Edge excitations of paired fractional quantum Hall states

M. Milovanović and N. Read
Departments of Physics and Applied Physics, P.O. Box 208284, Yale University, New Haven, Connecticut 06520
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The Hilbert spaces of the edge excitations of several ''paired'' fractional quantum Hall states, namely, the Pfaffian, Haldane-Rezayi, and 331 states, are constructed and the states at each angular momentum level are enumerated. The method is based on finding *all* the zero-energy states for those Hamiltonians for which each of these known ground states is the exact, unique, zero-energy eigenstate of lowest angular momentum in the disk geometry. For each state, we find that, in addition to the usual bosonic charge-fluctuation excitations, there are fermionic edge excitations. The wave functions for each case have a similar form, related to Slater determinants, and the edge states satisfy a ''projection rule,'' that the parity of the number of fermions added to the edge equals the parity of the charge added. The edge states can be built out of quantum fields that describe the fermions, in addition to the usual scalar bosons (or Luttinger liquids) that describe the charge fluctuations. The fermionic fields in the Pfaffian and 331 cases are a noninteracting Majorana (i.e., real Dirac) and Dirac field, respectively. For the Haldane-Rezayi state, the field is an anticommuting scalar. For this system, we exhibit a chiral Lagrangian that has manifest SU(2) symmetry, but breaks Lorentz invariance, because of the breakdown of the spin-statistics connection implied by the scalar nature of the field and the positive-definite norm on the Hilbert space. Finally, we consider systems on a cylinder, where the fluid has two edges, and construct the sectors of zero-energy states, discuss the projection rules for combining states at the two edges, and calculate the partition function for each edge excitation system at finite temperature in the thermodynamic limit. The corresponding theory for the hierarchy and its generalizations is also given. It is pointed out that the conformal field theories for the edge states are examples of orbifold constructions. Two appendixes contain technical details.

I. INTRODUCTION

The theory of the excitations at the edge of an incompressible fractional quantum Hall state[1] has undergone extensive development since its beginnings a few years ago.[2–5] In the integer quantum Hall effect, that is, when the bulk fluid fills an integer number of Landau levels, the edge excitations are essentially single electrons occupying single-particle edge states[6] that propagate in one direction along the edge and correspond to the classical skipping orbits. There is one ''channel'' of such edge states for each filled Landau level; each channel can be considered as a more or less noninteracting, one-dimensional, unidirectional Fermi sea. In the fractional effect, the edge excitations, like the bulk states, are highly correlated and cannot be described by single-electron states. The basic variables are density fluctuations, which propagate in one direction along the edge. The quantum field theory, which describes these, is a chiral Luttinger liquid.[2,5] In the simplest case, that of the Laughlin states at filling factors $1/q$, q odd, this density mode is the only low-energy excitation at the edge. In the special case of the integer effect at filling factor 1, this is equivalent, via bosonization, to the single-electron, Fermi-sea description.[3] The theory was soon extended[2,5] to the edge excitations of the hierarchy theory,[1] which yields an incompressible ground state in the bulk for all rational filling factors p/q with q odd. In general, some chiral conformal field theory, which generalizes the chiral Luttinger liquid, is expected to describe the low-energy, long-wavelength excitations. It was predicted, in Ref. 7, that this theory would, in general, be the same as the conformal field theory, the correlation functions of which reproduce the bulk wave functions, and which describes such universal properties of the bulk states as the statistics of their fractionally charged excitations, as it does for the hierarchy states. This deep connection implies that the properties of the edge states are not only of interest in their own right, but they can also be used to probe the properties of the underlying bulk state. Effects in tunneling into or between edge states have been the subject of various works.[8,9]

In this paper, we wish to extend the theory of edge states to cover some other interesting states that have been proposed and which do not fit into the hierarchy scheme. In particular, there are (i) the Haldane-Rezayi (HR) (Ref. 10) state, proposed to explain the plateau observed at $\nu=5/2$ in terms of a half filling of the first excited Landau level, in which the electrons have no net polarization; (ii) the Moore-Read (Pfaffian) state,[7] again for a half-filled Landau level, but this time with spin-polarized or spinless electrons. They are ground states of electron systems with special short-range interactions, described later in this paper. The nature of the observed 5/2 state remains controversial (other suggestions include an alternative spin-singlet state,[11] which we believe to be a spin-singlet generalized hierarchy state), while the Pfaffian has been proposed as an explanation for a $\nu=1/2$ plateau in double layer systems,[12] though theoretical calculations do not support this suggestion. Instead, they suggest[13] that the ground state there is (iii) the so-called 331 state. Since this is constructed as a two-component generalization of the Laughlin states, it is part of the generalized hierarchy,[14] but since it can also be interpreted, like the HR and Pfaffian states, as a ''paired'' state, it will be natural to include it here.

