Positive-operator-valued measures in the Hamiltonian formulation of quantum mechanics

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(Received 16 January 2015; published 12 June 2015)

In the Hilbert space formulation of quantum mechanics, ideal measurements of physical variables are discussed using the spectral theory of Hermitian operators and the corresponding projector-valued measures (PVMs). However, more general types of measurements require the treatment in terms of positive-operator-valued measures (POVMs). In the Hamiltonian formulation of quantum mechanics, canonical coordinates are related to PVM. In this paper the results of an analysis of various aspects of applications of POVMs in the Hamiltonian formulation are reported. Several properties of state parameters and quantum observables given by POVMs or represented in an overcomplete basis, including the general Hamiltonian treatment of the Neumark extension, are presented. An analysis of the phase operator, given by the corresponding POVMs, in the Hilbert space and the Hamiltonian frameworks is also given.

DOI: 10.1103/PhysRevA.91.062114 PACS number(s): 03.65.Fd, 03.65.Sq

I. INTRODUCTION

The Hamiltonian formulation of quantum mechanics (HQM) [1–4] provides an alternative mathematical formulation that is equivalent to the more standard one based on Hilbert spaces and has proven to be useful in discussing such issues as nonlinear constraints [5,6] the geometry of entanglement [2], the classical limit [7,8], hybrid quantum-classical systems [9–11], and nonlinear and stochastic generalizations of quantum mechanics (QM) [1,2,12]. In the Hamiltonian formulation quantum pure states are represented by points of an appropriate smooth manifold $\mathcal{M}$ and the quantum dynamics is represented by a Hamiltonian flow on $\mathcal{M}$. In order to formulate probabilistic aspects of QM and in particular describe ideal measurements in the sense of von Neumann, the manifold $\mathcal{M}$ is equipped with a Riemannian metric. Standard postulates of QM about states, observables, and dynamics are formulated in terms of notions associated directly with a Hamiltonian dynamical system on $\mathcal{M}$, without any reference to the Hilbert space formulation.

An ideal measurement of a quantum observable, represented in the von Neumann scheme by a Hermitian operator and its spectral projector-valued measure (PVM), is in the Hamiltonian framework formulated using quadratic functions of canonical coordinates on $\mathcal{M}$, their critical values and critical points, and the Riemannian metric on $\mathcal{M}$. However, there are legitimate questions that can be asked about the preparation of a quantum system that cannot be cast into the von Neumann ideal measurement conception [13–15]. Data about the system can be collected that cannot be obtained as eigenvalues of an appropriate self-adjoined operator. On the other hand, such sets of data do satisfy certain conditions, such as covariance with respect to some natural transformations [13,14], which justify association of such data with certain physical quantities. Important examples of such data sets, like those related to polarization or the phase of quantum motion, are conveniently described by positive-operator-valued measures (POVMs) instead of PVMs. Another instance where the use of POVMs appears most naturally is in the context of approximate or indirect measurements or joined measurements of canonically related observables. It is our goal to formulate and analyze important properties of POVMs in the framework of the Hamiltonian formulation and thus prepare the way for the Hamiltonian formulation of the generalized measurement.

The paper is organized as follows. In the next section we provide a brief presentation of the Hamiltonian formulation of QM, insisting on its independence from the Hilbert space formulation. Section III is devoted to an abstract treatment in the framework of the Hamiltonian formulation of various questions related to the use of the POVM, with all considerations restricted to a finite-dimensional state space. In particular, we discuss in detail the kinematical and the dynamical aspects of the Hamiltonian analog of the Neumark extension for a POVM. In Sec. IV we treat in detail the example of a POVM corresponding to the phase of quantum motion. The Neumark extension of the phase POVM in the Hilbert space formulation is derived and the corresponding Hamiltonian formulation is presented. Section V provides a brief summary.

II. BASICS OF THE HAMILTONIAN FORMULATION

The Hamiltonian formulation of quantum mechanics is formally rather similar to the standard theory of Hamiltonian dynamical systems as it is used in classical mechanics [16]. The additional features are related to the statistical properties of quantum systems. Pure states of a quantum system are in the HQM mathematically represented by points of a smooth manifold with Kahler structure $(\mathcal{M},G,\Omega,J)$, where $\mathcal{M}$ is a smooth manifold admitting a Riemannian G and symplectic Ω structures and J is a map on the tangent space $T\mathcal{M}$ satisfying $G(X,Y)=\Omega(X,JY)$. One refers to $(\mathcal{M},G,\Omega,J)$ as the quantum phase space. In fact, in the case of systems with a finite N-dimensional Hilbert space $\mathcal{H}^N$, the Hamiltonian formulation is given using $\mathcal{M}=\mathbf{R}^{2N}$ with the standard Riemannian, symplectic, and complex structures. On the other hand, phase spaces for systems with infinite-dimensional Hilbert spaces can be considered as direct sums of an even number of real infinite-dimensional vector

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spaces. In this and the next section we restrict our attention to the finite-dimensional cases. In any case, a pure quantum state is represented by an equivalence class of points in $\mathcal{M}$. Nevertheless, we almost always refer to the points of $\mathcal{M}$ as the quantum states neglecting the fact that many points correspond to physically the same quantum pure state.

The symplectic two-form Ω associates a Hamiltonian vector field X_f with a sufficiently smooth function f on $\mathcal{M}$ by the formula

$$\Omega(X,X_f) = df(X), \tag{1}$$

where X is a vector field on $\mathcal{M}$. Thus, any smooth function generates a symplectic transformation. The symplectic structure also defines a Poisson bracket between smooth functions f and g on $\mathcal{M}$,

$$\{f,g\} = \Omega(X_f,X_g), \tag{2}$$

where X_f, X_g are Hamiltonian vector fields corresponding to f,g.

The Euclidian space $\mathbf{R}^{2N}$ admits global canonical coordinates $(q,p) \equiv \{q_i,p_j; i,j = 1,2,\ldots,N\}$, which satisfy $\{q_i,p_j\} = \delta_{ij}$ and $\{q_i,q_j\} = \{p_i,p_j\} = 0$. In the classical mechanical application of Hamiltonian dynamics any real smooth function of (q,p) represents a physical variable. Quantum mechanics is characterized also by the metric structure and therefore the set of physical variables of a quantum system, defined as generators of transformations that preserve the typical structures Ω and G, is different: Only real quadratic functions of the form

$$f(q,p) = \sum_{ij}^{N} f_{ij}^1(q_i q_j + p_i p_j) + f_{ij}^2 q_i p_j, \tag{3}$$

where f_{ij}^1 are real symmetric and f_{ij}^2 are real antisymmetric, are assumed to be related to quantum physical observables. The most important property of such quadratic functions is that the corresponding Hamiltonian vector fields generate symplectic maps that preserve the Riemannian structure. Thus, the physical variables generate transformations that preserve the two constituting structures of the quantum phase space $(\mathcal{M},G,\Omega,J)$. It is important to stress that, contrary to the case of classical mechanics, not all values of a function (3) representing a physical variable can be obtained as a result of quantum measurements of this physical variable. Possible results of measurements in the Hamiltonian formulation will be discusses shortly.

