Towards the Hall viscosity of the Fermi-liquid-like phase at the filling factor of 1/2

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We discuss the Berry curvature calculations of the Hall viscosity for the Fermi-liquid-like state, i.e., a Fermi-liquid state of underlying composite particles of the Hall system. We conclude, within assumptions made, that in the linear response, with small deformation of the system and in the thermodynamic limit, the Hall viscosity takes the value characteristic for the Laughlin states. We present arguments that the value is the same even for general deformations in the same limit.

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I. INTRODUCTION

The Hall viscosity[1–4] may represent an additional invariant by which we can characterize quantum Hall states. It was calculated[2,4] for Laughlin states, with the general filling factor $\nu=1/m$, where m=odd integer, and this includes the state of a filled lowest Landau level (LLL) ($m=1$). The calculated value is one and the same irrespective of m. These are model states, and the question is whether the Hall viscosity will stay the same if we modify the model Hamiltonians for which these states are exact zero-energy states. (In other words we would like to know whether, just in the case of the Hall conductance, we have some kind of quantization, invariance upon details of the electron system.) Related to this are the following questions: How can we characterize the Hall viscosity of gapless states, i.e., ground states that we associate with gapless quantum Hall phases? What are the values of Hall viscosity and do they also exhibit invariance making the characterization of the gapful, Laughlin, and other states by Hall viscosity less connected with its topological character? In this respect we may ask what is the value of the Hall viscosity for the Fermi-liquid-like state[5,6] at the filling factor of 1/2, whether it is equal to the Laughlin value? Or perhaps should it be the limiting value of Hall viscosities of the states of Jain series that leads to the Fermi-liquid-like state? The Hall viscosity is proportional to the mean orbital spin, a property associated with coordinate transformations in two dimensions, per particle in the quantum Hall state.[4] Because of the coupling of the orbital spin to the curvature of the surface,[7] we can read off its value, $S/2$, from the relationship between the total flux (the number of flux quanta) through the system, N_s, and number of particles, N, on a sphere: $N_s=\nu^{-1}N-S$ for a given state. The quantity S, also known as the shift that characterizes the state, and in the case of Jain series, $\nu=p/(2p+1)$ (where $p=1,2,3,\ldots$), it grows with p as $S=2p+1$. Given that the Hall viscosity is proportional to the shift times the density of the system, i.e., filling factor, the limiting value of the Jain series will diverge linearly with p as the filling factor converges to 1/2, while the system possibly converges to the Fermi-liquid-like state.

In this paper we will discuss the Hall viscosity of the Fermi-liquid-like state and phase[5,6] at $\nu=1/2$. We will consider a wave function based on the mean-field solution of the Halperin-Lee-Read theory, but at the filling factor $\nu=1$, i.e., the Fermi-liquid-like state of bosons.[8] We use the wave function that besides the Fermi sea of underlying composite quasiparticles incorporates also zero-point plasmon fluctuations, i.e., the complete Laughlin-Jastrow part. (It can be characterized also as the unprojected to the LLL wave function for the ground state introduced in Ref. 6, but at $\nu=1$, i.e., the Fermi-liquid-like state of bosons.) The bosonic and fermionic states share the same underlying Fermi-liquid-like physics. The bosonic version will facilitate the calculations of the Berry curvature and its coefficient, the Hall viscosity, in the case of these gapless systems. In the introductory Sec. II we will review the basic ansatz for calculating the Hall viscosity in the case of quantum Hall states that was used in Refs. 1 and 9. In Sec. III the Hall viscosity of free Fermi gas is discussed as a step towards the calculation for the Fermi-liquid-like state. In Sec. IV the Hall viscosity as a response to a small deformation of the Fermi-liquid-like state is discussed (a) when the system is quasi-one-dimensional, (b) in the case when the Fermi surface is rectangular, and finally (c) in the case of interest, i.e., when the Fermi surface is isotropic and circular. The next section, Sec. V, discusses the Hall viscosity of a system under a general deformation, and Sec. VI discusses the importance of the inversion symmetry for the neutral part as an effective symmetry for composite fermions (CFs) that is present in the systems with rectangular shape, which are deformed. Section VII contains a discussion of results and conclusions.

II. HALL VISCOSITY OF QUANTUM HALL STATES

The basic approach to the Hall viscosity that was first formulated in Ref. 1 relies on calculating the Berry curvature of shear deformations of the ground state that is adiabatically transformed. This means an assumption is made that the state is nondegenerate along the process. This can be assured if the system has a gap which is the characteristic of usual quantum Hall states. The shear deformations are examined by following how the quantum liquid is spread out in the deformed geometry of a torus [we will discuss the boundary conditions (BCs) shortly]—see Fig. 1.

The parameters V and $\tau=\tau_1+i\tau_2$ describe the deformation from the reference point $V=1$ and $\tau=i$. The approach used in Ref. 1 is to stay in the coordinate space that we begin with, i.e., with "old" coordinates $(x,y)\in[0,1]\times[0,1]$, but study the solutions of a deformed Hamiltonian. This Hamiltonian is the usual local Hamiltonian in "new" coordinates (x',y')

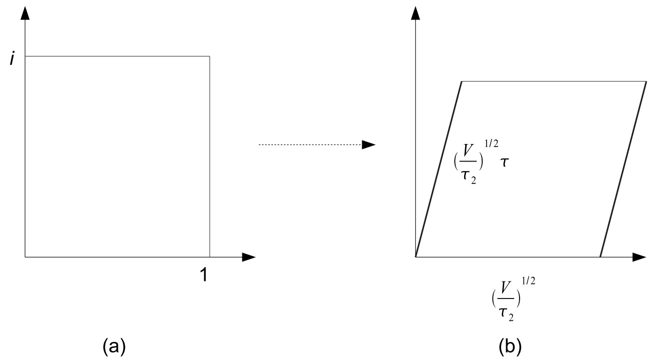

FIG. 1. Deformed torus.

but now expressed in terms of the old coordinates using the coordinate transformation $x'+iy'=(V/\tau_2)^{1/2}(x+\tau y)$, associated with the deformation. The usual (periodic) boundary conditions are applied. This leads to a deformed ground state Ψ which should be used in the formula for Hall viscosity with the Berry curvature,

$$\eta^A = 2\,\mathrm{Im}\langle \partial_{\tau_1}\Psi|\partial_{\tau_2}\Psi\rangle, \tag{1}$$

calculated at the reference point $V=1$ and $\tau=i$. To recover the physical units we should multiply with $\hbar/L_xL_y$, where L_x and L_y are in a general case the lengths associated with a rectangular system. In the following we will study such a general geometry in which the deformations from a rectangle with lengths L_x and L_y are made—see Fig. 2.