The terminology ''paired'' state must be interpreted carefully. It is an old idea that Laughlin's states might be generalized if the electrons are first grouped into clusters of m particles (such as pairs, $m=2$) and the resulting objects of charge m then form a Laughlin state. While this may be possible, and was apparently part of the idea of HR (Ref. 10), it is not quite what we have in mind. In terms of the now-popular composite fermions, which consist of an electron plus an even number q of attached vortices, and which at filling factor $1/q$ behave as particles in zero net field,[15] the paired states are obtained by forming a BCS-type paired ground-state wave function, rather than a Fermi sea (the real-space wave functions are given in Sec. II). This was first pointed out in Ref. 7, and has been discussed more recently in Refs. 16 and 17. We note that it is not clear that these two procedures lead to equivalent states. In particular, the first idea seems to lead to a prediction of abelian statistics for fractionally charged quasiparticles,[12] while the second has been connected with nonabelian statistics.[7] The comparison of the two procedures hinges on the question of whether the two operations, of grouping particles into pairs, and of attaching an appropriate number of vortices, commute. In any case, they do both lead to the result that quasiparticles have charges in multiples of $1/2q$ rather than $1/q$,[7] and, in the latter procedure, to the existence of BCS-type (composite) fermionic excitations obtained by breaking pairs; these are expected to have a gap in their spectrum. In this paper, we will see that the gap for the latter excitations goes to zero at the edge, and the fermions appear as gapless edge excitations, in addition to the usual bosonic charge fluctuations. The fermions can be described by quantum field theories, which are related to relativistic conformal field theories (CFT's); in the cases of the Pfaffian and 331 states, these are the chiral versions of familiar Majorana and Dirac fermions, respectively.

The goal of this paper is to understand the structure of the Hilbert spaces, and the field theories, of the edge excitations of these paired states. Only closed systems are considered, with the fluid in the form of either a droplet with one edge, or an annulus with two oppositely moving edges. Knowledge of these field theories provides the necessary background for the study of the tunneling and other properties of these states, which might be useful as a diagnostic for the nature of the bulk ground state.

There is some previous work on the edge states considered here. Wen[18] has provided numerical evidence for decoupled Majorana fermions at the edge of the Pfaffian state. Wen, Wu, and Hatsugai[19] studied the edge excitations of the HR and other states, using techniques developed by Wen and Wu,[20] for applying operator product expansions in CFT to FQHE wave functions along the lines of Ref. 7. They obtained analytical results for the edge states, but were not able to show either the completeness or the linear independence of their states, though the dimensions of the spaces were confirmed numerically for the lowest excitations.

In the remainder of this paper, we will first write (in Sec. II), for the Pfaffian, HR, and 331 states, wave functions for edge excitations that are zero-energy eigenstates of the appropriate Hamiltonian and manifestly allow an interpretation as decoupled systems of fermion and the usual boson excited states at the edge. These states appear to be linearly independent, and it remains to check that we have obtained all the edge (or zero energy) states. This is proved in Appendix A. Linear independence is confirmed, up to the eighth level of excited states for the Pfaffian, and the sixth level for HR, by direct construction in Appendix B. For the Pfaffian, this reproduces and extends Wen's[18] numerical results, and analogous results for the HR state by Wen, Wu, and Hatsugai.[19] Our method has the advantage over the computer calculations of being *analytic* and valid for any number of electrons. In Sec. III, we present a $1+1$ fermion field theory for the neutral part of the spectrum of the HR state; the SU(2) symmetry of the system is explicit in this construction. The exponents in the singularity in the electron occupation number at the edge are predicted. We also present a CFT, the correlators of which reproduce the *bulk* wave functions in the manner of Ref. 7. Finally, in Sec. IV, we consider systems on a cylinder, with two edges, which gives further information about the structure of the systems. This information, a complete description of the number of states at each excitation level in the thermodynamic limit, is conveniently expressed as a partition function similar to the usual grand canonical one. The results are interpreted using the CFT idea of an orbifold construction. We emphasize that our basic results are derived without the use of CFT, which is needed only in Sec. III and at the end of Sec. IV, where the results are compared with CFT's.

