Fractionalization in dimerized graphene and graphene bilayer

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We propose that a natural setting for the fractionalization, introduced in recent proposals in dimerized graphene, would be certain systems with excitonic instabilities. We demonstrate this by an example of graphene bilayer.

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I. INTRODUCTION

Fractionally charged objects came into quantum-field theory and physics by the work of Jackiw and Rebbi.[1] The phenomenon is associated with Dirac fermions in the presence of nontrivial configurations—vortices of condensed Bose field. The associated fermionic bound states (zero energy modes) are the reason for the fractional charge of the vortices. Recent proposals[2–6] for fractionalization in dimerized graphene are based on this mechanism. The assumed variation in hopping parameters (i.e., dimerization) couples in the manner of a Bose field to electrons in graphene monolayer behaving as Dirac fermions. The proposals lack microscopic Hamiltonian based foundations and lead, for example, depending on which macroscopic description of the fractional objects, i.e., Bose field configuration, is assumed, to different conclusions for statistics as we will show below. Mechanism for dimerization in graphene monolayer (Kekule modulation) is proposed[2] but further physical introduction and inclusion of vortices in graphene monolayer is not clear how to achieve.[2] As already pointed out in Ref. 2, the search for bound states (zero modes in a Dirac-type equation) formally looks like solving Bogoliubov–de Gennes equation of p-wave superconductor but the difference is that the charge is conserved. This motivates our proposal that a natural setting for finding fractional-charge objects would be p-wave excitonic superfluid in which, by way of binding and condensing particles and holes (not particles and particles as in a superconductor), total charge is conserved. Such a system, i.e., p-wave-like excitonic instability, is possible under general conditions in Bernal stacked graphene bilayer,[7,8] as shown in Ref. 9. It should be noted that at present experiments do not reveal Dirac fermionlike spectrum in the graphene bilayer, consistent with noninteracting picture that we have from Refs. 7 and 8, and the question is how to induce sufficiently strong interlayer interactions (small interlayer hopping) at large bias that would lead to the excitonic instability. But once we have it the general discussion in Ref. 9 points out to a p-wave-like gap function that carries the phase which a single particle accumulates in the presence of the Dirac nodal points of the graphene monolayers. Our work will discuss the nature of the vortex solutions with associated bound charged states (as opposed to neutral states that we would have in a superconductor) that makes vortices fractionally charged in this graphene bilayer excitonic superfluid.

In Sec. II we will review the recent proposals[2–6] for fractionalization. There is explicit discrepancy in assigning statistics to the fractional objects between Refs. 3 and 5. To clarify that situation we will rederive the result for statistics in Ref. 3 by emphasizing the point character of the assumed objects. In that respect the derivation differs from Ref. 3 (see also Ref. 11). This will then allow us to compare the two descriptions of essentially different objects with the same fractional charge from Refs. 3 and 5. Section III will be devoted to the bilayer graphene as a stage for charge fractionalization. First we will discuss the difference between the bound states of vortices of p-wave superconductor and p-wave excitonic superfluid. There we will also discuss ways to incorporate vortex singularity into Bogoliubov–de Gennes equations which will then facilitate our discussion of the vortices associated with specific p-wave-like gap function and excitonic condensate of the graphene bilayer that can arise under special conditions.[9]

II. FRACTIONALIZATION IN DIMERIZED GRAPHENE

The effective (low-energy) Hamiltonian in the presence of the Kekule deformation of a graphene monolayer is[2]

$$\mathcal{H} = \int d^2 r \Psi^+(r) \mathcal{K}_D \Psi(r), \tag{1}$$

with

$$\Psi^+(r) = [u_b^+(r) u_a^+(r) v_a^+(r) v_b^+(r)]$$

and

$$\mathcal{K}_D = \begin{pmatrix} 0 & -2i\partial_z & \Delta(r) & 0 \\ -2i\partial_{\bar{z}} & 0 & 0 & \Delta(r) \\ \bar{\Delta}(r) & 0 & 0 & 2i\partial_z \\ 0 & \bar{\Delta}(r) & 2i\partial_{\bar{z}} & 0 \end{pmatrix},$$

where u_a, v_a and u_b, v_b denote the electronic (effective-Dirac) variables of the two triangular sublattices, a and b, respectively, of the honeycomb graphene lattice. The usual Kekule texture we get with $\Delta(r) = \Delta_0$. With no texture we have two cones in the spectrum, $\epsilon_\pm(\vec{p}) = \pm|\vec{p}|$, and, with the Kekule texture, mass gaps are opening: $\epsilon_\pm(\vec{p}) = \pm\sqrt{|\vec{p}|^2 + |\Delta_0|^2}$, in the single-particle spectra.