If we consider a gapped system of noninteracting electrons that fill the LLL to get the Hall viscosity, we have to sum the contributions from each single-particle state in the LLL. The wave functions that describe the way how a single-particle wave function is changed as the geometry of the finite-volume system is varied (Fig. 2) are

$$\Psi_j = \sum_{k=-\infty}^{+\infty} \exp\left\{ i\frac{(X_j+kL_y)x}{l_B^2} + i\tau\frac{(X_j+kL_y-y)^2}{2l_B^2}\right\},$$
$$X_j = \frac{2\pi l_B^2 j}{L_x}. \tag{2}$$

We did not include the normalization of each wave function that is labeled by an integer $j=0,\ldots,N_s-1$, where $N_s=L_xL_y/2\pi l_B^2$, i.e., the number of flux quanta through the system; in our case this number is equal to the number of electrons ($N_s=N$). The coordinates x and y are old coordinates

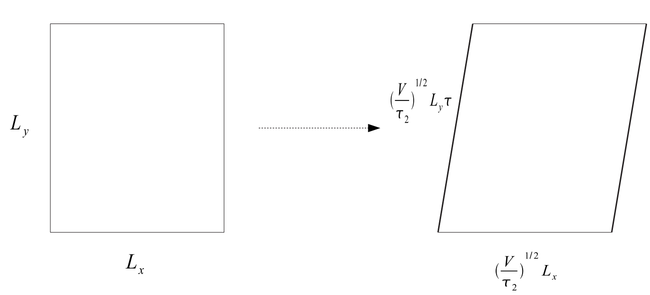

FIG. 2. Deformed rectangle.

and the wave functions satisfy the ordinary periodic BC (PBC) in the x direction,

$$\Psi_j(x+L_x) = \Psi_j(x), \tag{3}$$

and the magnetic BC in the y direction,

$$\Psi_j(y+L_y) = \exp\left\{ i\frac{L_y x}{l_B^2}\right\}\Psi_j(y). \tag{4}$$

We will relax[9] the demand for the magnetic BC [or we may take the large-L_y limit[10] in Eq. (2)]. The wave function becomes simpler in this cylinder geometry:

$$\Psi_j = \exp\left\{ i\frac{X_j x}{l_B^2} + i\tau\frac{(X_j-y)^2}{2l_B^2}\right\}\frac{(\tau_2)^{1/4}}{(l_B\sqrt{\pi})^{1/2}}, \tag{5}$$

where we included the normalization. We have a set of orthonormal wave functions which can be used for the calculation of the Berry curvature as a sum of contributions of each single-particle state. As we prove[3] in Appendix A, if the wave function is nonanalytical, i.e., nonholomorphic in τ variable only in its normalization, it can be expressed as $\Psi=\frac{1}{\sqrt{Z}}f(\tau,x,y)$, and its contribution to the Hall viscosity is

$$\frac{\hbar}{L_xL_y}\frac{1}{2}\left(\frac{\partial^2}{\partial\tau_1^2}+\frac{\partial^2}{\partial\tau_2^2}\right)\ln Z, \tag{6}$$

where the evaluation is done at $\tau=i$ point in the τ space. Specifying to our set [Eq. (5)], the sum of all contributions is

$$\eta^A = \frac{\hbar n}{4}, \tag{7}$$

where $n=N/L_xL_y=1/2\pi l_B^2$ is the density of the system. Therefore, we recovered the well-known result [for $\nu=1$ quantum Hall effect (QHE)] using the cylinder geometry, and it will be the same even if we were applying the so-called "thin-torus limit," i.e., cylinder limit for which $L_x \to 0$.

III. HALL VISCOSITY OF FREE FERMI GAS

Classically and in the adiabatic response theory it is expected[1] that the system with time-reversal symmetry does not have the asymmetric (Hall) viscosity. We study the asymmetric viscosity of the free Fermi gas in the following. We will use the Berry curvature formula to calculate the Hall viscosity even for this system assuming that in the adiabatic response, when we probe a finite fraction, i.e., small finite system, the ground state stays nondegenerate as τ is varied at least for a small interval before a reconfiguration of the Fermi surface. In the case of free Fermi gas the tiny gap $\Delta \sim 1/L^2$, where L is the length of the system, $L=\max\{L_x,L_y\}$ keeps the filling of the Fermi see intact at least for values of V, τ_1, and τ_2 in the neighborhood of $V=1$ and $\tau=i$. Therefore, because the Hall viscosity is the Berry curvature at a specific point in the parameter space and not an integral of it in the same space, the demand for the ground state being nondegenerate can be relaxed to the same requirement in the neighborhood of the unperturbed point. In fact the linear-response theory leads to the Berry curvature formula.[9]

In our case of the free Fermi gas we need a small enough system. We will adopt the approach in Ref. 1 and study the deformed Hamiltonian:

$$H = -\frac{1}{V\tau_2}[|\tau|^2\partial_x^2 - 2\tau_1\partial_x\partial_y + \partial_y^2], \quad (8)$$

on space $(x,y)\in[0,L_x]\times[0,L_y]$, with periodic boundary conditions. We seek the solutions in the form

$$\exp\{ik_x x' + ik_y y'\} = \exp\left\{ i(k_x x + k_x\tau_1 y + k_y\tau_2 y)\sqrt{\frac{V}{\tau_2}}\right\}, \quad (9)$$

as we demand that locally we have the same equation irrespective whether we work with old or new coordinates. The eigenvalues are $\epsilon(\mathbf{k})\sim k_x^2+k_y^2$. But the demand for PBCs and orthogonality leads to

$$k_x = \frac{2\pi}{L_x}\sqrt{\frac{\tau_2}{V}}m, \quad k_y = 2\pi\frac{1}{\sqrt{V\tau_2}}\left(\frac{n}{L_y} - \frac{m}{L_x}\tau_1\right), \quad (10)$$

where m and n are integers. Therefore, the quantized energy levels are

$$\epsilon(\mathbf{k}) = \epsilon(n,m) = \frac{(2\pi)^2}{V\tau_2}\left[\tau_2^2\frac{m^2}{L_x^2} + \left(\frac{n}{L_y} - \frac{m}{L_x}\tau_1\right)^2\right]. \quad (11)$$

Although the deformation τ modifies the eigenvalues, eigenstates are independent of it, and this leads to zero value for Berry curvature and the Hall viscosity as expected in a time-reversal-invariant system. As we recovered the result that is valid for a system of any size, we will use the obtained description and formulas even in the large-N limit in the following.