II. EDGE STATES OF A DISK

A. Edge states of the Laughlin state

In this section, we will first review what is known about the edge states of the Laughlin states, emphasizing points that will help us in studying other FQHE systems. We then turn to results for the Pfaffian, HR, and 331 states.

Considering first the interior of a system, i.e., in the thermodynamic limit where the edge disappears to infinity, or in a system filling a finite but closed geometry such as the sphere, a FQH system at a given rational filling fraction possesses, by definition, a unique ground state (except for some global phenomena in the case of surfaces of nontrivial topology) and a gap for all excited states of the same or higher density than this ground state. In a number of interesting cases (with the particles confined to a fixed Landau level, usually the lowest), a model short-range repulsive interaction can be found for which this ground state has a wave function that is known exactly, and is a zero-energy eigenstate. Therefore, turning to a QH droplet in a plane, with these interactions there is always a possibility of excitations of zero energy, which are ''inflations'' of the densest zero-energy state of the system. Hence a zero-energy state always involves some deformation of the edge compared with the ground state.

The simplest examples of these systems are modeled in terms of Haldane's pseudopotentials[21] by the Hamiltonian (we work in the lowest Landau level throughout)

$$H=\sum_{l=0}^{q-1} \mathscr{V}_l \sum_{i<j} \mathscr{P}_l^{ij}, \tag{2.1}$$

where $\mathscr{V}_l$ are positive constants (pseudopotentials) and $\mathscr{P}_l^{ij}$ is the projection operator onto the relative angular momentum state of angular momentum l, for particles i and j. The densest zero-energy eigenstate of H (2.1), that is the one with the lowest total angular momentum (= total degree of the polynomial part), is the Laughlin state (here in the symmetric gauge)[22]

$$\Psi_L(z_1,\ldots,z_N)=\prod_{i<j}(z_i-z_j)^q\exp\left[-\tfrac{1}{4}\sum|z_i|^2\right]. \tag{2.2}$$

If the Laughlin state is multiplied by any symmetric polynomial in the z_i's, it will still be a zero-energy state.[21] We will elaborate a little on this point.

The Laughlin quasihole operator

$$\prod_{i=1}^{N}(z_i-w), \tag{2.3}$$

multiplied into Ψ_L, produces a quasihole located at w, and many-quasihole states can be obtained by repeated use of this operator. All such states are clearly zero-energy excitations for (2.1), because the wave function still vanishes as $(z_i-z_j)^q$ as $z_i\to z_j$. The operator can be viewed as a generating function, through the expansion

$$\prod_{i=1}^{N}(z_i-w)=e_N+\cdots+(-w)^{N-2}e_2+(-w)^{N-1}e_1+(-w)^N, \tag{2.4}$$

for the elementary symmetric polynomials,

$$e_n=\sum_{1\leqslant i_1<i_2<\cdots<i_n\leqslant N}z_{i_1}\cdots z_{i_n}. \tag{2.5}$$

The symmetric polynomials in the z_i form a closed set under the operations of taking sums, differences, and products, so they form a ring with unit element 1; allowing linear combinations with arbitrary complex coefficients, they form an algebra. All such polynomials, and therefore all zero-energy states, are obtained as linear combinations of products of the e_n, i.e., the e_n's form a basis for the algebra of symmetric polynomials. The e_n's can be obtained by integrating the quasihole operator, times a suitable factor, over all w; thus all zero-energy states can also be obtained as linear combinations of products of integrals of quasihole operators acting on Ψ_L. Single Laughlin quasihole states reconstructed from the e_n's *via* (2.4) are not orthogonal and may be considered as coherent states, similar to those of a single electron in the lowest Landau level, which can be expanded in angular momentum eigenstates. Likewise here, the e_n are operators that increase the angular momentum of the electrons by n. Also, as operators, they clearly commute.