A. Charge fractionalization

Let us assume a vortex structure in the complex parameter $\Delta(\vec{r})$ (Ref. 2):

$$\Delta(\vec{r}) = \Delta(r)\exp\{-in\theta\}, \tag{2}$$

where polar coordinates are used. We seek solutions for electronic states of the dimerized graphene in the presence of this structure with zero energy. The equations that follow from Eq. (1), in the case of sublattice a, are

$$\partial_z u + i\Delta(\vec{r})v = 0,$$

$$i\bar{\Delta}(\vec{r})u - \partial_{\bar{z}} v = 0, \tag{3}$$

and with the exchange $z \leftrightarrow \bar{z}$, they are also valid in the case of sublattice b.

In the polar coordinates we have

$$\exp\{-i\theta\}\left(\partial_r - \frac{i}{r}\partial_\theta\right)u(\vec{r}) + i\exp\{-in\theta\}\Delta(r)v(\vec{r}) = 0,$$

$$i\exp\{in\theta\}\bar{\Delta}(r)u(\vec{r}) - \exp\{i\theta\}\left(\partial_r + \frac{i}{r}\partial_\theta\right)v(\vec{r}) = 0. \tag{4}$$

In order to separate angular dependence, we substitute $u(\vec{r}) = u_0 \exp\{-im\theta\}u(r)$ and $v(\vec{r}) = v_0 \exp\{-il\theta\}v(r)$ to have

$$\left(\partial_r - \frac{m}{r}\right)u(r) + i\Delta(r)\frac{v_0}{u_0}v(r) = 0,$$

$$\left(\partial_r - \frac{l}{r}\right)v(r) + i\bar{\Delta}(r)\frac{u_0}{v_0}u(r) = 0, \tag{5}$$

if $l = n-1-m$. Furthermore, if we take $i\Delta(r)\frac{v_0}{u_0} = f(r) \equiv |\Delta(r)|$ that fixes the ratio $\frac{v_0}{u_0}$, the radial problem is reduced to solving

$$\left(\partial_r - \frac{m}{r}\right)u(r) + f(r)v(r) = 0,$$

$$\left(\partial_r - \frac{n-1-m}{r}\right)v(r) + f(r)u(r) = 0. \tag{6}$$

There are two linearly independent solutions to the equations. The behavior at large r is, apart from powers of r, $\exp\{\mp\mu r\}$. Since the solution must be normalizable, only one is acceptable. At the origin, the asymptotes that follow, with $f(r) \sim f_0 r^{|n|}$,

$$u(r) = u_1 r^m + u_2 r^{|n|+n-m},$$

$$v(r) = v_1 r^{|n|+1+m} + v_2 r^{n-1-m}. \tag{7}$$

To have single-valued nonsingular solutions at the origin, we have to demand that m is an integer and

$$n - 1 \geq m \geq 0. \tag{8}$$

The solutions that we get are similar but not the same as in Ref. 10. There, superconducting couplings in the Dirac Lagrangian induce different signs in the angular dependence $u \sim \exp\{-im\theta\}$, $v \sim \exp\{i(n-1-m)\theta\}$ which then guarantees one angular-momentum eigenstate per any odd value of the vorticity, $v = -n$. In our case only $v = -1$ vorticity solution is an angular-momentum eigenvalue: $m = 0$. In this case the radial problem is simplified and the explicit solution is

$$\Psi(\vec{r}) = C\begin{bmatrix} 0 \\ i\exp\{i\alpha\} \\ 1 \\ 0 \end{bmatrix}\exp\left\{-\int_r^r f(r')dr'\right\}, \tag{9}$$

where α is a constant defined by $\Delta(r) = |\Delta(r)|\exp\{i\alpha\} = f(r)\exp\{i\alpha\}$ and C is a normalization constant. If we do not demand that the solutions have to be eigenstates of angular momentum, the condition [Eq. (8)] ensures that we have n zero mode solutions in the case of the vortex with vorticity $v = -n$ (n is positive).[2] Thus negative vorticity vortex states exist only on sublattice a, and very similar analysis shows that only positive vorticity vortex states exist on sublattice b, in which case again there are as many zero modes as the value of vorticity.[2]