IV. HALL VISCOSITY OF FERMI-LIQUID-LIKE STATE

We will consider the bosonic Fermi-liquid-like state at the filling factor $\nu=1$. Therefore, we study the wave function

$$\Psi = \Psi_L^{\nu=1}\,\mathrm{Det}(\exp\{i\vec{k}_i\vec{r}_j\}), \quad (12)$$

which is not normalized, and $\Psi_L^{\nu=1}$ is the Slater determinant of lowest Landau level single-particle states [Eq. (2) or Eq. (5) with $\tau=i$] and $\mathrm{Det}(\exp\{i\vec{k}_i\vec{r}_j\})$ is the Slater determinant of free waves whose wave vectors $\vec{k}$'s fill a two-dimensional Fermi sea. Equation (12) describes the Fermi-liquid-like state of bosons as both determinants are antisymmetric under exchange of particles, and the complete function is symmetric. This state was introduced mainly to understand the Fermi liquid at 1/2, as it is believed that the exact fermionic Chern-Simons transformation that relates the two states does not change the main characterization of the states. We also study this wave function in an expectation that our conclusions will not depend on the kind of the Laughlin-Jastrow factor in the wave function. Because we did not include the projection to the LLL, we study the unprojected to LLL wave function. We will assume the following evolution of the wave function under deformation τ. Each factor will evolve according the deformed single-particle Hamiltonians: (a) the one with magnetic field as in Avron *et al.* with magnetic boundary conditions in the case of the evolution of the part that "sees" the magnetic field, i.e., the Slater determinant of lowest Landau level single-particle states and (b) the Hamiltonian given in Eq. (8) with PBCs that governs the evolution of the part with plane waves. Therefore, we assume separate evolutions that we know very well. As we study the small deformations of a rectangular system and plane waves do not depend on it, the most important question is what is the shape of the Fermi sea of the unperturbed finite Fermi system in a rectangular geometry. This is a difficult question, although we believe that in the thermodynamic limit the Fermi sea will assume its circular isotropic shape. The question has to be resolved only by studying the full interacting system in the LLL. Here, we will be studying the Hall viscosity (A) in a limiting case of thin torus (cylinder), (B) of a system with rectangular shape of its Fermi surface, and then reach the conclusion for the value of the Hall viscosity in the case (C) which is an isotropic circular Fermi surface as a starting (unperturbed) point.

Before that we will analyze the Berry curvature formula for the wave function in Eq. (12) with the assumed evolution in general terms. Because the part with Slater determinant of free waves does not depend on τ under deformation and $\Psi_L^{\nu=1}$ will depend in the holomorphic way, the expression for the Hall viscosity is again

$$\eta_A = \frac{\hbar}{L_xL_y}\frac{1}{2}\Delta\ \ln Z, \quad (13)$$

where Z is the norm of the deformed wave function [defined in Eq. (12) at $\tau=i$] and the derivatives are calculated at $V=1$ and $\tau=i$.

A. Thin-cylinder limit

In the thin-cylinder limit the system is much longer in one direction than the other. Then because of the PBC in the x direction we have the quantization of the momentum as before

$$X_j = \frac{2\pi l_B^2}{L_x}j. \quad (14)$$

Here, $j=0,\ldots,N_s-1$, where $N_s=L_xL_y/2\pi l_B^2$. Now we take the $L_x\to 0$ limit along $L_y\to\infty$ to keep N_s constant. For the Fermi-liquid-like state that means that the neutral fermions in the k space will form a line along the y direction with two Fermi points instead of a circle (line) for a Fermi surface. In real space that is described by the following wave function:

$$\prod_{i<j}\sin\left\{\frac{\pi}{L_y}(y_i - y_j)\right\}, \quad (15)$$

where we assumed an odd number of electrons. Notice that there is no x dependence. Therefore, when we ask for the norm of the complete wave function (with the Laughlin-Jastrow factor at $\nu=1$-Vandermonde determinant) we get

$$Z=\prod_{i=1}^{N}\int dy_i \sum_{\sigma\in S_N}\exp\{-\tau_2(y_i-k_{\sigma(i)})^2\}\prod_{k<l}\sin^2\left\{\frac{\pi}{L_y}(y_k-y_l)\right\}. \tag{16}$$

Under translations of y variables,

$$Z=\prod_{i=1}^{N}\int dy_i \exp\{-\tau_2 y_i^2\}\sum_{\sigma\in S_N}\prod_{k<l}\sin^2\left\{\frac{\pi}{L_y}(y_k-y_l+k_{\sigma(k)}-k_{\sigma(l)})\right\}. \tag{17}$$

Due to the Gaussian factors in y integration for $\tau_2\sim 1$ we can assume that relevant values of y's in the product are $y_i \lesssim l_B$, $\forall i\in[1,N]$. Because $|k_{\sigma(k)}-k_{\sigma(l)}|\geq 2\pi l_B^2/L_x$, when $L_x\to 0$ we can neglect the presence of y's in the sine functions, and due to the scaling $\tau_2 y_i\to y_i$ we recover the result for the Hall viscosity identical to the integer quantum Hall effect at $\nu=1$.

B. System with a rectangular shape of its Fermi surface

The Fermi gas with a rectangular shape of its Fermi surface may be rather artificial, but as we already discussed (a) this shape may appear in small systems with rectangular boundaries and (b) the conclusions reached and constructions applied to this system will serve as a stage for discussing the problem with rectangular boundaries in the thermodynamic limit and circular shape of the Fermi surface.

Let us assume that we have a Fermi surface of a rectangular shape where, for simplicity, we take that the length and width are the same and proportional to $\sqrt{N}\in\mathbb{Z}$. [To retain PBCs (instead of antiperiodic BCs) we may demand that $\sqrt{N}$ is an odd number.] The ground-state function of the ideal gas has to be an eigenvector under inversion symmetry: $y_i\to -y_i$ and $x_i\to x_i$, $\forall i$ (or $y_i\to y_i$ and $x_i\to -x_i$, $\forall i$) and that constrains its form to two possibilities:

$$(1)\quad \mathcal{A}\left\{\prod_{\text{over slices in k space}}\left\{\prod_{i<j;i,j\in\text{ slice}}\sin\left[\frac{\pi}{L_y}(y_i-y_j)\right]\cos\left[\frac{\pi}{L_x}(x_i-x_j)\right]\right\}\right\}, \tag{18}$$

where slices are lines in k space, along k_x and k_y directions, of length $\sqrt{N}$ each and to each one is assigned $\sqrt{N}$ number of particles (see Fig. 3). For a fixed $\sqrt{N}$ number of particles we have two slices or lines symmetrically positioned in k space around $k_x=-k_y$ line (Fig. 3). So as a first step we divide particles in $\sqrt{N}$ slices (lines and groups), and at the end we antisymmetrize ($\mathcal{A}$) that construction in the curly braces, which represents a particular division into $\sqrt{N}$ groups. See an example with four particles in Appendix B. [We introduced slicing in k space, although at this point it seems redundant—only division in $\sqrt{N}$ groups and later antisymmetrization is all that is in Eq. (18); slicing in k space is helpful to introduce and analyze more general Fermi surfaces as we will see later on.]