It is convenient to introduce the set of complex coefficients π_{lm} such that

$$\begin{aligned} f(q,p) &= \sum_{ij}^{N} f_{ij}^1\left(q_i^2 + p_i^2\right) + f_{ij}^2 q_i p_j \\ &= \sum_{lm}^{N} \pi_{lm}(q_l - ip_l)(q_m + ip_m). \end{aligned} \tag{4}$$

Obviously, one has

$$\mathrm{Re}\pi_{ij} = f_{ij}^1, \quad \mathrm{Im}\pi_{ij} = -f_{ij}^2/2. \tag{5}$$

In fact, π_{lm} form an $N \times N$ Hermitian matrix, so there is a Hermitian operator $\hat{F}$ on a Hilbert space $\mathcal{H}^N$ and a proper basis $|e_l\rangle$, $l = 1,2,\ldots,N$, such that

$$\pi_{lm} = \langle e_l|\hat{F}|e_m\rangle. \tag{6}$$

Here the Hermitian scalar product between two vectors $\langle\psi_1|\psi_2\rangle$ is related to the metric and symplectic structures on $\mathcal{M}$ by $\langle\psi_1|\psi_2\rangle = G(\psi_1,\psi_2)/2 + i\Omega(\psi_1,\psi_2)/2$, where on the right-hand side we identified $\mathbf{R}^{2N}$ with its tangent space.

Using the proper basis $|e_l\rangle$, one associates a Hilbert space vector $|\psi_{qp}\rangle \in \mathcal{H}^N$ with the point $m \in \mathbf{R}^{2N}$ parametrized by the canonical coordinates values $\{q_l = m_l, p_l = m_{l+N}; l = 1,2,\ldots,N\}$. The relation is

$$|\psi\rangle = \sum_{l}^{N}(q_l + ip_l)|e_l\rangle. \tag{7}$$

The operator $\hat{F}$ in (6) is given in terms of this proper basis by

$$\hat{F} = \sum_{ij}^{N} \pi_{ij}|e_i\rangle\langle e_j| \tag{8}$$

and the quadratic function

$$F(q,p) = \langle\psi_{qp}|\hat{F}|\psi_{qp}\rangle \tag{9}$$

is to be interpreted as the quantum expectation of the quantum observable $\hat{F}$ in the state $|\psi_{qp}\rangle$. The Poisson bracket between two quadratic functions F_1 and F_2 is related to the quadratic function corresponding to the commutator between the corresponding operators $\hat{F}_1$ and $\hat{F}_2$:

$$\frac{1}{i}\langle\psi|[\hat{F}_1,\hat{F}_2]|\psi\rangle = \{F_1,F_2\}, \tag{10}$$

where, as before, $F_i(\psi) = \langle\psi|\hat{F}_i|\psi\rangle$.

The kinematic part of the Hamiltonian formulation of QM will be referred to as the quantum phase-space formulation. The dynamics of a quantum system is in the HQM given by the abstract Hamiltonian equations

$$\dot{m} = X_H(m), \tag{11}$$

where $m \in \mathcal{M}$ and X_H is the Hamiltonian vector field corresponding to the function $H(m) = \langle\psi_m|\hat{H}|\psi_m\rangle$, where $\hat{H}$ is the Hamiltonian of the system. In the complex space the corresponding equation is the Schrödinger equation

$$i\hbar|\dot{\psi}(t)\rangle = \hat{H}|\psi(t)\rangle. \tag{12}$$

In canonical coordinates Eq. (11) is

$$\begin{aligned} \dot{q}_i &= \{H(q,p),q_i\} = \frac{\partial H(q,p)}{\partial p_i}, \\ \dot{p}_i &= \{H(q,p),p_i\} = -\frac{\partial H(q,p)}{\partial q_i}, \quad i = 1,2,\ldots,N. \end{aligned} \tag{13}$$

Information about the state of a quantum system is obtained by performing operations with the considered systems and possibly additional systems. A quite restricted class of such operations is the ideal measurements in the sense of von Neumann. In the Hilbert space formulation, the data collected by such a measurement involve spectral decomposition of an appropriate Hermitian operator, i.e., involve an appropriate PVM. In the Hamiltonian formulation of quantum mechanics, the data collected by such measurements involve only the functions of the form (9) and the Riemannian structure G. A full description of the von Neumann measurement of an observable with a possibly degenerate and continuous spectrum in the Hamiltonian framework is discussed in [1]. As an illustration, we recapitulate the case of an observable with a discrete nondegenerate spectrum. Possible results of a measurement from this class are exhausted by the critical values of a function of the form (9). We denote these critical values and the corresponding critical points from $\mathbf{R}^{2N}$ by $F_{0,i}$, $i = 1,2,\ldots,N$, and $X_{F,i}$, respectively. For the system in a state $m_{in} \equiv (q_{in}, p_{in})$ the probability of the measurement result $F_{0,i}$ is given by $G(m_{in}, X_{F,i})$, where $X_{F,i}$ is the ith critical point of F. In terms of the quadratic function (9), the possible results of an F measurement are obtained by diagonalization of the Hermitian matrix π^f_{ij}. Spectral decomposition of the operator $\hat{F}$ corresponds to the harmonic-oscillator representation of the quadratic function $f(q',p') = \sum_l \pi^f_l (q'^2_l + p'^2_l)$, where the sum goes over distinct eigenvalues π^f_l of the matrix π^f_{ij} and $\{q'_l, p'_l\}$ denote here the real and imaginary parts of the eigenvalues of $\hat{F}$. The transformation from (q,p) to (q',p') coordinates is of course canonical and isometric.

However, the most general class of quantum operations that can be used to obtain information about a quantum state is in the Hilbert space formulation described by POVMs [14,15]. A description of such a generalized measurement process in terms of a POVM involves two important mathematical properties of POVMs. (i) For all purposes related to quantum information processing, a POVM can always be given in terms of an overcomplete basis (Davis theorem [15,17]). (ii) A POVM can be obtained by projecting a PVM acting in a larger Hilbert space (Neumark theorem [15,18]). The Hamiltonian formulation of these properties of POVMs is the topic of this paper.

III. POSITIVE-OPERATOR-VALUED MEASURES AND FUNCTIONS REPRESENTING PHYSICAL VARIABLES

Consider an overcomplete set of vectors $\{|b_i\rangle, i \in I\}$ where the index i can be discrete $i = 1,2,\ldots,M > N$ or continuous. It is sometimes convenient to use a multidimensional or complex index set I, for example, in the case of ordinary coherent states. The one-dimensional (1D) projectors $|b_i\rangle\langle b_i|$ are not mutually orthogonal. Such an overcomplete set provides a POVM $\{\hat{B}_i = |b_i\rangle\langle b_i|, i \in I\}$. Such a POVM can be used, instead of a PVM, to define a Hermitian operator or to represent a Hermitian operator as a function of the corresponding $2M$ noncanonical variables. Furthermore, there is a Hilbert space $\bar{\mathcal{H}}$, a PVM $\{\hat{P}_i\}$, and a projector $\hat{\Pi}$, $\bar{\mathcal{H}} \to \mathcal{H}$, such that a POVM $\{\hat{B}_i\}$ is given by $\hat{B}_i = \hat{\Pi}\hat{P}_i\hat{\Pi}$. We analyze the formulation and consequences of these facts in the Hamiltonian framework of QM. In the abstract treatment of this section we restrict our attention to the case when the index i, enumerating the vectors of the overcomplete basis, is discrete and finite. The case when the index enumerating the overcomplete basis is real and continuous but bounded will be treated in detail in the next section using the example of a POVM associated with the quantum phase. Another common example of an overcomplete basis and the corresponding POVM with an interesting Hamiltonian formulation, which, however, will not be treated here, is provided by the coherent states of a single linear harmonic oscillator on $L_2(\mathbf{R})$. The index set here is the complex plane $\mathbf{C}$. The Neumark extension of this POVM is the PVM given by the multiplication operator on the Hilbert space $L_2(\mathbf{R}^2)$. The Hamiltonian formulations of the original and the extended system with constraints both involve infinite-dimensional Hamiltonian systems.