As long as w lies well inside the disk of radius $\sqrt{2q(N-1)}$ formed by the electrons, say more than one magnetic length from the edge, it represents the position of the quasihole. As $w\to\infty$, the disturbance in density, due to the quasihole, approaches the edge and eventually, in the limit, becomes trivial. The leading corrections to this limit are then the terms $e_1,e_2,\ldots$, so that e_n with larger angular momentum represent larger distortions of the edge. This is borne out by the energetics, if we introduce a term λM into the Hamiltonian, where M is the operator representing total angular momentum about the origin. Since rotations are a symmetry of the unperturbed problem, this term merely splits the degeneracy of the different angular momentum eigenstates. Then it is clear that states $e_n\Psi_L$ have energy increasing linearly with n and since gapless excitations can occur only near the edge of an incompressible QH state, we can again conclude that these states are edge excitations. This can be verified in detail for the $q=1$ case, which is a filled Landau level.[3,2] The same expansion for these coherent states with more than one quasihole gives for each linearly independent bosonic state a corresponding edge state. By the arguments above, these must span the full space of edge excitations.

One may count the number of states at each increased angular momentum, as follows. Note that to describe *edge* states, we consider the limit $N\to\infty$, with $\Delta M=M-M_0$ fixed [M_0 is the angular momentum in the ground state, and $M_0=qN(N-1)/2$ in the Laughlin state]. Bulk states are obtained either by fixing w's in quasihole states or by applying e_n's with n of order N, and then $N\to\infty$ in either case. Since ΔM in an edge state $\Pi_\alpha e_{n_\alpha}\Psi_L$ is $\Delta M=\Sigma_\alpha n_\alpha$, the total number of states at ΔM is $p(\Delta M)$, the number of ways ΔM can be divided ("partitioned") into positive integer parts, the sum of which is ΔM. However, their meaning is clearest if we use a different basis for the algebra of symmetric polynomials, namely, the sums of powers

$$s_n=\sum_i z_i^n \tag{2.6}$$

(these are not all independent for finite N, but must all be used as $N\to\infty$), which are one body operators, and, up to constant factors, can be viewed as the positive angular momentum components of the change in density at the edge from the ground state, projected into the space of zero-energy states. Thus, the edge states are built up out of repeated applications of density operators, which do behave as boson creation operators (i.e., they commute) for n positive. The components of the projected density with n negative act as destruction operators and with a suitable normalization, the algebra of independent simple harmonic oscillators is obtained, or equivalently, the abelian (U(1)) analog of a Kac-Moody algebra. This point of view is extensively discussed in Refs. 2, 3 and 5.

The partition function can be obtained from Euler's generating function, which is an infinite series in an indeterminate x:

$$\mathscr{Z}(x)\equiv 1+\sum_{\Delta M=1}^{\infty}p(\Delta M)x^{\Delta M}\equiv\prod_{n=1}^{\infty}(1-x^n)^{-1}, \tag{2.7}$$

where we used the binomial expansion

$$(1-x^n)^{-1}=1+x^n+x^{2n}+\ \ldots\ . \tag{2.8}$$

We recognize $\mathscr{Z}(x)$ as the statistical-mechanical partition function of a chiral Bose field, that is a collection of simple harmonic oscillators of frequencies $n\omega$, $n=1, 2, \ldots$, if we set $x=\exp(-\omega/k_BT)$ (and $\hbar=1$). For the convenience of the reader, at later points in this paper, we include a table of the partition function $p(\Delta M)$ for small ΔM:

ΔM	1	2	3	4	5	6	7	8
dim$=p(\Delta M)$	1	2	3	5	7	11	15	22

In addition to these states, we can obtain states with a net charge added to the edge, either by changing the electron number, which of course can only give integral charges, or by adding quasiholes at the center of the disk, which allows the charge effectively added to the edge to be a multiple of $1/q$. The wave functions of such a state, with a positive charge m/q added at the edge, are the same as the above, except that a factor $\Pi_i z_i^m$ is included. The partition function for the states in each such sector is the same as for $m=0$; these again represent density fluctuations on top of the state, which now has charge m/q added. The states with different electron number will, in the following, generally be found to play a role in the structure of the theory. Whether the states with quasiholes added at the center should be viewed as part of the edge theory is somewhat a matter of taste; they could alternatively be viewed as a part of the more general theory of bulk and edge excitations together.