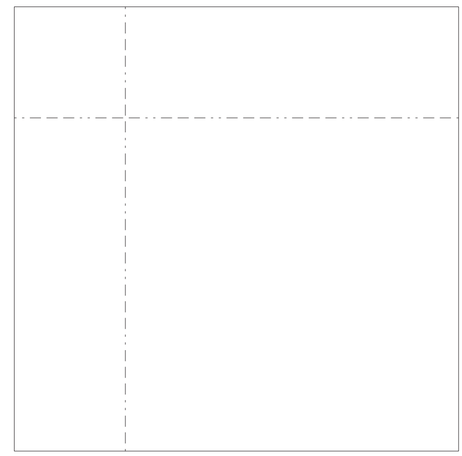

FIG. 3. Rectangular Fermi surface and two slices with the same group of particles.

The second possibility (2) is with x's and y's interchanged. If the width and length are not the same, for example, $L_y>L_x$ then for a single slice of $\sqrt{N}(L_y/L_x)$ integers (integers denote particles of the particular slice or group), S_y, we have to symmetrize, in addition, smaller slice of $S_x\subset S_y$ integers, with $\sqrt{N}(L_x/L_y)$ of them, i.e.,

$$\prod_{i<j;i,j\in S_y}\sin\left\{\frac{\pi}{L_y}(y_i-y_j)\right\}S\left\{\prod_{k<l;k,l\in S_x}\cos\left[\frac{\pi}{L_x}(x_k-x_l)\right]\right\}, \tag{19}$$

so that the x part is also symmetric under permutations inside S_y.

Then our norm, i.e., Z, for the compressible quantum Hall state at $\nu=1$ becomes a sum of terms, each representing two fixed permutations σ,σ' of N integers as in the following:

$$\prod_{i=1}^{N}\int dy_i\prod_{l=1}^{N}\exp\left\{-\frac{\tau_2}{2}(y_l-k_{\sigma(l)})^2\right\}\prod_{p=1}^{N}\exp\left\{-\frac{\tau_2}{2}(y_p-k_{\sigma'(p)})^2\right\}\prod_{\text{over slices}}\left\{\prod_{k<l;k,l\in\text{ slice}}\sin\left[\frac{\pi}{L_y}(y_k-y_l)\right]\right\}\prod_{\text{over slices}}'\left\{\prod_{p<q;p,q\in\text{ slice}}\sin\left[\frac{\pi}{L_y}(y_p-y_q)\right]\right\}, \tag{20}$$

where we suppressed (did not write) the part that corresponds to x integration. (Note that now $\sigma \neq \sigma'$ in general due to the x dependence of the wave function describing Fermi sea.) Now we can shift each y_i by $(k_{\sigma(i)}+k_{\sigma'(i)})/2 \equiv \bar{k}_i$ and estimate how the scaling with τ_2 can be affected with the Fermi sea part. The factors that come out, $\exp\{-(\tau_2/4)(k_{\sigma(i)} - k_{\sigma'(i)})^2\}$ (besides the Gaussians in y's), will suppress the contributions of terms [Eq. (20)] for which σ and σ' differ too much and in the following we will assume $|\sigma(i)-\sigma'(i)| \ll L_x/l_B$. With this in mind we concentrate on a single slice:

$$\prod_{k<l;k,l\in \text{ slice}} \sin\left\{\frac{\pi}{L_y}(y_k - y_l + \bar{k}_k - \bar{k}_l)\right\}. \tag{21}$$

Now we ask again the question when $|\bar{k}_k-\bar{k}_l| \lesssim l_B$. Because in this case L_x is not small we can have $\bar{k}_i \lesssim l_B$ for $\sigma(i) \lesssim L_x/l_B$ and an estimate can be that this can happen for all pairs $\sigma(l)$ and $\sigma(k)$ in Eq. (21) for which $\sigma(l), \sigma(k) \lesssim L_x/l_B$, and we might think that there are $N_p \approx (L_x/l_B)(L_x/l_B-1)/2$ of them. But this is an overestimate for the construction in Eq. (18) because by making division in slices we do not have the factor $\sin\{(\pi/L_y)(y_k-y_l)\}$ for each pair of particles. As we do not have as many pairs as $N(N-1)/2$, but because of slicing only around $\sim L_x \times L_y^2/l_B^3$ [or $N^{3/2}(L_y/L_x)$], N_p should be reduced[11] by L_x and therefore is not on the order of N, which would pose a problem in the scaling argument for the Hall viscosity and influence its final value. Therefore, we argued that we can model the contribution of each term as in Eq. (20) with $\sigma \approx \sigma'$ as

$$\sim \frac{1}{\tau_2^{N/2+\alpha}} I(L_x, L_y), \tag{22}$$

where $\alpha \lesssim L_x/l_B$. Even if the specific value of α depends on the choice of grouping of particles for slicing in Eq. (18), by extracting a leading contribution in N we can recover the same result for the Hall viscosity as before. We in fact are taking the large-N limit before the limit $\tau \to i$, which is allowed[12] but implies the same value of the Hall viscosity only in this limit. More precisely, if we do not assume the effective reduction of each $\sin\{(\pi/L_y)(y_k-y_l+\bar{k}_k-\bar{k}_l)\}$ to either $\sin\{(\pi/L_y)(y_k-y_l)\} \sim (\pi/L_y)(y_k-y_l)$ or $\sin\{(\pi/L_y)(\bar{k}_k - \bar{k}_l)\}$, in the end the term in Eq. (20) can be expressed as a series with each member of the form as in Eq. (22), where again $\alpha \lesssim L_x/l_B$, and the argument follows. This is possible because for any $|\bar{k}_k-\bar{k}_l| \lesssim l_B$ and, as we have due to the shifts and Gaussians $|y_k-y_l| \lesssim l_B$, we can approximate

$$\sin\left\{\frac{\pi}{L_y}(y_k - y_l + \bar{k}_k - \bar{k}_l)\right\} \approx \frac{\pi}{L_y}(y_k - y_l + \bar{k}_k - \bar{k}_l), \tag{23}$$

and an expansion in the differences in y's, i.e., $(\pi/L_y)(y_k - y_l)$ follows.

C. System with circular Fermi surface

Our expectation is that the composite fermions will make an isotropic circular Fermi surface even in the thermodynamic limit of the system with rectangular boundaries. Nevertheless, in this case we have to demand that the ground state of the system retains the inversion symmetry of the system in its neutral sector, i.e., that the Fermi part of the ground-state wave function is an eigenvector under $y_i \to -y_i$ and $x_i \to x_i$, $\forall i$, transformation. The rectangular shape is a feature of the system on which the shear transformation is applied. Such a ground-state wave function can be constructed by a generalization of the construction in the previous case (B) given in Eq. (18). Now the two slices along k_x and k_y directions are as in Fig. 4. A group of particles, its number equal to the length of the two slices, is assigned to them. As we sweep the whole circle with these two slices we make a certain division of all particles into groups that correspond to slices, and therefore to make the wave function antisymmetric under particle exchange, we need an overall antisymmetrization as in Eq. (18). The same arguments as in the case with rectangular Fermi surface can be applied here and lead to the conclusion that the Hall viscosity of the Fermi-liquid-like quantum Hall state is unaffected by the presence of the Fermi see in the ground-state function. Namely, all estimates that we did in the rectangular case will be modified by geometrical factors that will not affect the conclusion on the leading behavior in the thermodynamic limit. To illustrate what we mean by geometrical factors let us consider a Fermi surface that is a square with $\sqrt{N}$ length of each side. In that case the number of pairs is $\sqrt{N} \times \sqrt{N}(\sqrt{N} - 1)/2$, but in the case of a circle, which delineates the same volume equal to N, we will have for the same quantity $(16/\pi^{3/2})N^{3/2} + 4N$, i.e., the same leading behavior $\sim N^{3/2}$ up to a numerical-geometrical factor.