A. Positive-operator-valued measures as a set of dependent coordinates

A set of canonical coordinates (q,p) is uniquely related to a basis of mutually orthogonal vectors, i.e., with a PVM. On the other hand, an overcomplete basis $\{|b_j\rangle\}$ with the index j continuous or discrete with $\max j = M > N$ can be used to associate with each vector in the $2N$-dimensional space a set of $2M > 2N$ real numbers. Thus, the overcomplete basis provides $2M$ parameters to characterize the points from $\mathbf{R}^{2N}$. Obviously, the values of these parameters on $\mathbf{R}^{2N}$ cannot be linearly independent.

Consider a set of vectors $\mathcal{B} = \{|b_j\rangle\}_{j=1}^M$ $(M \geqslant N)$ generating a resolution of unity in $\mathcal{H}^N$,

$$\sum_{j=1}^{M} |b_j\rangle\langle b_j| = \mathbb{I}_N. \tag{14}$$

The set $\mathcal{B}$ could be a proper basis $(M = N)$ where the vectors are necessarily mutually orthogonal, but could also be an overcomplete basis $(M > N)$, when at least two of the vectors are not orthogonal. In any case, a (normalized) state $|\psi\rangle$ from the Hilbert space $\mathcal{H}^N$ can be expanded, using (14), as

$$|\psi\rangle = \sum_{j=1}^{M} c_j |b_j\rangle = \sum_{j=1}^{M} (q_j + ip_j)/\sqrt{2\hbar}\,|b_j\rangle, \tag{15}$$

with real q_j and p_j. Coefficients (q_j, p_j) are uniquely defined if and only if the vectors $|b_j\rangle$ form a proper basis. On the other hand, if the resolution of unity (14) is overcomplete, i.e., if some of the $|b_j\rangle$ are not mutually orthogonal, the coefficients (q_j, p_j) satisfying (15) are not unique, but any such set of $2M$ coefficients satisfies the relations

$$\begin{aligned} q_i &= \sum_j q_j \mathrm{Re}\langle b_i|b_j\rangle - p_j \mathrm{Im}\langle b_i|b_j\rangle, \\ p_i &= \sum_j p_j \mathrm{Re}\langle b_i|b_j\rangle + q_j \mathrm{Im}\langle b_i|b_j\rangle. \end{aligned} \tag{16}$$

Obviously, if the basis $\mathcal{B}$ is proper then Eq. (16) are reduced to trivial identities and the explicit expressions for the coefficients are

$$q_j = \mathrm{Re}\langle b_j|\psi\rangle, \quad p_j = \mathrm{Im}\langle b_j|\psi\rangle. \tag{17}$$

However, if the basis is overcomplete the relations are nontrivial and express the nonuniqueness of the expansion (15). The general explicit form of the coefficients in this case is given later in (22).

The coordinate form of the abstract Schrödinger equation (12) or equivalently of the abstract Hamilton equations (11), corresponding to the general set $\mathcal{B}$ satisfying (14), is equivalent to the set of equations

$$G_{\mathcal{B}}\begin{bmatrix}\dot{\boldsymbol{q}}\\ \dot{\boldsymbol{p}}\end{bmatrix}=\begin{bmatrix}\partial H/\partial \boldsymbol{p}\\ -\partial H/\partial \boldsymbol{q}\end{bmatrix}, \tag{18}$$

where the vectors of coordinates are given by $\boldsymbol{q}=[q_1,\ldots,q_M]^T$ and $\boldsymbol{p}=[p_1,\ldots,p_M]^T$, while $H(\boldsymbol{q},\boldsymbol{p})=\langle\psi_{q,p}|\hat{H}|\psi_{q,p}\rangle$. The Gram matrix of the set $\mathcal{B}$ can be cast into the real form

$$G_{\mathcal{B}}=\begin{bmatrix}\boldsymbol{g} & -\boldsymbol{\pi}\\ \boldsymbol{\pi} & \boldsymbol{g}\end{bmatrix}, \tag{19}$$

where matrices $\boldsymbol{g}$ and $\boldsymbol{\pi}$ have the elements $g_{jk}=\mathrm{Re}\langle b_j|b_k\rangle$ and $\pi_{jk}=\mathrm{Im}\langle b_j|b_k\rangle$, respectively. If $\mathcal{B}$ is a proper basis ($M=N$), then $G_{\mathcal{B}}$ becomes an identity matrix of dimension $2N$ and Eq. (18) assumes the form of the Hamilton equations in a canonical basis

$$\begin{bmatrix}\dot{\boldsymbol{q}}\\ \dot{\boldsymbol{p}}\end{bmatrix}=\begin{bmatrix}\partial H/\partial \boldsymbol{p}\\ -\partial H/\partial \boldsymbol{q}\end{bmatrix}. \tag{20}$$

Consider now the case when the set $\mathcal{B}$ is overcomplete ($M>N$). Due to the overcompleteness of the basis $\mathcal{B}$, there are $M-N$ nontrivial zero-valued complex linear combinations of basis vectors or equivalently $2(M-N)$ real linear combinations

$$G_{\mathcal{B}}\begin{bmatrix}\boldsymbol{x}^{(k)}\\ \boldsymbol{y}^{(k)}\end{bmatrix}=0,\quad G_{\mathcal{B}}\begin{bmatrix}-\boldsymbol{y}^{(k)}\\ \boldsymbol{x}^{(k)}\end{bmatrix}=0,\quad k=1,\ldots,M-N, \tag{21}$$

where $\boldsymbol{x}^{(k)}=[x_1^{(k)},\ldots,x_M^{(k)}]^T$ and $\boldsymbol{y}^{(k)}=[y_1^{(k)},\ldots,y_M^{(k)}]^T$ are $M-N$ independent solutions of (21). Thus, the general form of the (q_j,p_j) coefficients satisfying (15) is

$$\begin{bmatrix}\boldsymbol{q}\\ \boldsymbol{p}\end{bmatrix}=\begin{bmatrix}\mathrm{Re}\langle\boldsymbol{b}|\psi\rangle\\ \mathrm{Im}\langle\boldsymbol{b}|\psi\rangle\end{bmatrix}+\sum_{k=1}^{M-N}\left(a_k\begin{bmatrix}\boldsymbol{x}^{(k)}\\ \boldsymbol{y}^{(k)}\end{bmatrix}+b_k\begin{bmatrix}-\boldsymbol{y}^{(k)}\\ \boldsymbol{x}^{(k)}\end{bmatrix}\right). \tag{22}$$