It can be argued[2] that the charge bound to a vortex of vorticity $v = 1$ is $-\frac{e}{2}$. We have to study the change in the local density of states of the Dirac Hamiltonian [Eq. (1)] in the presence of mass twist in Eq. (2). Because of the sublattice symmetry[2] to any negative eigenstate of the Dirac kernel, $\Psi_{-\epsilon}(r)$ corresponds to a positive-energy state $\Psi_\epsilon(r)$, related to $\Psi_{-\epsilon}(r)$ by a unitary transformation. Hence the local density of states,

$$\nu(r,\epsilon) = \sum_{\epsilon'} \Psi^+_{\epsilon'}(r)\Psi_{\epsilon'}(r)\delta(\epsilon - \epsilon'), \tag{10}$$

is symmetric with respect to zero energy. Demanding the conservation of the total number of states after the inclusion of the mass twist, we get

$$\int d\vec{r}\left\{2\int_{-\infty}^{0^-}\delta\nu(\vec{r},\epsilon)d\epsilon + |\Psi_0(\vec{r})|^2\right\} = 0, \tag{11}$$

where $\Psi_0(\vec{r})$ stands for the single zero mode. Its normalization to one leads to

$$\int d\vec{r}\int_{-\infty}^{0^-}\delta\nu(\vec{r},\epsilon)d\epsilon = -\frac{1}{2}, \tag{12}$$

so the net charge difference is $-\frac{e}{2}$.

B. Statistics

The introduced system can be in short described as Dirac electrons in the presence of a twisted mass,[3]

$$\mathcal{L} = \bar{\Psi}(i\gamma_\nu\partial_\nu + \Delta\exp\{i\gamma_5\phi\})\Psi, \tag{13}$$

where γ_ν, $\nu = 0,1,2$ are 4×4 Dirac matrices in the Weyl representation. The problem may be reformulated by dividing the vortex excitations into two groups, + and −, according to + and − values of vorticity corresponding to singularities in ϕ_+ and ϕ_-, respectively, where $\phi = \phi_+ + \phi_-$, and introducing gauge fields,[11]

$$a_\mu = \frac{1}{2}(\partial_\mu\phi^+ - \partial_\mu\phi^-),$$

$$b_\mu = \frac{1}{2}(\partial_\mu \phi^+ + \partial_\mu \phi^-). \tag{14}$$

We denote by $\mathrm{sgn}(m_3)$ (Ref. 3) a quantity that at the singular point of any vortex takes + or − depending whether the charge of the vortex is of sublattice a or b kind.[12] $\mathrm{sgn}(m_3)\frac{\partial\times b}{\pi}$ represents vortex excitation electric charge current,

$$\tilde{j}^\mu = \frac{\mathrm{sgn}(m_3)}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\mu\partial_\lambda\phi. \tag{15}$$

$\mathrm{sgn}(m_3)$ is necessary as we found out that vortex excitations with both positive (+1) and negative (−1) vorticities that live on sublattices b and a, respectively, possess the zero mode, which, unoccupied by electron, represents $-\frac{e}{2}$ absence of charge.

$\mathrm{sgn}(m_3)\frac{\partial\times a}{\pi}$ represents current of axial charge (valley index[3]) associated with vortex excitations (normalized by vortex axial charge)—$\tilde{j}_5^\mu$. The sign of axial charge in the density current $\tilde{j}_5^\mu$ comes from $\mathrm{sgn}(m_3)$. This comes from the correspondence (see Appendix) in the sign of the sublattice density difference $\bar{\Psi}\gamma_3\Psi$ [our states are eigenstates of $\gamma_0\gamma_3$ (Ref. 4)] and the sign of the expectation value of $\bar{\Psi}\gamma_0\gamma_5\Psi$, which is the axial charge density. The expectation value of the axial charge is $\pm\frac{1}{2}$ like that of ordinary charge.

Then a simple statement follows for the topological part of the effective action in the dual representation of the theory, i.e., in terms of vortices instead of Dirac particles (electrons). [In the dual picture elementary (2π) fluxes of gauge fields are particles and the gauge fields represent particle background.] The form of the topological part of the action is

$$\frac{\mathrm{sgn}(m_3)}{2\pi}a(\partial\times b) - \frac{1}{2}a\tilde{j}^\mu - \frac{1}{2}b\tilde{j}_5^\mu. \tag{16}$$

We just encoded in Eq. (16) the expressions of our currents found previously. There is overall $\frac{1}{2}$ factor because of the value of the charge of the vortices that must couple as $-\frac{1}{2}A^\mu\tilde{j}_\mu$ to the external field if we introduce it $(a\to a+A)$.