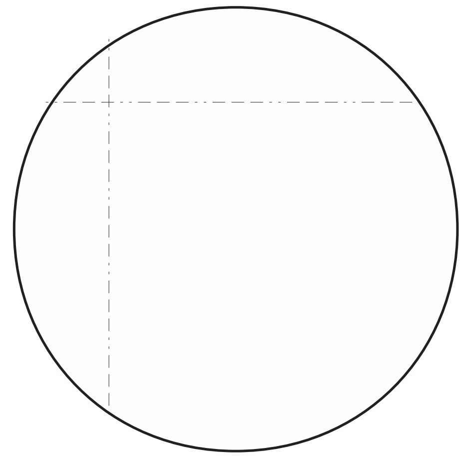

FIG. 4. Circular Fermi surface and two slices with the same group of particles.

Therefore, we argued that, also in this case with assumed unperturbed Fermi surface of circular shape, the norm of the deformed wave function [defined at $\tau = i$ in Eq. (12)] in the neighborhood of $\tau = i$ can be written as a sum of terms, each of the form as in Eq. (22), i.e.,

$$Z \approx \sum_{\sigma \in S_N} \frac{1}{\tau_2^{N/2+\alpha_\sigma}} I_\sigma(L_x, L_y), \tag{24}$$

where $\alpha_\sigma \lesssim L_x/l_B$. Therefore, in the thermodynamic limit $(N \to \infty, L_x \sim \sqrt{N})$ we have the same leading behavior in the exponent of τ_2 equal to $N/2$, which as in the integer ($\nu=1$) quantum Hall case will lead to the value

$$\eta_A = \frac{\hbar}{L_x L_y}\frac{1}{2}\Delta \ln Z|_{\tau_2=1} = \frac{\hbar n}{4}, \qquad (25)$$

the same as in the case of the Laughlin states.

V. HALL VISCOSITY AT $\tau \neq i$

The most important question when considering the question of the Hall viscosity for the Fermi-liquid-like state is whether it is dependent on τ or maybe it is independent of the geometry (τ), which is a remarkable property of the integer quantum Hall state[1] at $\nu=1$ and other quantum Hall states[4] that exhibit Hall conductance plateaus. When considering arbitrary τ we have to start with the deformed Fermi surface as it follows from the deformed dispersion relation in Eq. (11). To simplify the notation we will take $L_x=L_y=L$ or, in general, that m and n carry factors connected with the lengths and may not be integers. Therefore, we write the dispersion relation $\epsilon(\tau)$ as

$$\epsilon(\tau) = \frac{(2\pi)^2}{VL^2\tau_2}[\tau_2^2 m^2 + (n-\tau_1 m)^2], \qquad (26)$$

or if we absorb the scaling factor $f(\tau)=(2\pi)^2/VL^2\tau_2$ and define $e(\tau)$ as $\epsilon(\tau)=f(\tau)e(\tau)$, we may focus on the dispersion relation expressed as

$$e(\tau) = \tau_2^2 m^2 + (n-\tau_1 m)^2. \qquad (27)$$

The equation $\epsilon_F=e_F f$, where ϵ_F is the Fermi energy, defines the (deformed and scaled) Fermi surface, i.e.,

$$e_F \equiv e = \tau_2^2 m^2 + (n-\tau_1 m)^2, \qquad (28)$$

which is illustrated in Fig. 5.

We find that the maximum values of m and n (that belong to points on the Fermi surface) are $m_{\max}=\sqrt{e}/\tau_2$ and $n_{\max}=(\sqrt{e}/\tau_2)|\tau|$, respectively. For $m=-\Delta$ we have the corresponding $n=-\tau_1\Delta \pm \sqrt{e-\tau_2^2\Delta^2}$ and for $n=\Delta$ we have $m=(\tau_1\Delta \pm \sqrt{e-\tau_2^2\Delta^2})/(|\tau|^2)$. This implies that if we keep $|\tau|=1$ we would have the same length for the corresponding two slices along k_x and k_y directions that we introduced before. To simplify the discussion in the following we will assume that this is the case, i.e., that due to $|\tau|=1$ we have the symmetry under inversion around the axis defined by $n=\tau_1 m$.

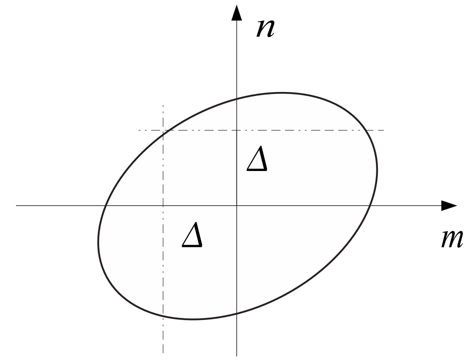

FIG. 5. Deformed Fermi surface and two slices with the same group of particles.

As our deformed Hamiltonian in Eq. (8) has the symmetry under simultaneous transformations $\tau_1 \to -\tau_1$ and $y\to -y$, $x\to x$ or $\tau_1\to-\tau_1$ and $y\to y$, $x\to -x$ our dispersion relation [Eq. (11)] has the same symmetry and the corresponding Fermi surface as well. This symmetry has to exist in the ground state, which has to accommodate to the deformed rectangular shape for $\tau\neq i$. For $\tau=i$ the symmetry can be identified as the inversion symmetry around the x or y axis that has to be generalized to the case with $\tau\neq i$ for which we need to include also $\tau_1\to-\tau_1$ transformation. With this in mind we can come up with a ground-state wave function that will have this symmetry in the Fermi part under simultaneous transformations in coordinate and momentum space. Using the slice decomposition that is illustrated in Fig. 5 for the deformed Fermi surface and with the simplifying assumption $|\tau|=1$, the Fermi part will look like

$$\mathcal{A}\left\{\prod_{\text{over slices in k space}}\left[\prod_{i<j;i,j\in\text{ slice}}\exp\left(ik_y^c\sum_{i\in\text{ slice}} y_i\right)\sin\left(\frac{\pi}{L_y}(y_i-y_j)\right)\exp\left(ik_x^c\sum_{i\in\text{ slice}} x_i\right)\cos\left(\frac{\pi}{L_x}(x_i-x_j)\right)\right]\right\}, \qquad (29)$$

where slices in the k_x and k_y directions correspond to the manner shown in Fig. 5; to each slice in k_y corresponds the slice of the same length in k_x corresponding to the same group of particles. The exponentials with k_x^c and k_y^c carry the momentum $\vec{k}^c=(k_x^c,k_y^c)$, which is due to the deformation of the Fermi surface and the absence of the inversion symmetries around k_x and k_y axis. The momentum $\vec{k}^c$ lies along the new symmetry line, i.e., $k_y^c=\tau_1 k_x^c$, and represents the momentum of the center of the mass of the particles that belong to the particular slice. $\mathcal{A}$ in Eq. (29) is again the overall antisymmetrization that will bring all possible assignments of particles into slices in the final form of the wave function. The construction when x's and y's (k_x^c's and k_y^c's) are interchanged is also possible and we will discuss that case later.