The matrix $G_{\mathcal{B}}$ is singular and Eq. (18) cannot be cast into the canonical form of Hamiltonian equations. The parameters $\{q_j,p_j\}$ do not form a set of canonical coordinates on $\mathbf{R}^{2N}$. In fact, (18) has the equivalent form

$$\begin{bmatrix}\dot{\boldsymbol{q}}\\ \dot{\boldsymbol{p}}\end{bmatrix}=G_{\mathcal{B}}^{(-1)}\begin{bmatrix}\partial H/\partial \boldsymbol{p}\\ -\partial H/\partial \boldsymbol{q}\end{bmatrix}+\sum_{k=1}^{M-N}\left(\lambda_k\begin{bmatrix}\boldsymbol{x}^{(k)}\\ \boldsymbol{y}^{(k)}\end{bmatrix}+\mu_k\begin{bmatrix}-\boldsymbol{y}^{(k)}\\ \boldsymbol{x}^{(k)}\end{bmatrix}\right), \tag{23}$$

where λ_k and μ_k are arbitrary real numbers and $G_{\mathcal{B}}^{(-1)}$ is the Moore-Penrose pseudoinverse. The terms in (23) under the sum do not influence the evolution of the state $|\psi\rangle$. In anticipation of the Hamiltonian treatment, in the next section the numbers λ_k and μ_k can be considered as corresponding to the gauge degrees of freedom. Fixing their values would give additional $2(M-N)$ constraints and yield one possible solution. For example, the natural gauge could be $\lambda_k=0$ and $\mu_k=0$ for $k=1,\ldots,M-N$.

B. Hamiltonian formulation of the Neumark extension

We shall first briefly recapitulate the Hilbert space formulation of the Neumark extension, introducing the appropriate notation at the same time. It will then be demonstrated that the Hamiltonian description of the relation between the Neumark extension and the original system is in fact given in terms of a reduction of the Hamiltonian systems with primary constraints. i.e., with gauge degrees of freedom.

Let $\{|e_k\rangle\}_{k=1}^N$ be a proper orthonormal basis of $\mathcal{H}^N$. Then one has

$$|b_j\rangle=\sum_{k=1}^N\beta_{kj}|e_k\rangle,\quad j=1,\ldots,M, \tag{24}$$

with $\beta_{kj}=\langle e_k|b_j\rangle$. Using (14) we get the relations

$$\sum_{j=1}^M\beta_{kj}\beta_{k'j}^*=\sum_{j=1}^M\langle e_k|b_j\rangle\langle b_j|e_{k'}\rangle=\langle e_k|e_{k'}\rangle=\delta_{k'k}. \tag{25}$$

The relation means that we have a set $\{\boldsymbol{b}_k=[\beta_{k1},\ldots,\beta_{kM}]^T\}_{k=1}^N$ of N orthonormal vectors from $\mathbb{C}^M$. We can choose $M-N$ auxiliary vectors $\{\boldsymbol{b}_{N+1},\ldots,\boldsymbol{b}_M\}$ such that $\{\boldsymbol{b}_k\}_{k=1}^M$ is an orthonormal basis of $\mathbb{C}^M$. Now let us consider an enlarged Hilbert space

$$\mathcal{H}^M=\mathcal{H}^N\oplus\mathcal{H}^\perp, \tag{26}$$

where $\mathcal{H}^\perp=\mathcal{S}(\{|e_k\rangle\}_{k=N+1}^M)$ with orthonormal auxiliary basis states $\{|e_k\rangle\}_{k=N+1}^M$ and $\mathcal{S}$ denoting the span. The states $\{|B_j\rangle=\sum_{k=1}^M\beta_{kj}|e_k\rangle\}_{j=1}^M$ are also orthonormal. Hence, an arbitrary normalized state $|\Psi\rangle\in\mathcal{H}^M$ has the unique expansion

$$|\Psi\rangle=\sum_{j=1}^M(Q_j+iP_j)/\sqrt{2\hbar}\,|B_j\rangle, \tag{27}$$

with real Q_j and P_j that can be regarded as a pair of canonical coordinates on the extended phase space $\mathbf{R}^{2M}$. Define the projector operator by

$$\hat{\Pi}|e_k\rangle=|e_k\rangle,\quad k=1,\ldots,N \tag{28a}$$

$$\hat{\Pi}|e_k\rangle=0,\quad k=N+1,\ldots,M. \tag{28b}$$

This leads to

$$\hat{\Pi}|B_j\rangle=|b_j\rangle \tag{29}$$

and

$$\hat{\Pi}|\Psi\rangle=\sum_{j=1}^M(Q_j+iP_j)/\sqrt{2\hbar}\,|b_j\rangle, \tag{30}$$

which is of the same form as (15). In other words, the PVM given by the proper basis $\{|B_j\rangle\}$ in $\mathcal{H}^M$ is the Neumark extension of the POVM given on $\mathcal{H}^N$ by $\{|b_j\rangle\}$).

Strictly speaking, the Neumark theorem is not concerned with the dynamics, i.e., Hamiltonians, on $\mathcal{H}^N$ versus that on $\mathcal{H}^M$. Nevertheless, it is natural to require that $\hat{H}$ on $\mathcal{H}^N$ and the corresponding $\hat{H}_{ex}$ on $\mathcal{H}^M$ satisfy the following condition: All states from $\mathcal{H}^M$ that are projected onto the same state $|\psi(t_0)\rangle$ in $\mathcal{H}^N$ evolve during $t-t_0$ into the states that are all projected onto the same state $|\psi(t)\rangle$. This is the case if

$$\hat{H}_{ex}=\hat{\Pi}^{-1}\hat{H}\hat{\Pi}. \tag{31}$$

In anticipation of the Hamiltonian formulation, expectation values of $\hat{H}_{ex}$ in $|\Psi\rangle$ and $\hat{H}$ in $|\psi\rangle=\hat{\Pi}|\Psi\rangle$ are related by

$$\begin{aligned} H_{ex}(Q,P) &= \langle \Psi_{Q,P}|\hat{H}_{ex}|\Psi_{Q,P}\rangle \\ &= \sum_{j,j'}(Q_j - iP_j)(Q_{j'} + iP_{j'})\langle B_j|\hat{\Pi}^{-1}\hat{H}\hat{\Pi}|B_{j'}\rangle \\ &= \sum_{j,j'}(Q_j - iP_j)(Q_{j'} + iP_{j'})\langle b_j|\hat{H}|b_{j'}\rangle \\ &= H(Q_j,P_j), \end{aligned} \quad (32)$$

where (Q,P) in $H_{ex}(Q,P)$ and in $H(Q,P)$ are the same numbers but are treated as values of independent coordinates on $\mathbf{R}^{2M}$ or dependent parameters on $\mathbf{R}^{2N}$, respectively.