If we introduce gauge fields

$$R = a + b, \quad L = a - b, \tag{17}$$

we arrive at the following form of the Lagrangian:

$$\frac{\mathrm{sgn}(m_3)}{8\pi}(R\,\partial R - L\,\partial L) - R\left(\frac{\tilde{j}^+}{2}\right) - L\left(\frac{\tilde{j}^-}{2}\right), \tag{18}$$

where

$$\tilde{j}^+ = \frac{\tilde{j}+\tilde{j}_5}{2} \tag{19}$$

and

$$\tilde{j}^- = \frac{\tilde{j}-\tilde{j}_5}{2}, \tag{20}$$

which are the currents of the good quantum number—vorticity as opposed to charge $\tilde{j}$ and axial $\tilde{j}_5$ currents. Considering the Aharonov-Bohm phases for encircling quasiparticles around each other, we easily and clearly get that quasiparticles, $\tilde{j}^+$ and $\tilde{j}^-$ have semionic statistics among themselves and their mutual statistics is trivial.

Therefore through a straightforward analysis of quasiparticles as point particles, i.e., singularities of phase—vortex solutions, we come to the conclusion that they obey semionic statistics and their theory is the doubled Chern-Simons (CS), i.e., $\mathrm{U}(1)_2\times\overline{\mathrm{U}(1)}_2$—see Ref. 13, as found in Ref. 3. On the other hand in Ref. 5 the quasiparticles were viewed as extended objects—meron configurations of field $\vec{n}$, $\vec{n}^2=1$, and the conclusion was that they possess quarton statistics. If we apply a simplification that vector $\vec{n}$ is always in the x-y plane except at the center of excitation, we will find that, following the arguments in Ref. 5, the excitation possesses semionic statistics in accordance with Ref. 3.

Therefore the discrepancy we find between the two approaches is in their implicitly assumed description of the vortices which may be connected to different regimes (phases) of some well-defined microscopic system. Either the screening is large enough and we can have free vortices (but created in pairs because of the charge conservation), such as free quasiparticles in a quantum Hall system (that is why we used doubled CS theory), or we have ordinary neutral superfluid in which vortices are extended objects and cannot exist deconfined but bound in pairs.

III. FRACTIONALIZATION IN BILAYER GRAPHENE

The question is what are the physical systems that may support fractionalization; obviously dimerized graphene is a hypothetical system. The bosonic degrees of freedom that we need can come as a result of electronic correlations, most notably excitonic and superconducting. From these, only excitonic conserve charge and may produce in their defects charged zero modes (as opposed to neutral zero modes in superconductors) that we need to have fractionalization.

In order to facilitate the discussion of the excitonic instability and its zero modes in graphene bilayer, we will first discuss zero modes in the case of p-wave superconducting and excitonic systems.

(1) The effective BCS Hamiltonian for the quasiparticles is

$$H = \sum_k \xi_k c_k^+ c_k + \frac{1}{2}(\Delta_k^* c_{-k} c_k + \Delta_k c_k^+ c_{-k}^+). \tag{21}$$

Introducing the Bogoliubov transformation,

$$\alpha_k = u_k c_k - v_k c_{-k}^+, \tag{22}$$

which should diagonalize the Hamiltonian into $H=\Sigma E_k\alpha_k^+\alpha_k+\mathrm{const}$, implies Bogoliubov–de Gennes equations,

$$E_k u_k = \xi_k u_k - \Delta_k^* v_k, \tag{23}$$

$$E_k v_k = -\xi_k v_k - \Delta_k u_k. \tag{24}$$

If $\Delta_k=\Delta(k_x-ik_y)$ and $\xi_k\approx-\mu(\mu>0)$, in the long-distance approximation, the equations for zero mode(s) become