The complete deformed wave function for the Fermi-liquid-like state has the Gaussian factors of the form

$$\exp\left\{-\frac{\tau_2}{2}(y_i - k_{\sigma(i)})^2\right\}, \tag{30}$$

which enters the integral for Z. If τ_2 is small y_i can fluctuate being less localized with the Gaussian. So the relevant interval of y_i values in the integral becomes larger and the sequence of approximations for two particles, beginning with the corresponding term in the product in the integral [Eq. (21)],

$$\sin\left\{\frac{\pi}{L_y}(y_k - y_l + \bar{k}_k - \bar{k}_l)\right\} \approx \sin\left\{\frac{\pi}{L_y}(y_k - y_l)\right\} \approx \frac{\pi}{L_y}(y_k - y_l), \tag{31}$$

is more likely to be allowed. Each term like this will contribute $1/\sqrt{\tau_2}$ when the scaling $\sqrt{\tau_2}y_i \to y_i, \ \forall i$ is applied. If, for small enough τ_2, we assume that for each pair we can do this approximation, in addition to the overall exponent, which we get by the change of variables in the y integration, of order N, we will get another contribution of order $N^{3/2}$ that would lead to the divergence of the Berry curvature and therefore for finite τ_2 of the Hall viscosity. This is certainly an overestimate, but the possibility of divergence seems lurking. Applying arguments similar to the one in Sec. IV B we come to an estimate that the number of relevant pairs is around $\sim(L_x/l_B)(1/\tau_2)$. Therefore, only for strong deformations for which $\tau_2 \sim \frac{1}{\sqrt{N}}$ we may expect the departure of the value for the Hall viscosity from the one of Laughlin states. These arguments cannot be precisely quantified but suggest that the Hall viscosity of the Fermi-liquid-like state may depend on the value of τ for very large deformations. But as we do apply the large-N limit to recover the Laughlin state value for the Hall viscosity as $\tau \to i$ and because here relevant τ_2 is on the order of $\frac{1}{\sqrt{N}}$, we can expect the same Hall viscosity value in the same limit for any finite τ_2. Therefore, the feature of the Laughlin states that their Hall viscosity is independent of τ may stand as a reflection of their true topological nature due to the comparison with the Fermi-liquid-like state that can recover the same value only in large-N limit.

The precise estimate of how the Hall viscosity depends on τ (in the case of the Fermi-liquid-like state) is hard to get also because of the exponentials with $\vec{k}^c$ that carry dependence on y_i. (We have to keep in mind that the scaling $\sqrt{\tau_2}y_i \to y_i, \ \forall i$ is a purely mathematical transformation of variables under the Z integral and does not affect k variables.) In the argument above we assumed $\Sigma_{i\in S}y_i \approx 0$ for each slice S, which might not be the case.

VI. INVERSION SYMMETRY AND HALL VISCOSITY

For $\tau = i$ we view the inversion symmetry as the symmetry under transformations $y_i \to -y_i$ and $x_i \to x_i$, $\forall i$, around the x axis or $y_i \to y_i$ and $x_i \to -x_i$, $\forall i$, around the y axis. For $\tau \neq i$, as we already noted, it can be generalized by adding $\tau_1 \to -\tau_1$ transformation. The symmetry has to be incorporated in the ground-state wave function, more precisely in its neutral part, when we discuss the system with rectangular shape (or deformed rectangular shape, $\tau \neq i$) and our aim is the calculation of the Hall viscosity.

In the case of the Fermi-liquid-like state two constructions stand out at $\tau = i$ (and their generalizations for $\tau \neq i$) for the Fermi part:

$$\text{(a)} \quad \mathcal{A}\left\{\prod_{\text{over slices in k space}}\left[\prod_{i<j;i,j\in \text{slice}} \sin\left(\frac{\pi}{L_y}(y_i - y_j)\right)\cos\left(\frac{\pi}{L_x}(x_i - x_j)\right)\right]\right\}, \tag{32}$$

with the notation that we explained previously, and

$$\text{(b)} \quad \mathcal{A}\left\{\prod_{\text{over slices in k space}}\left[\prod_{i<j;i,j\in \text{slice}} \cos\left(\frac{\pi}{L_y}(y_i - y_j)\right)\sin\left(\frac{\pi}{L_x}(x_i - x_j)\right)\right]\right\}. \tag{33}$$

They are explicitly invariant under the inversion symmetry transformations. The constructions are valid for both circular and rectangular Fermi surfaces. In Appendix B we display the functions (a) and (b) in the case of four particles. In that case it can be easily shown that the state—construction that is $\mathcal{A}\{\sin\{(\pi/L_y)(y_1 - y_2)\}\cos\{(\pi/L_y)(y_3 - y_4)\}\cdots\}$—is identical to zero. As the square of the inversion symmetry is equal to identity in general we expect that wave functions (a) and (b) represent two degenerate ground states and two independent sectors of the Fermi liquid.

Throughout the paper we discussed case (a) for the Fermi-liquid-like state and concluded that, in the thermodynamic limit, around $\tau = i$ the Hall viscosity is equal to the one of Laughlin states and that our expectation is that for general $\tau \neq i$ this will still be true in the same limit. If we try similar arguments in case (b) we can come to the expectation that, due to the cosine functions in the dependence on y's, no change in the overall scaling with τ_2 will occur, and for this construction the Hall viscosity is independent of τ and equal to the one of the Laughlin states.

The single-particle Hamiltonian describes that the evolution of the part of the Fermi-liquid-like state that sees magnetic field is not invariant under the inversion symmetry and the "true" energetics of the problem at $\nu = 1$ (fermionic at $\nu = 1/2$) will certainly differentiate between the two possibilities for the ground state: with Fermi parts (a) and (b). We expect that the construction with Fermi part (a), irrespective whether the ground state is nondegenerate or degenerate, will make a ground state as it can smoothly evolve from the thin-torus limit and its Fermi part [Eq. (15)] when the gauge is fixed, so that the Gaussians are along the y axis. The construction with Fermi part (b) may appear as an additional sector.