We now present the phase-space formulation of the Neumark extension. Consider the space $\mathbf{R}^{2M}$ as a symplectic manifold of a Hamiltonian system with the canonical coordinates denoted by $\{(Q_j,P_j);\, j = 1,2,\ldots,M\}$. The natural conditions $\langle e_k|\Psi(t)\rangle = 0$ for $k = N+1,\ldots,M$ are implemented as constraints on the phase space $\mathbf{R}^{2M}$. Explicitly, the $2(M-N)$ constraints are

$$\phi_k(\boldsymbol{Q},\boldsymbol{P}) \equiv \sum_{j=1}^{M}\left(\beta^R_{kj}Q_j - \beta^I_{kj}P_j\right) = 0, \quad (33a)$$

$$\pi_k(\boldsymbol{Q},\boldsymbol{P}) \equiv \sum_{j=1}^{M}\left(\beta^I_{kj}Q_j + \beta^R_{kj}P_j\right) = 0, \quad (33b)$$

where $\beta^R_{kj} = \mathrm{Re}\beta_{kj}$ and $\beta^I_{kj} = \mathrm{Im}\beta_{kj}$. The constraints satisfy the Poisson brackets

$$\{\phi_k,\pi_{k'}\}_{Q,P} = \mathrm{Re}\sum_{j=1}^{M}\beta^*_{k'j}\beta_{kj} = \delta_{k'k}, \quad (34a)$$

$$\{\phi_k,\phi_{k'}\}_{Q,P} = \{\pi_k,\pi_{k'}\}_{Q,P} = \mathrm{Im}\sum_{j=1}^{M}\beta^*_{k'j}\beta_{kj} = 0. \quad (34b)$$

In the general case of arbitrary constraints, the constrained manifold need not be symplectic and need not support a Hamiltonian system. However, in our case (33), the matrix of Poisson brackets between the constraints (34) is nonsingular, i.e., the constraints are primary, and therefore the manifold determined by the constraints is also symplectic. The symplectic structure on the constrained manifold is given by the Dirac-Poisson bracket on $\mathbf{R}^{2N}$ [19,20],

$$\begin{aligned} \{f_1,f_2\}_{\mathbf{R}^{2N}} &= \{f_1,f_2\}_{\mathbf{R}^{2M}} + c\sum_{m,n}^{2(M-N)}\{F_n,f_1\}_{\mathbf{R}^{2M}} \\ &\quad \times \{F_m,F_n\}^{-1}_{\mathbf{R}^{2M}}\{F_m,f_2\}_{\mathbf{R}^{2M}}, \end{aligned} \quad (35)$$

where f_1,f_2 are functions on the constrained manifold, the symbols $F_n,F_m,m,n = 1,2,\ldots,2(M-N)$ denote the constraints (33), and the Poisson brackets on the right-hand side are the canonical brackets on $\mathbf{R}^{2M}$. The general formula (35) in the notation (33) assumes the explicit form

$$\{f_1,f_2\}_{\mathbf{R}^{2N}} = \{f_1,f_2\}_{Q,P} - \sum_{k=N+1}^{M}\left(\frac{\partial f_1}{\partial\phi_k}\frac{\partial f_2}{\partial\pi_k} - \frac{\partial f_2}{\partial\phi_k}\frac{\partial f_1}{\partial\pi_k}\right). \quad (36)$$

Consider now the relation between the Hamiltonian function $H(Q,P)$ as a function of the dependent parameters on $\mathbf{R}^{2N}$, i.e., as the Hamiltonian of the system given on $\mathbf{R}^{2N}$, and the Hamiltonian system on $\mathbf{R}^{2M}$ with the Hamiltonian $H(Q,P)$ [where (Q,P) are now independent and canonical on $\mathbf{R}^{2M}$] with imposed constraints (33). In the case of general constraints they can be incorporated into the dynamics using the standard Dirac approach [19,20]. Namely, the total Hamilton function has the form

$$\begin{aligned} H_T &= H + \sum_{k=N+1}^{M}(\phi_k\lambda_k - \pi_k\mu_k), \\ H_T &= H + \sum_{k=N+1}^{M}(\phi_k\{\pi_k,H\}_{Q,P} - \pi_k\{\phi_k,H\}_{Q,P}), \end{aligned} \quad (37)$$

where the appropriate Lagrange multipliers λ_k,μ_k have been determined from the compatibility conditions and using (34). However, if the constraints are such that the constrained manifold is symplectic, as they are in our case, then the general procedure of constructing the Hamiltonian on the constrained manifold can be bypassed. In fact, in this case the Hamiltonian of the system on the constrained manifold is simply obtained as a restriction of the Hamiltonian on $\mathbf{R}^{2M}$ on the constrained manifold $\mathbf{R}^{2N}$. This is precisely the relation between the expectation values of the Hamiltonian operators (32) introduced within the treatment of the Neumark extension.

What has been demonstrated is that the $2M$ real state parameters $\{Q_j,P_j;\, j = 1,2,\ldots,M\}$ given by the POVM, i.e., by the overcomplete set $\{|b_j\rangle\langle b_j|;\, j = 1,2,\ldots,2M\}$ in $\mathcal{H}^N$, can be considered as parameters on $\mathbf{R}^{2N}$ or equivalently as canonical coordinates of an extended Hamiltonian system on $\mathbf{R}^{2M}$ with imposed primary constraints. We see that the overcomplete description given by a POVM involves in the Hamiltonian formulation the existence of constraints, i.e., the gauge degrees of freedom, and the corresponding reduction of an extended Hamiltonian system. This is yet another example of the insights into the quantum-mechanical formalism provided by the Hamiltonian formulation.

C. Functions associated with POVMs

In this section we derive several simple but useful formulas.

1. Functions corresponding to PVMs or POVMs

As before, $\mathcal{B} = \{|b_k\rangle;\, k = 1,2,\ldots,M \geqslant N\}$ denotes an arbitrary set of vectors, with the corresponding set of 1D projectors $\{\hat{P}_k = |b_k\rangle\langle b_k|\}$. Using another set $\mathcal{B}' = \{|b'_l\rangle;\, l = 1,2,\ldots,M' \geqslant N\}$, with the corresponding set $\{\hat{P}_l = |b'_l\rangle\langle b'_l|\}$ that satisfies (14), each of the projectors $|b_k\rangle\langle b_k|$ is associated with a quadratic function of the parameters (q'_l,p'_l) provided by $\mathcal{B}'$. Thus, corresponding to the set $\{\hat{P}_k\}$ is the set of quadratic functions

$$P_k(q',p') = \sum_{lm}^{M'}\pi^k_{lm}(q'_l - ip'_l)(q'_m + ip'_m), \quad (38a)$$

where

$$\pi^k_{lm} = \langle b'_l|\hat{P}_k|b'_m\rangle, \quad k = 1,2,\ldots,M. \quad (38b)$$

It is our goal to obtain explicit conditions that distinguish the sets of coefficients π^k_{lm} in (38), given by $\mathcal{B}$ and $\mathcal{B}'$ representing a PVM and/or a POVM. In this setup, N is the dimension of the Hilbert space, $M \geqslant N$ is the number of vectors in the set $\mathcal{B}$ whose properties such as the resolution of unity and orthogonality are to be studied, and $M' \geqslant N$ is the number of vectors in the complete (or overcomplete) set $\mathcal{B}'$ that is used to associate functions with projectors from the set $\mathcal{B}$.