$$-\mu u - \Delta(-i)\partial_{\bar{z}} v = 0, \tag{25}$$

$$\mu v - (-i)\Delta\partial_z u = 0. \tag{26}$$

We may ask for zero modes that exist in vortex solutions for which we demand $v(\theta+2\pi)=-v(\theta)$ and $u(\theta+2\pi)=-u(\theta)$ in polar coordinates.[14] By solving Eqs. (25) and (26) with $\Delta=$const, we are neglecting the short (small radial) distance details of the solution. We seek the solution in the following form:

$$u = \frac{u(r)}{\bar{z}^l}, \quad v = \frac{v(r)}{z^k}, \tag{27}$$

and the equations that we get are

$$-\mu\frac{u(r)}{\bar{z}^l} + i\Delta\frac{1}{z^k}e^{i\theta}\partial_r v(r) = 0, \tag{28}$$

$$\mu\frac{v(r)}{z^k} + i\Delta\frac{1}{\bar{z}^l}e^{-i\theta}\partial_r v(r) = 0. \tag{29}$$

To separate angular and radial dependences, we must have $k=l=\frac{1}{2}$ and the equations are reduced to

$$-\mu u(r) + i\Delta\partial_r v(r) = 0, \tag{30}$$

$$\mu v(r) + i\Delta\partial_r u(r) = 0. \tag{31}$$

If $u(r)=u_0 f(r)$ and $v(r)=v_0 f(r)$, the equations reduce to a single one,

$$\mu f(r) + \Delta\partial_r f(r) = 0, \tag{32}$$

if $\frac{u_0}{v_0}=-i$. Therefore our solutions can be cast in the following form:

$$u = ie^{\pi/4}\frac{f(r)}{\sqrt{\bar{z}}}, \quad v = e^{i(\pi/4)}\frac{f(r)}{\sqrt{z}}, \tag{33}$$

where $f(r)$ is of the simple radial dependence $\sim e^{-\mu/\Delta r}$ for $\mu=$const.

The usual approach[15,16] is to model order parameter with the vortex singularity, i.e., to take, instead of Δ_k, $e^{i(\theta/2)}\Delta_k e^{i(\theta/2)}$ in our case. This, symmetrized with respect to phase expression, is used to ensure the antisymmetry of the order parameter, i.e., in which the term $\int d\vec{r}\Psi^+(\vec{r})\Delta_k\Psi^\dagger(\vec{r})$ in the Bogoliubov Hamiltonian is well defined and consistent with the anticommutativity of Fermi operators when phase is coordinate dependent.

From the approach we used in getting the zero mode, we can turn to the usual approach by the simple phase transformation $u\to e^{-i(\theta/2)}u$, $v\to e^{i(\theta/2)}v$, so that at the end our zero mode solution has the components:

$$u = ie^{i(\pi/4)}\frac{f(r)}{\sqrt{r}}, \quad v = e^{i(\pi/4)}\frac{f(r)}{\sqrt{r}}. \tag{34}$$

Now the quasiparticle operator for the zero energy state can be written as

$$\gamma_0^+ = \int d^2\vec{r}[u(\vec{r})c^+(\vec{r}) + v(\vec{r})c(\vec{r})], \tag{35}$$

and immediately we can conclude that $\gamma_0^+=\gamma_0$ for our solution, i.e., it represents neutral Majorana mode.

We should notice that, with respect to the Dirac problem in the dimerized graphene here, in the latter approach, momentum operators are together, in the same term, with order parameter and phase singularity in the Hamiltonian. On the other hand, in the case of the excitonic problem which may be related and become a physical realization of dimerized graphene lattice,[2] we have a different basic Hamiltonian,

$$H = \sum_k E_k(\beta_k^+\beta_k - \gamma_k^+\gamma_k) - \sum_k(\Delta_k\beta_k^+\gamma_k + \Delta_k^*\gamma_k^+\beta_k). \tag{36}$$

Introducing

$$B_k = u_k\beta_k - v_k\gamma_k, \tag{37}$$

$$C_k = v_k\beta_k + u_k\gamma_k, \tag{38}$$

which should diagonalize the Hamiltonian into $H=\Sigma_k\epsilon_k(B_k^+B_k - C_k^+C_k)$, we get

$$\epsilon_k u_k = E_k u_k + v_k\Delta_k^*, \tag{39}$$

$$\epsilon_k v_k = E_k v_k - u_k\Delta_k. \tag{40}$$

For the zero modes ($\epsilon_k=0$) simple redefinition $v_k\to -v_k$ transforms the equations into the equations of the superconducting problem. We will assume $\Delta_k=\Delta(k_x-ik_y)$ and $E_k=\epsilon=$const>0. Then for the zero modes we have