VII. DISCUSSION AND CONCLUSIONS

In this paper we discussed the Berry curvature calculations of the Hall viscosity for the unprojected to the LLL wave function of the Fermi-liquid-like state. We concluded that in the linear response, with small deformation of the system and in the thermodynamic limit, the Hall viscosity takes the value characteristic for the Laughlin states [Eq. (7)]. We presented arguments that the value is the same even for general deformations in the same limit.

The preprint in Ref. 13 appeared very recently when we were in the process of finishing of the present paper. There the claim is made, on the basis of the Berry curvature formula applied to the wave functions in the LLL (or projected to a definite LL), that at $\nu=1/2$, irrespective whether the state is incompressible or not, if the Hamiltonian is particle-hole symmetric, the Hall viscosity acquires the Laughlin value. The value is the same irrespective of the deformation (τ). Although our analysis is on the unprojected (to the LLL) wave function of Fermi-liquid-like state, we agree on the value of the Hall viscosity for the state at $\nu=1/2$ in the thermodynamic limit. For a general τ it is surprising that the same value of the Hall viscosity is maintained in the LLL,[13,14] and somehow it has to be reconciled with the expected quantization in the incompressible states. A way out is to claim that the Fermi-liquid-like state has the dissipative (symmetric) viscosity nonzero,[4] but still the quantization of the Hall viscosity for the compressible state in the LLL undermines our expectation that in the Hall viscosity we have yet another characteristic of the incompressible quantum Hall states that is quantized, i.e., has a constant value under small changes (perturbations) in the Hamiltonian. (In other words even gapless phases may have an invariant such as the Hall viscosity.)

On the other hand the Fermi-liquid-like state and the firmly established phase[5] at $\nu=1/2$ may be viewed as some kind of a critical state where effective particle and hole physics and two Jain's sequences of particle states (from $\nu=1/3$) and hole states (from $\nu=2/3$) meet. The situation is somewhat similar or reminiscent of the graphene and the critical behavior of the neutrality point of the Dirac fermions.[15] Nevertheless, it looks conclusive[13] that no critical behavior in the case of the state at $\nu=1/2$ and the Hall viscosity is expected.

Our study shows that the Hall viscosity of the unprojected Fermi-liquid-like state at general τ may deviate from the Laughlin state value for a finite number of particles. Maybe the behavior for a finite number of particles cannot be explained by noninteracting or weakly interacting CF physics if we stay in the LLL as it involves higher LL physics, i.e., all energy scales, which may reflect its critical nature.

In the adiabatic transport theory that we apply our basic assumption is that flux changing excitations are not relevant or higher in energy for the calculation of the Hall viscosity. The result of Ref. 13 seems to give credence to this approach. Within assumptions made, we established that the Fermi-liquid-like state in the thermodynamic limit in the linear response has the value of Hall viscosity equal to the value of Laughlin states. We hope that our analysis will help further elucidation of the problem and the search for the final answer.

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APPENDIX A

We will prove formula (6) for the ground-state function that is holomorphic in τ variable except for the normalization, which is the case also with the Laughlin wave function. The normalized wave function is $\Psi_0=\Psi_L/\sqrt{Z}$, where Ψ_L depends on particle coordinates and τ only. We want to calculate

$$\mathrm{Im}\frac{\partial\langle\Psi_0|}{\partial\tau_1}\frac{\partial|\Psi_0\rangle}{\partial\tau_2}. \tag{A1}$$

First we have for τ_i, with $i=1,2$,

$$\frac{\partial|\Psi_0\rangle}{\partial\tau_i}=\frac{1}{\sqrt{Z}}\frac{\partial|\Psi_L\rangle}{\partial\tau_i}-\frac{1}{2}\frac{\partial\ln Z}{\partial\tau_i}|\Psi_0\rangle. \tag{A2}$$

Therefore,

$$\mathrm{Im}\frac{\partial\langle\Psi_0|}{\partial\tau_1}\frac{\partial|\Psi_0\rangle}{\partial\tau_2}=\mathrm{Im}\left\{\frac{1}{Z}\frac{\partial\langle\Psi_L|}{\partial\tau_1}\frac{\partial|\Psi_L\rangle}{\partial\tau_2}-\frac{1}{2}\frac{1}{\sqrt{Z}}\frac{\partial\ln Z}{\partial\tau_1}\langle\Psi_0|\frac{\partial|\Psi_L\rangle}{\partial\tau_2}-\frac{1}{2}\frac{1}{\sqrt{Z}}\frac{\partial\ln Z}{\partial\tau_2}\frac{\partial\langle\Psi_L|}{\partial\tau_1}|\Psi_0\rangle\right\}. \tag{A3}$$

Here, $|\Psi_L\rangle$ is holomorphic in τ; therefore,

$$\frac{\partial|\Psi_L\rangle}{\partial\bar{\tau}}=\frac{\partial|\Psi_L\rangle}{\partial\tau_1}+i\frac{\partial|\Psi_L\rangle}{\partial\tau_2}=0,\qquad\frac{\partial|\Psi_L\rangle}{\partial\tau}=\frac{\partial|\Psi_L\rangle}{\partial\tau_1}-i\frac{\partial|\Psi_L\rangle}{\partial\tau_2}=0. \tag{A4}$$

Then,

$$(1)\quad\frac{\partial\langle\Psi_L|}{\partial\tau}\frac{\partial|\Psi_L\rangle}{\partial\bar{\tau}}=\frac{\partial\langle\Psi_L|}{\partial\tau_1}\frac{\partial|\Psi_L\rangle}{\partial\tau_1}+\frac{\partial\langle\Psi_L|}{\partial\tau_2}\frac{\partial|\Psi_L\rangle}{\partial\tau_2}+i\frac{\partial\langle\Psi_L|}{\partial\tau_1}\frac{\partial|\Psi_L\rangle}{\partial\tau_2}-i\frac{\partial\langle\Psi_L|}{\partial\tau_2}\frac{\partial|\Psi_L\rangle}{\partial\tau_1}=0. \tag{A5}$$

It follows that

$$\frac{\partial\langle\Psi_L|}{\partial\tau_1}\frac{\partial|\Psi_L\rangle}{\partial\tau_1}+\frac{\partial\langle\Psi_L|}{\partial\tau_2}\frac{\partial|\Psi_L\rangle}{\partial\tau_2}-2\,\mathrm{Im}\frac{\partial\langle\Psi_L|}{\partial\tau_1}\frac{\partial|\Psi_L\rangle}{\partial\tau_2}=0, \tag{A6}$$

(2) $$\frac{\partial \ln Z}{\partial \tau_2} = \frac{1}{Z}\left(\langle \Psi_L | \frac{\partial |\Psi_L\rangle}{\partial \tau_2} + \frac{\partial \langle \Psi_L |}{\partial \tau_2} |\Psi_L\rangle \right) = -\frac{1}{\sqrt{Z}} 2\, \mathrm{Im} \langle \Psi_L | \frac{\partial |\Psi_L\rangle}{\partial \tau_1}, \quad \text{(A7)}$$