If the set of functions (38) corresponds to either a PVM or a POVM, an analog of the condition (14) must be satisfied by the coefficients π^k_{lm}. Furthermore, if the set corresponds to a PVM, then the condition of mutual orthogonality of the involved projectors has its analog in terms of the coefficients π^k_{lm}.

2. Resolution of unity in terms of π^k_{lm} coefficients

Consider the scalar product $\langle b'_l|b'_m\rangle$ between arbitrary two-vectors from $\mathcal{B}'$. The condition (14) on $\mathcal{B}$ would imply

$$\langle b'_l|b'_m\rangle = \langle b'_l|\sum_k^M \hat{P}_k|b'_m\rangle = \sum_k^M \langle b'_l|\hat{P}_k|b'_m\rangle = \sum_k^M \pi^k_{lm}, \quad l,m = 1,2,\ldots,M'. \tag{39}$$

Thus, if $\mathcal{B}$ is complete, then

$$\sum_k^M \pi^k_{lm} = \langle b'_l|b'_m\rangle, \quad l,m = 1,2,\ldots,M'. \tag{40}$$

Obviously, if the coordinates (q'_l,p'_l) in the set of functions (38) are associated with an orthogonal (and complete) basis, then

$$\sum_k^M \pi^k_{lm} = \delta_{lm}. \tag{41}$$

It is equally simple to show that if (40) is true, then the set $\mathcal{B}$ satisfies (14). This follows from the equalities in the reverse order of (39) and from the fact that if an operator has all matrix elements between vectors from a complete (or overcomplete) set equal to zero, then it is the zero operator, i.e., it annihilates each vector from the Hilbert space. Thus, the set of functions given by (38) satisfies (40) if and only if the set of projectors $\{\hat{P}_k = |b_k\rangle\langle b_k|; k = 1,2,\ldots,M \geqslant N\}$ generates a resolution of unity (14).

3. Orthogonality of two projectors in terms of π^k_{lm} coefficients

The orthogonality of projectors $\hat{P}_k$ and $\hat{P}_{k'}$ is equivalent to

$$\hat{P}_k\hat{P}_{k'} = \delta_{kk'}\hat{P}_{k'}. \tag{42}$$

Using arbitrary set $\mathcal{B}'$ satisfying (14), the condition (42) implies the following conditions on all pairs of coefficients $\pi^k_{lm},\pi^{k'}_{lm}$:

$$\sum_l^{M'} \pi^k_{l'l}\pi^{k'}_{ll''} = \delta_{kk'}\pi^k_{ll''}, \quad l,l' = 1,2,\ldots,M; \quad k,k' = 1,2,\ldots,M. \tag{43}$$

Observe that the two conditions (43) and (40) are based only on the assumption that $|\psi\rangle = \sum_l^{M'}(q'_l + ip'_l)|b'_l\rangle$, which is true since $\mathcal{B}'$ satisfies (14).

The two criteria (43) and (40) taken together imply that the set of functions $\mathcal{P}^k = \sum_{l,m}^{M'} \pi^k_{lm}(q'_l - ip'_l)(q'_m + ip'_m)$, $k = 1,2,\ldots,M$, represents a PVM if all pairs $\pi^{k_1}_{lm},\pi^{k_2}_{lm}$ $(k_1,k_2 = 1,2,\ldots,M)$ correspond to orthogonal projectors, i.e., satisfy (43), and if the condition (40) is satisfied. If only the condition (40) is satisfied but there is a pair of π^{k_1},π^{k_2} violating (43), then the set of functions corresponds to a POVM. Furthermore, the parameters appearing as the arguments in the considered functions are canonical, i.e., the basis is proper orthonormal if (41) is satisfied.

Let us also briefly discuss the notion of orthogonality of the quadratic functions representing observables in a proper basis. Consider two operators $\hat{A}_1$ and $\hat{A}_2$. The operators are orthogonal if

$$\mathrm{Tr}[\hat{A}_1\hat{A}_2] = 0. \tag{44}$$

In terms of the coefficients π^k_{ij} in the quadratic functions corresponding to $\hat{A}_1,\hat{A}_2$ the previous condition is written as

$$\mathrm{Tr}[\hat{A}_1\hat{A}_2] = \sum_i^N \langle a_i|\hat{A}_1\hat{A}_2|a_i\rangle = \sum_{ii'}^N \langle a_i|\hat{A}_1|a'_i\rangle\langle a'_i|\hat{A}_2|a_i\rangle = \sum_{i,i'}^N \pi^1_{ii'}\pi^2_{i'i} = 0. \tag{45}$$

Thus, it makes sense to call two quadratic functions of the form (9) orthogonal if

$$\sum_{ij}^N \pi^1_{ij}\pi^2_{ji} = 0. \tag{46}$$

The condition (46) supplies us with the notion of orthogonality between two quadratic functions solely in terms of these functions, with no reference to the analogous Hilbert space formulation.

An alternative criterion for orthogonality of two 1D projectors in terms of the associated quadratic function, i.e., in terms of π^1_{ij},π^2_{ij}, is obtained from the fact that orthogonal 1D projectors commute and the relation (10). In fact,

$$\langle[\hat{P}_\mu,\hat{P}_\nu]\rangle = \delta_{\mu,\nu} = i\{P_\mu,P_\nu\} = i\sum_{ij}\langle a_k|e_\mu\rangle\langle e_\mu|a_l\rangle\langle a_k|e_\nu\rangle\langle e_\nu|a_l\rangle \times[-i\{p_i,q_j\} - i\{q_i,p_j\}] = 2\sum_k \pi^\mu_{ii}\pi^\nu_{ii}. \tag{47}$$

If the two 1D projectors are orthogonal, the sum of the products of the diagonal coefficients in the corresponding quadratic functions is zero.

4. Relations between functions representing an operator given by PVMs or by POVMs

A Hermitian operator can be defined using a proper basis $|e_i\rangle$, $i = 1,2,\ldots,N$, or an overcomplete basis $|b_l\rangle$, $l = 1,2,\ldots,M$. Similarly, the operator can be represented as a quadratic function of $2N$ canonical variables (q,p) using the proper basis or as a quadratic function of the $2M$ noncanonical parameters (Q,P). Relations between different representations are given by the simple formula

$$\pi_{lm} = \sum_{ij} a_{ij}\langle b_l|e_i\rangle\langle e_j|b_m\rangle, \tag{48}$$

where π_{lm} $(l,m = 1,2,\ldots,M)$ and a_{ij} $(i,j = 1,2,\ldots,N)$ are the coefficients in representations given by the overcomplete and the proper orthogonal basis, respectively.

The inverse relation expressing a_{ij} $(i,j = 1,2,\ldots,N)$ in terms of π_{lm} $(l,m = 1,2,\ldots,M)$ reads

$$a_{ij} = \sum_{lm} \pi_{lm}\langle e_i|b_l\rangle\langle b_m|e_j\rangle. \tag{49}$$

In formulas (48) and (49) the Hermitian scalar product could be replaced by the combination of the Riemannian scalar product and the symplectic skew product, expressing the relations entirely in terms of objects appearing in the Hamiltonian formulation. However, the corresponding transformations are not canonical.