$$\epsilon u + \Delta(-i)\partial_{\bar{z}} v = 0, \tag{41}$$

$$\epsilon v - \Delta(-i)\partial_z u = 0. \tag{42}$$

The equations are the same as in Eqs. (25) and (26). Again, we may ask for the zero modes that exist in vortex solutions for which we demand $v(\theta+2\pi)=-v(\theta)$ and $u(\theta+2\pi)=-u(\theta)$ in polar coordinates and the answer would be the same. But if we want to stay in the language of the order parameter, we may model it, in the presence of a vortex, as $\Delta_k e^{i\theta}$ with no symmetrization as was necessary in the superconducting problem. In this case the solutions become

$$u = ie^{i(\pi/4)}e^{i(\theta/2)}\frac{f(r)}{\sqrt{r}}, \quad v = e^{i(\pi/4)}e^{i(\theta/2)}\frac{f(r)}{\sqrt{r}}, \tag{43}$$

which are not satisfactory for the electronic wave functions because they are not single valued while the singularity is borne by the order parameter. If we choose in the order parameter $e^{-i(\theta/2)}\Delta_k e^{i(\theta/2)}$, our solutions are represented by

$$u = ie^{i(\pi/4)}e^{i\theta}\frac{f(r)}{\sqrt{r}}, \quad v = e^{i(\pi/4)}\frac{f(r)}{\sqrt{r}}, \tag{44}$$

and this, although not an angular-momentum eigenstate, is the solution that describes a charged mode because $u^*\neq v$ as a crucial difference with respect to the superconducting case. The excitonic Bogoliubov transformation mixes the same kind of charged operators, leading to charged zero modes as

in the dimerized graphene problem. Although the precise form of the equations for the zero modes is not the same in the two cases (excitonic p-wave system and dimerized graphene), they are very similar in the long-distance regime and should lead to the same conclusion about the charge fractionalization.

(2) For the case of bilayer graphene excitonic instability described in Ref. 9, we have in the limit of small interlayer hopping

$$E_k \approx V\left[1 - \frac{1}{2}\left(\frac{\epsilon_k}{t_\perp}\right)^2\right] \tag{45}$$

for the energy in Eqs. (39) and (40). Here $t_\perp$ is the interlayer hopping parameter, $\pm V$ is the bias that causes excess electrons and holes in the lower and upper graphene layers, respectively, and ϵ_k is the bare kinetic energy for which $\epsilon_k \sim k$. The gap function is of the following form, near two nodal points:[9]

$$\Delta_k = i(k_x \mp ik_y)|k|\Delta, \tag{46}$$

where Δ is a positive constant. This is the result in Ref. 9. There, by assuming that, in the graphene bilayer, excitonic instability is formed by the lower conduction and upper valence band, and in the Hartree-Fock setup and inclusion of short-range interactions, it was found explicitly (in an algebraic treatment and without further approximations) that the gap function has to have a phase,

$$\Delta_k \sim \frac{i(k_x \pm ik_y)}{k}, \tag{47}$$

near the two nodal points. Therefore the p-wave-like form of the excitonic instability in graphene bilayer is rather general. Further approximations that were done then in Ref. 9 in solving the gap equation brought the expression [Eq. (46)] that we assume and use in this section.

We will write E_k as $E_k = V - \delta k^2$ with also δ being a positive constant. Then Eq. (39) with $\epsilon_k = 0$ becomes

$$\left(V + \delta\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\right)u + \Delta(-i)(-i)\sqrt{\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}}(-i)\partial_{\bar{z}}v = 0, \tag{48}$$

$$\left(V + \delta\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\right)v - \Delta(i)(i)\sqrt{\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}}(-i)\partial_z u = 0. \tag{49}$$