(3) $$\frac{\partial \ln Z}{\partial \tau_1} = \frac{1}{Z}\left(\langle \Psi_L | \frac{\partial |\Psi_L\rangle}{\partial \tau_1} + \frac{\partial \langle \Psi_L |}{\partial \tau_1} |\Psi_L\rangle \right) = \frac{1}{\sqrt{Z}} 2\, \mathrm{Im} \langle \Psi_L | \frac{\partial |\Psi_L\rangle}{\partial \tau_2}. \quad \text{(A8)}$$

From Eqs. (A7) and (A8) it follows that

$$\mathrm{Im} \frac{\partial \langle \Psi_0 |}{\partial \tau_1} \frac{\partial |\Psi_0\rangle}{\partial \tau_2} = \mathrm{Im}\left\{ \frac{1}{Z} \frac{\partial \langle \Psi_L |}{\partial \tau_1} \frac{\partial |\Psi_L\rangle}{\partial \tau_2} \right\} - \frac{1}{4}\left(\frac{\partial \ln Z}{\partial \tau_1} \right)^2 - \frac{1}{4}\left(\frac{\partial \ln Z}{\partial \tau_2} \right)^2. \quad \text{(A9)}$$

Because

$$\frac{\partial^2 \ln Z}{\partial \tau_i^2} = \frac{\partial}{\partial \tau_i} \frac{1}{Z} \frac{\partial Z}{\partial \tau_i} = -\frac{1}{Z^2}\left(\frac{\partial Z}{\partial \tau_i} \right)^2 + \frac{1}{Z} \frac{\partial^2 Z}{\partial \tau_i^2}, \quad \text{(A10)}$$

and because of Eq. (A6),

$$\Delta Z = \frac{\partial^2}{\partial \tau \partial \bar{\tau}} Z = \frac{\partial \langle \Psi_L |}{\partial \bar{\tau}} \frac{\partial |\Psi_L\rangle}{\partial \tau} = \frac{\partial \langle \Psi_L |}{\partial \tau_1} \frac{\partial |\Psi_L\rangle}{\partial \tau_1} + \frac{\partial \langle \Psi_L |}{\partial \tau_2} \frac{\partial |\Psi_L\rangle}{\partial \tau_2} + 2\, \mathrm{Im} \frac{\partial \langle \Psi_L |}{\partial \tau_1} \frac{\partial |\Psi_L\rangle}{\partial \tau_2} = 4\, \mathrm{Im} \frac{\partial \langle \Psi_L |}{\partial \tau_1} \frac{\partial |\Psi_L\rangle}{\partial \tau_2}, \quad \text{(A11)}$$

we have

$$\mathrm{Im} \frac{\partial \langle \Psi_0 |}{\partial \tau_1} \frac{\partial |\Psi_0\rangle}{\partial \tau_2} = \frac{1}{4} \Delta \ln Z. \quad \text{(A12)}$$

APPENDIX B

For four particles the construction in Eq. (18) or Eq. (32) is

$$\Psi_b = \sin\left\{\frac{\pi}{L_y}(y_1 - y_2)\right\} \sin\left\{\frac{\pi}{L_y}(y_3 - y_4)\right\} \cos\left\{\frac{\pi}{L_x}(x_1 - x_2)\right\} \cos\left\{\frac{\pi}{L_x}(x_3 - x_4)\right\} - \sin\left\{\frac{\pi}{L_y}(y_1 - y_3)\right\} \sin\left\{\frac{\pi}{L_y}(y_2 - y_4)\right\} \cos\left\{\frac{\pi}{L_x}(x_1 - x_3)\right\} \cos\left\{\frac{\pi}{L_x}(x_2 - x_4)\right\} + \sin\left\{\frac{\pi}{L_y}(y_1 - y_4)\right\} \sin\left\{\frac{\pi}{L_y}(y_2 - y_3)\right\} \cos\left\{\frac{\pi}{L_x}(x_1 - x_4)\right\} \cos\left\{\frac{\pi}{L_x}(x_2 - x_3)\right\}, \quad \text{(B1)}$$

and the one in Eq. (33) is

$$\Psi_a = \cos\left\{\frac{\pi}{L_y}(y_1 - y_2)\right\} \cos\left\{\frac{\pi}{L_y}(y_3 - y_4)\right\} \sin\left\{\frac{\pi}{L_x}(x_1 - x_2)\right\} \sin\left\{\frac{\pi}{L_x}(x_3 - x_4)\right\} - \cos\left\{\frac{\pi}{L_y}(y_1 - y_3)\right\} \cos\left\{\frac{\pi}{L_y}(y_2 - y_4)\right\} \sin\left\{\frac{\pi}{L_x}(x_1 - x_3)\right\} \sin\left\{\frac{\pi}{L_x}(x_2 - x_4)\right\} + \cos\left\{\frac{\pi}{L_y}(y_1 - y_4)\right\} \cos\left\{\frac{\pi}{L_y}(y_2 - y_3)\right\} \sin\left\{\frac{\pi}{L_x}(x_1 - x_4)\right\} \sin\left\{\frac{\pi}{L_x}(x_2 - x_3)\right\}, \quad \text{(B2)}$$

i.e., with x's and y's interchanged. The wave functions Ψ_a and Ψ_b can be represented by their configurations in k space. Below, each configuration describes the placements of four fermions in the corners of a square that correspond to the allowed values of four momenta:

$$\Psi_{a(b)} = +\frac{3|1}{2|4} \pm \frac{1|3}{4|2} \pm \frac{2|4}{3|1} + \frac{4|2}{1|3} \mp \frac{4|1}{2|3} - \frac{1|4}{3|2} - \frac{2|3}{4|1} \mp \frac{3|2}{1|4} - \frac{2|1}{3|4} \mp \frac{1|2}{4|3} \mp \frac{3|4}{2|1} - \frac{4|3}{1|2} \pm \frac{4|1}{3|2} + \frac{1|4}{2|3} + \frac{3|2}{4|1} \pm \frac{2|3}{1|4} + \frac{2|1}{4|3} \pm \frac{1|2}{3|4} \pm \frac{4|3}{2|1} + \frac{3|4}{1|2} \mp \frac{3|1}{4|2} - \frac{1|3}{2|4} - \frac{4|2}{3|1} \mp \frac{2|4}{1|3}. \quad \text{(B3)}$$

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[10] The limit will require additional assumptions in our arguments; and, if not stated otherwise, we take $L_y \sim L_x$ in the following to simplify the discussion.
[11] The probability for two particles to pair is $\sim l_B/L_x$.
[12] Although the thermodynamic limit may bring divergences in $\ln Z$ that grow faster than N (Z does not have in general the meaning of a statistical partition function), by decoupling the τ_2 dependence in the large-N limit ($\frac{\alpha}{N} \sim 0$) what is left becomes independent of τ_2 and does not influence the final value for the Hall viscosity.
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