IV. RELEVANT EXAMPLE: THE PHASE

The phase of quantum motion is an observable physical quantity that is naturally expressed using an appropriate POVM (see [21] and references therein). For our purpose it is enough to discuss the phase POVM in the case of the simplest quantum systems with finite Hilbert spaces and with a nondegenerate energy spectrum. In this section we first illustrate the construction of the relevant nonorthogonal basis and the POVM in the Hilbert space formulation. We then present the corresponding Neumark extension. The Hilbert space analysis will be followed by the corresponding Hamiltonian treatment.

A. Phase POVMs and the Neumark extension

Consider an N_1-dimensional Hilbert space $\mathcal{H}_1$ with an arbitrary proper basis denoted by $|n\rangle_1$, $n = 1,2,\ldots,N_1$. With this basis one associates an infinite set of vectors parametrized by an angle $\varphi \in [0,2\pi)$ defined as

$$|\varphi\rangle_1 = \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{N} e^{ik_n\varphi}|n\rangle_1, \quad \varphi \in [0,2\pi), \tag{50}$$

where $k_n \in \mathbf{Z}$ are integers. In the general construction presented here these integers are arbitrary. However, if the constructed POVM is to correspond to the phase, then the integers are to be precisely the nondegenerate and discrete energy eigenvalues of the considered system.

A collection of operators defined as

$$\hat{\Theta}_1(\varphi_1,\varphi_2) = \int_{\varphi_1}^{\varphi_2}\hat{\mathcal{P}}_1(\varphi)d\varphi \tag{51}$$

and

$$\hat{\mathcal{P}}_1(\varphi) \equiv |\varphi\rangle_{11}\langle\varphi| = \frac{1}{2\pi}\sum_{n=1}^{N}\sum_{m=1}^{N} e^{i(k_n-k_m)\varphi}|n\rangle_{11}\langle m| \tag{52}$$

forms a resolution of unity, i.e.,

$$\hat{\Theta}_1(0,2\pi) = \hat{I}, \tag{53}$$

but the operators associated with disjoined subsets of $\varphi \in [0,2\pi)$ are not orthogonal. Thus the collection (52) forms a POVM. As pointed out, if the integers k_n coincide with the energy eigenvalues, the collection of operators (51) satisfies the so-called covariance condition

$$\exp -ia\hat{H}\hat{\Theta}_1(\varphi_1,\varphi_2)\exp ia\hat{H} = \hat{\Theta}_1((\varphi_1+a)\mathrm{mod}2\pi,(\varphi_2+a)\mathrm{mod}2\pi). \tag{54}$$

This fact justifies the association of the POVM (52) with the data corresponding to the phase of the quantum motion.

In order to formulate the Neumark extension of the phase POVM one needs an appropriate Hilbert space $\mathcal{H}_2$ with dimension $N_2 > N_1$ and a projector-valued measure $\mathcal{P}_2(\varphi)$ with projectors onto orthogonal subspaces of $\mathcal{H}_2$ associated with disjoined intervals. Then the theorem claims that there is a projector $P_{2\to1}$ from $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that $P_{2\to1}\hat{\Theta}_2P_{2\to1}$ is isomorphic to $\hat{\Theta}_1$.

For the case of the POVM given by (52) the Hilbert space $\mathcal{H}_2$, the PVM $\hat{\mathcal{P}}_2$, and the projector $P_{2\to1}$ are given as follows. The Hilbert space $\mathcal{H}_2$ is in fact the complex vector space of square integrable functions on the interval $(0,2\pi)$. The coordinate representation is determined by generalized vectors $|\varphi\rangle_2$ and ${}_2\langle\varphi|\varphi'\rangle_2 = \delta(\varphi-\varphi')$ and a proper basis $|k\rangle_2$ is given by

$${}_2\langle\varphi|k\rangle_2 \equiv \psi_k(\varphi) = \frac{1}{\sqrt{2\pi}}e^{-ik\varphi}, \quad k \in \mathbf{Z}. \tag{55}$$

The proper basis with orthogonal generalized vectors

$$|\varphi\rangle_2 = \frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{+\infty} e^{ik\varphi}|k\rangle_2 \tag{56}$$

is used to define the PVMs

$$\hat{\Theta}_2(a,b) = \int_b^b \hat{\mathcal{P}}_2 d\varphi, \tag{57}$$

$$\hat{\mathcal{P}}_2(\varphi) = |\varphi\rangle_{22}\langle\varphi| = \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty}\sum_{k'=-\infty}^{+\infty} e^{i(k-k')\varphi}|k\rangle_{22}\langle k'|. \tag{58}$$

The relevant projector $P_{2\to1}$ is defined as

$$\hat{P}_{2\to1} = |q_1\rangle_{22}\langle q_1| + |q_2\rangle_{22}\langle q_2| + \cdots + |q_N\rangle_{22}\langle q_N|, \tag{59}$$

where $|q_i\rangle_2$ are vectors (56) with $k = |q_i\rangle$. It follows that

$$\hat{P}_{2\to1}\hat{\mathcal{P}}_2(\varphi)\hat{P}_{2\to1} = \hat{P}_{2\to1}|\varphi\rangle_{22}\langle\varphi|\hat{P}_{2\to1} = \frac{1}{2\pi}\sum_{i=1}^{N}\sum_{i'=1}^{N} e^{i(q_i-q_{i'})\varphi}|q_i\rangle_{22}\langle q_{i'}|, \tag{60}$$

which is isomorphic to the measure $\hat{\mathcal{P}}_1(\varphi)$. This is the Neumark theorem for the POVM given by (52).

B. The phase in the Hamiltonian formulation

The Hamiltonian formulation of the original quantum system on the finite-dimensional Hilbert space $\mathcal{H}_1$ is given on the finite even-dimensional phase space $\mathcal{M}_1$. The Neumark extension involves a Hamiltonian system on an infinite-dimensional symplectic manifold $\mathcal{M}_2$, which is a direct sum of two real vector spaces of square integrable functions on $[0,2\pi)$.

The phase POVM involves an overcomplete set of vectors $\{|\varphi\rangle\}$ indexed by the continuous index $\varphi \in [0,2\pi)$. Consequently, the expansion of an arbitrary $|\psi\rangle \in \mathbf{C}^N$,

$$|\psi\rangle = \int d\varphi[q(\psi;\varphi) + ip(\psi;\varphi)]|\varphi\rangle, \tag{61}$$

generates functional parameters $[q(\psi;\varphi), p(\psi;\varphi)]$ of points $\psi \in \mathbf{C}^N$. On the other hand, the proper energy basis $\{|n\rangle_1\}$ (with eigenvalues $k_n \neq k_{n'}, n \neq n'$) generates, via

$$|\psi\rangle = \sum_n^N [q_n(\psi) + ip_n(\psi)]|n\rangle_1, \tag{62}$$

$2N$ canonical coordinates $\{q_n(\psi), p_n(\psi)\}$ of a point ψ indexed by discrete and finite n. All relevant formulas from Sec. III involve either of the expressions

$$\langle\varphi_1|\varphi_2\rangle = \frac{1}{2\pi}\sum_n^N \exp ik_n(\varphi_2 - \varphi_1) \tag{63}$$

or

$$\langle n|\varphi\rangle = \frac{1}{\sqrt{2\pi}} \exp ik_n\varphi \tag{64}$$

and the real and the imaginary parts thereof. For example, the arbitrary vector $|\Psi\rangle_2$ from $\mathcal{H}_2$ is expended as

$$|\Psi\rangle_2 = \int_0^{2\pi} d\varphi[Q(\varphi) + iP(\varphi)]|\varphi\rangle_2, \tag{65}$$

where the conditions $\langle e_l|\Psi\rangle_2 = 0$, $l \neq k_n$, and $n = 1,\ldots,N$ obtain the explicit form

$$\int_0^{2\pi} d\varphi[Q(\varphi) + iP(\varphi)]\langle e_l|\varphi\rangle_2 = \int_0^{2\pi} d\varphi[Q(\varphi) + iP(\varphi)]\frac{1}{\sqrt{2\pi}}e^{il\varphi} = 0. \tag{66}$$