To consider the vortex solution we have to set $e^{-i(\theta/2)}\Delta_k e^{i(\theta/2)}$ instead of Δ_k, which if we want to absorb the phase into v and u leads to slightly different equations due to the presence of the k^2 terms that we also must keep in this long-distance analysis. Because

$$e^{i(\theta/2)}\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}e^{-i(\theta/2)} = e^{i(\theta/2)}\frac{\partial}{\partial z}e^{-i(\theta/2)}e^{i(\theta/2)}\frac{\partial}{\partial \bar{z}}e^{-i(\theta/2)} = \left(\frac{\partial}{\partial z} - \frac{1}{2z}\right)\left(\frac{\partial}{\partial \bar{z}} + \frac{1}{2\bar{z}}\right), \tag{50}$$

the equations for the vortex solution become

$$\left[V + \delta\left(\frac{\partial}{\partial z} - \frac{1}{2z}\right)\left(\frac{\partial}{\partial \bar{z}} + \frac{1}{2\bar{z}}\right)\right]u + \Delta(+i)\sqrt{\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}}\partial_{\bar{z}}v = 0, \tag{51}$$

$$\left[V + \delta\left(\frac{\partial}{\partial z} - \frac{1}{2z}\right)\left(\frac{\partial}{\partial \bar{z}} + \frac{1}{2\bar{z}}\right)\right]v + \Delta(-i)\sqrt{\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}}\partial_z u = 0, \tag{52}$$

with the requirement $u(\theta+2\pi) = -u(\theta)$ and $v(\theta+2\pi) = -v(\theta)$. If we seek solutions in the following form [see Eq. (27)],

$$u = \frac{u(r)}{\bar{z}^l}, \quad v = \frac{v(r)}{z^k}, \tag{53}$$

we face the problem of properly performing the operations with the square-root operator, $\sqrt{\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}}$. Its action we will define on the space of monomials in z and $\bar{z}$, for which we have

$$\sqrt{z\frac{\partial}{\partial z}}\sqrt{\bar{z}\frac{\partial}{\partial \bar{z}}}z^n\bar{z}^m = \sqrt{n}\sqrt{m}z^n\bar{z}^m. \tag{54}$$

We will assume the asymptotic behavior of $u(r)$ and $v(r)$ as $u(r) \sim e^{-\lambda r}$ and $v(r) \sim e^{-\lambda r}$, and justify it in the end. Then, for example, in the case of $v(r)$ we have

$$\begin{aligned}
&\sqrt{\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}}\frac{1}{z^k}e^{i\theta}\partial_r v(r) \\
&= -\lambda v_0\sqrt{\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}}\left\{\frac{1}{z^k}\frac{z^{1/2}}{\bar{z}^{1/2}}\sum_{n=0}^{\infty}\frac{1}{n!}(-\lambda)^n z^{n/2}\bar{z}^{n/2}\right\} \\
&= -\lambda v_0\frac{1}{\sqrt{z\bar{z}}}\sqrt{z\frac{\partial}{\partial z}}\sqrt{\bar{z}\frac{\partial}{\partial \bar{z}}}\sum_{n=0}^{\infty}\frac{1}{n!}(-\lambda)^n z^{(n+1)/2-k}\bar{z}^{(n-1)/2}
\end{aligned} \tag{55}$$

$$= -\lambda v_0\frac{1}{\sqrt{z\bar{z}}}\sum_{n=0}^{\infty}\frac{1}{n!}(-\lambda)^n \tag{56}$$

$$\times\sqrt{\frac{n+1}{2} - k}\sqrt{\frac{n-1}{2}}z^{(n+1)/2-k}\bar{z}^{(n-1)/2}. \tag{57}$$

Therefore Eq. (51) becomes

$$\left[V + \delta\left(\frac{\partial}{\partial z} - \frac{1}{2z}\right)\left(\frac{\partial}{\partial \bar{z}} + \frac{1}{2\bar{z}}\right)\right]u(r) + \Delta i(-\lambda)v_0\frac{1}{\sqrt{z\bar{z}}}\sum_{n=0}^{\infty}\frac{1}{n!}\times(-\lambda)^n\sqrt{\frac{n+1}{2} - k}\sqrt{\frac{n-1}{2}}z^{(n+1)/2-k}\bar{z}^{(n-1)/2+l} = 0. \tag{58}$$

