar product between two QDF pure states is preserved. Therefore, the standard no-cloning argument from linear QM applies. Thus, cloning of quantum states is impossible within the specific HHM with Hamiltonian (6), and with classical DF assuming the role of the cloning machine. Here we have an example of a theory that does not admit cloning of pure quantum states, but whose natural extension that includes ensembles admits superluminal communication.

IV. DISCUSSION

Remark 1: Physical interpretation and consequences. Cloning is commonly considered as an information-processing task. From this point of view, the problem formulated in Sec. II and discussed in Sec. III is rather formal and is concerned with an idealized system that could never occur in information-processing protocols with real systems. Pure states of isolated systems and their idealized evolution are only probabilistically related to information and its processing. Therefore, the relation between the system's states and information must be probabilistic, and the processing of such information necessary involves stochastic perturbations. This has been analyzed in the standard QM [28]. The question of cloning in real, experimentally available systems was not studied in the present publication but is important in analyzing the fundamental and practical consequences. In order to do that, one needs to use probability ensembles, represented by distributions on the relevant phase spaces and stochastic evolution equations. We believe that only with such an analysis could one attempt to draw conclusions as to the physical consistency of the nonlinear HQM and HHM.

Remark 2: Cloning vs superluminal signaling. It is well known that, if cloning would be possible in the standard QM then, also in the framework of this theory, it would also be possible to communicate information at superluminal speed. It has also been claimed that the condition of no superluminal signaling puts an upper bound on the fidelity of cloning, in effect excluding the perfect cloning in QM [29]. The condition of no superluminal signaling is in Ref. [29] expressed in terms of convex expansions of mixed states. In the opposite direction, it has been argued [2,17] that a nonlinear evolution of pure quantum states would enable signaling at superluminal speed. This is consistent with our results which show the possibility of cloning in NHQM. However, the argument does not exclude theories in which pure quantum states cannot be perfectly cloned, but the superluminal signaling is possible. Mean-field HHM with the special form of the Hamiltonian (6) is an example of such a theory.

Remark 3: Cloning in classical mechanics. It is commonly understood that perfect cloning of classical information contained in a classical pure state is possible. Of course, in order to discuss the possibility of cloning, one needs a precise definition of the state space and the type of dynamics characterizing the classical system. One formulation of the problem, particularly relevant in fundamental physics and for comparison with our results, is for the classical system modeled by using the framework of classical Hamiltonian dynamical systems. States of the system, the target, and the machine are described by the corresponding symplectic manifolds, their union is given by the Cartesian product and the symplectic structure on the total space is such that the symplectic structures on the components are obtained by the corresponding projections. It is known that the self-replication is not, but the cloning is possible by symplectic mappings on the total phase space, provided that the machine space has enough dimensions [9]. The proof of no self-replication is similar to the case in nonlinear quantum mechanics, presented in Sec. III. The possibility of cloning in Hamiltonian CM is established and discussed by concrete examples of symplectic cloning maps. It should be stressed that cloning is performed by linear symplectic mapping. On the other hand, cloning in NHQM and general HHM can be achieved by a symplectic map which must be nonlinear. This seems to be the crucial difference between the theories involving tensor or Cartesian products between the target and the object systems.

Remark 4: Cloning in classical statistical mechanics. Evolution of a probability distribution generated by a measure preserving mapping of a phase space is by definition linear and preserves the relative entropy between two distributions. These two properties, i.e., preservation of a nontrivial (quasi) distance between states and linearity are features of the Schrödinger evolution of pure quantum states. Also, the space of statistical states of a compound system, for example, $L^1(\mathcal{M}_1 \times \mathcal{M}_2)$ can be considered as the tensor product of $L^1(\mathcal{M}_1)$ and $L^1(\mathcal{M}_2)$. Thus, all three ingredients that are used in the standard proofs of no cloning in QM are also properties of classical statistical mechanics. Therefore, one expects, and it has been proved to be true [30], that cloning in classical statistical mechanics is impossible. Due to the creation of correlations between the subsystems, it is also possible to formulate the question of cloning in a more general way, more akin to the notion of broadcasting in QM. The answer to the question of possibility of broadcasting in Hamiltonian CM is also negative [30].

V. SUMMARY

We have analyzed the possibility of exact cloning of unknown quantum states in two types of nonlinear generalizations of quantum mechanics. Both types of generalizations were formulated as Hamiltonian dynamical systems on appropriate phase spaces. In the first type, which we called nonlinear Hamiltonian quantum mechanics (NHQM), the object, the target and the machine are treated as quantum systems, and it is shown that cloning can be realized by a nonlinear symplectic mapping. On the other hand, the process of self-replication, involving only the system and the target, cannot be realized by any symplectic transformation in NHQM. The other type of nonlinear generalizations of QM which we treated describes hybrid quantum-classical systems, again using the framework of Hamiltonian dynamical systems. Here, the object and the target are quantum, but the machine is a classical system. We show that there exists a nonlinear symplectic transformation which realizes the cloning operation. However, the cloning transformation cannot be realized in the Hamiltonian hybrid theory of the mean-field type, in which case the Hamiltonian must be a quadratic function of the quantum degrees of freedom and an arbitrary one of the classical degrees of freedom. It would be interesting to try to extend these results to the problem of broadcasting of mixed states in the nonlinear generalizations. This would require analysis of the Liouville evolution of densities and might result in possibility of broadcasting also in the mean-field Hamiltonian hybrid theory.

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