The functions $Q(\varphi), P(\varphi)$ are the canonical coordinates on $\mathcal{M}_2$. The constraints (33) are explicitly given by

$$\begin{aligned}
&\phi_l(Q(\varphi),P(\varphi)) \\
&= \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} d\varphi[\cos(l\varphi)Q(\varphi) - \sin(l\varphi)P(\varphi)] = 0, \\
&\pi_l(Q(\varphi),P(\varphi)) \\
&= \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} d\varphi[\sin(l\varphi)Q(\varphi) + \cos(l\varphi)P(\varphi)] = 0.
\end{aligned} \tag{67}$$

Variational derivatives of the constraints read

$$\begin{aligned}
\frac{\delta\phi_l}{\delta Q} &= \frac{1}{\sqrt{2\pi}}\cos(l\varphi), \quad \frac{\delta\phi_l}{\delta P} = -\frac{1}{\sqrt{2\pi}}\sin(l\varphi), \\
\frac{\delta\pi_l}{\delta Q} &= \frac{1}{\sqrt{2\pi}}\sin(l\varphi), \quad \frac{\delta\pi_l}{\delta P} = \frac{1}{\sqrt{2\pi}}\cos(l\varphi).
\end{aligned} \tag{68}$$

Poisson brackets between the constraints are, as in the general case (34),

$$\begin{aligned}
\{\phi_l,\pi_{l'}\}_{Q,P} &= \int_0^{2\pi} d\varphi\left(\frac{\delta\phi_l}{\delta Q}\frac{\delta\pi_{l'}}{\delta P} - \frac{\delta\phi_l}{\delta P}\frac{\delta\pi_{l'}}{\delta Q}\right) \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\varphi[\cos(l\varphi)\cos(l'\varphi) \\
&\quad + \sin(l\varphi)\sin(l'\varphi)] = \delta_{ll'}, \\
\{\phi_l,\phi_{l'}\}_{Q,P} &= \frac{1}{2\pi}\int_0^{2\pi} d\varphi[-\cos(l\varphi)\sin(l'\varphi) \\
&\quad + \sin(l\varphi)\cos(l'\varphi)] = 0, \\
\{\pi_l,\pi_{l'}\}_{Q,P} &= \frac{1}{2\pi}\int_0^{2\pi} d\varphi[\sin(l\varphi)\cos(l'\varphi) \\
&\quad - \cos(l\varphi)\sin(l'\varphi)] = 0.
\end{aligned} \tag{69}$$

The functions $\Theta_2^{(a,b)}(Q,P)$ of the canonical (Q,P) corresponding to the PVM $\hat{\Theta}_2(a,b)$ is given, after some computation, by the simple expression

$$\begin{aligned}
\Theta_2^{(a,b)}(Q,P) &= {}_2\langle\Psi|\int_a^b d\varphi|\varphi\rangle_{22}\langle\varphi|\Psi\rangle_2 \\
&= \int_a^b d\varphi[Q^2(\varphi) + P^2(\varphi)],
\end{aligned} \tag{70}$$

which is as expected for the coordinates corresponding to the eigenbases of $\hat{\Theta}_2$. The functions corresponding to the POVM $\hat{\Theta}_1(a,b)$ in the original space are by definition

$$P_1^{(a,b)}(Q,P) = {}_1\langle\Psi|\int_a^b d\varphi|\varphi\rangle_{11}\langle\varphi|\Psi\rangle_1. \tag{71}$$

Due to nonorthogonality of the vectors $|\varphi\rangle_1$, this expression cannot be significantly simplified. The explicit expression reads

$$\begin{aligned}
P_1^{(a,b)}(Q,P) &= \int_0^{2\pi} d\varphi\int_0^{2\pi} d\varphi''[Q(\varphi) - iP(\varphi)] \\
&\quad \times [Q(\varphi'') + iP(\varphi'')]\int_a^b d\varphi'{}_1\langle\varphi|\varphi'\rangle_{11}\langle\varphi'|\varphi''\rangle_1,
\end{aligned}$$

where

$${}_1\langle\varphi|\varphi'\rangle_{11}\langle\varphi'|\varphi''\rangle_1 = \frac{1}{2\pi}\sum_{n=1}^N e^{ik_n(\varphi'-\varphi)}\frac{1}{2\pi}\sum_{m=1}^N e^{ik_m(\varphi''-\varphi')}.$$

This expression results also from explicit substitution of the constraints (67) satisfied by (Q,P) into the expression (70).

V. SUMMARY

We have studied several questions related to the description and interpretation of POVMs in the Hamiltonian formulation of quantum mechanics. The topic is important from the point of view that considers the Hamiltonian formulation as independent and equivalent to the Hilbert space formulation, because the POVMs appear as a description of important quantum mechanical concepts, originally represented mathematically within the Hilbert space formulation. In particular, the POVMs appear in the treatment of approximate and indirect measurements and in the description of joint measurement of conjugate variables. Furthermore, a physically justified definition of certain observables requires the corresponding POVMs instead of standard representation via PVMs. As pointed out, if the Hamiltonian formulation is to be considered as a viable alternative approach to the mathematical formulation of quantum mechanics, it is important to analyze properties of representatives of the POVMs within the Hamiltonian approach.

In particular we have studied the properties of the sets of state coordinates corresponding in the Hamiltonian formulation to an overcomplete basis in the Hilbert space formulation. Coordinates in such a set are dependent and the relations can be treated as constraints on the Hamiltonian formulation in a larger phase space. We have demonstrated that the Hamiltonian treatment of systems with linear primary constraints corresponds to the Neumark extension and reduction. We have also provided the criteria that distinguish between objects representing POVMs from those of PVMs entirely within the Hamiltonian formulation. Finally, these abstract considerations have been illustrated using the example of a POVM corresponding to the phase of quantum motion. The Hilbert space formulation of the phase POVM and the corresponding Neumark extension was described first and then the corresponding Hamiltonian description was provided.

ACKNOWLEDGMENTS

This work was supported in part by the Ministry of Education and Science of the Republic of Serbia under Contracts No. 171006, No. 171017, No. 171020, No. 171038, and No. 45016 and by COST (Action No. MP1006).

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