To factor the angular dependence we choose $k = l = \frac{1}{2}$. Moreover, in the long-distance approximation, for which we take also $\sqrt{n(n-1)} \approx n$ in the series expansion, we have

$$[V + \delta\partial_r^2]u + i\Delta(-\lambda)^2 v(r) = 0, \tag{59}$$

$$[V+\delta\partial_r^2]v - i\Delta(-\lambda)^2 u(r) = 0. \tag{60}$$

This means

$$u_0[V+\delta\lambda^2] + i\Delta\lambda^2 v_0 = 0, \tag{61}$$

$$v_0[V+\delta\lambda^2] - i\Delta\lambda^2 u_0 = 0, \tag{62}$$

i.e., $\lambda_1^2=\frac{V}{-(\delta+\Delta)}$ or $\lambda_2^2=\frac{V}{\Delta-\delta}$. Both, $\lambda_1=\pm i\sqrt{\frac{V}{\delta+\Delta}}$, represent delocalized zero modes. For λ_2 that would be the case if $\delta>\Delta$, otherwise we have only one physical localized state with $e^{-\lambda_2 r}$, $\lambda_2=\sqrt{\frac{V}{\Delta-\delta}}$, decay function. Comparing with solutions in Ref. 9 for Δ for fixed parameters for the graphene bilayer, we find that for large enough V, which is the bias parameter, we can have the bound solution. Then the three zero modes carry fractional charge, $\frac{3}{2}$, which we get applying the same arguments that were given in Ref. 2 for the dimerized graphene (see Sec. II). Essential in these arguments is that the Dirac kernel has the correspondence of positive- and negative-energy solutions. In this case we also have the Dirac structure of the problem, with the nodal points, and the same correspondence.

Based on the Hartree-Fock[9] treatment of the excitonic problem we get three zero modes (for large enough V) and the question comes whether they will stay with further inclusion of interactions. In the case of three zero modes, the charge that they carry is mostly smeared out through the system (see the solutions above). If the interactions were able to split the two delocalized zero modes, we could have localized charge $\frac{1}{2}$ vortex excitation as in the case of dimerized graphene.[2] The conclusions about the topological terms and fractional statistics that we reached in the case of dimerized graphene will still hold. The interaction term that we have in mind would be $\sim V\rho_{k_0}\rho_{-k_0}$, where $k_0=|\lambda_1|$ and an excitonic coupling between the mode k_0 and $-k_0$ would produce necessary splitting. Otherwise (with no splitting) the situation is less clear but if we assume that the localized mode describes a missing of $\frac{e}{2}$ charge, the delocalized modes will describe additional neutral degrees of freedom (that may be occupied or unoccupied) that can further characterize the excitations.

The recent work, Ref. 17, which appeared while we were finishing this paper, concerns fractionalization in excitonic bilayer graphene that is not naturally (Bernal) stacked but consists of two parallel layers at some larger distance that leads to even number of zero modes due to the valley degeneracy and, therefore, no charge fractionalization. Still we cannot rule out, on the basis of the long-distance analysis of vortex solutions in the excitonic condensate that we presented, the same doubling in our case. This important question can be resolved only by a detailed numerical analysis of the bilayer graphene.

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APPENDIX

In the second-quantized formalism the zero mode solution in Eq. (9), of definite vorticity $v=-1$, contributes to the expansion of the Dirac field as a term equal to

$$\Psi_0(\vec{r}) = \left\{ \begin{bmatrix} 0 \\ i\exp\{i\alpha\} \\ 0 \\ 0 \end{bmatrix} c_0 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} c_0^+ \right\} C\exp\{-|\Delta|r\}, \tag{A1}$$

where we simplified the decay function by taking $|\Delta(r)|=|\Delta|=\text{const}$, and again the C is the normalization constant. Notice the absence of two different operators (one for particle, the other for hole) that we would have for a nonzero energy level. For the sake of argument, going in reverse, we can fix the normalization constant by demanding that the charge associated with the zero mode is $(-)\frac{1}{2}$, i.e.,

$$\int d^2\vec{r}\langle\Psi_0^+(\vec{r})\Psi_0(\vec{r})\rangle = -\frac{1}{2}. \tag{A2}$$

That would imply

$$\int d^2\vec{r}\langle\Psi_0^+(\vec{r})\gamma^5\Psi_0(\vec{r})\rangle = \frac{1}{2} \tag{A3}$$

for the expectation value of the axial charge, where we took

$$\gamma_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

in the Weyl representation. On the other hand our state is an eigenstate of the sublattice charge difference operator,

$$R = \alpha_3 = \gamma_0\gamma_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

so that

$$\int d^2\vec{r}\langle\Psi_0^+(\vec{r})\alpha_3\Psi_0(\vec{r})\rangle = \frac{1}{2} \tag{A4}$$

of the same sign as the expectation value of the axial charge. Both signs would reverse if the solution were with vorticity $v=1$ on sublattice b although the sign of the electric charge would remain the same.

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