B. Edge states of the Pfaffian state

The Pfaffian state,[7] for even particle number N, is defined by the wave function

$$\Psi_{\mathrm{Pf}}(z_1,\ldots,z_N)=\mathrm{Pf}\left(\frac{1}{z_i-z_j}\right)\prod_{i<j}(z_i-z_j)^q \times\exp\left[-\tfrac{1}{4}\sum|z_i|^2\right], \tag{2.9}$$

where the Pfaffian is defined by

$$\mathrm{Pf}M_{ij}=\frac{1}{2^{N/2}(N/2)!}\sum_{\sigma\in S_N}\mathrm{sgn}\sigma\prod_{k=1}^{N/2}M_{\sigma(2k-1)\sigma(2k)}$$

for an $N\times N$ antisymmetric matrix, the elements of which are M_{ij}. It is the lowest angular momentum ground state of the Hamiltonian,[12]

$$H=V\sum_{\langle ijk\rangle}\delta^2(z_i-z_j)\,\delta^2(z_i-z_k), \tag{2.10}$$

(where the sum is over distinct triples of particles) for the case $q=1$ and of similar three-body short-range interactions for $q>1$. The filling factor is $1/q$. The Pfaffian state is totally antisymmetric for q even, so could describe electrons, while for q odd, it describes charged bosons in a high magnetic field. Zero-energy quasihole excitations correspond to increasing the flux inside the area spanned by the fluid, as usual, but, in this case, the basic objects contain a half flux quantum each and must be created in pairs. A wave function for two quasiholes was proposed in Ref. 7; it is

$$\Psi(z_1,\ldots,z_N;w_1,w_2)=\frac{1}{2^{N/2}(N/2)!}\sum_{\sigma\in S_N}\mathrm{sgn}\sigma\frac{\Pi_{k=1}^{N/2}[(z_{\sigma(2k-1)}-w_1)(z_{\sigma(2k)}-w_2)+(w_1\leftrightarrow w_2)]}{(z_{\sigma(1)}-z_{\sigma(2)})\cdots(z_{\sigma(N-1)}-z_{\sigma(N)})} \times\prod_{i<j}(z_i-z_j)^q\exp\left[-\tfrac{1}{4}\sum|z_i|^2\right]. \tag{2.11}$$

It is clear that for $q=1$, the quasihole states are zero-energy eigenstates of H, (2.10); this also holds for the appropriate three body H for $q>1$. It will be seen that it is the pairing structure built into the ground state, which allows insertion of Laughlin-like factors, which act on only one member of each pair and hence effectively contain a half flux quantum each, unlike the usual Laughlin quasihole that corresponds to a full flux quantum. The same structure requires that quasiholes are made in pairs, since the wave function must be homogeneous. When quasiholes coincide, that is, when $w_1=w_2$, a Laughlin quasihole is recovered.

The multiquasihole states can be used to generate the edge spectrum of the Pfaffian state. We initially used such an approach, but now find it simpler to write down an ansatz which, we believe, in fact describes all the zero-energy states. We construct wave functions for N electrons (N odd or even), which we will interpret as having F fermions created at the edge:

$$\Psi_{n_1,\ldots,n_F}(z_1,\ldots,z_N)=\frac{1}{2^{(N-F)/2}(N-F)/2!}\sum_{\sigma\in S_N}\mathrm{sgn}\sigma\frac{\Pi_{k=1}^{F}z_{\sigma(k)}^{n_k}}{(z_{\sigma(F+1)}-z_{\sigma(F+2)})\cdots(z_{\sigma(N-1)}-z_{\sigma(N)})} \times\prod_{i<j}(z_i-z_j)^q\ \exp\left[-\tfrac{1}{4}\sum|z_i|^2\right]. \tag{2.12}$$

Here $0 \leq n_1 < n_2 < \cdots < n_F$, are a set of distinct nonnegative integers, and $N-F$ clearly must be even. The sum over permutations can be divided into a sum of terms in each of which the unpaired electrons [those with indices $\sigma(1)$ through $\sigma(F)$ in the above expression] are antisymmetrized among themselves. Then each term contains a Slater determinant in these coordinates, representing fermions in wave functions z^{n_k}, hence the stated conditions on the n_k's. The angular momentum of the states is

$$M=\sum_{k=1}^{F} n_k+\frac{1}{2}[qN(N-1)-(N-F)]. \tag{2.13}$$

Hence, the angular momentum relative to the ground state, $\Delta M = M - M_0$, is

$$\Delta M=\sum_{k=1}^{F} (n_k+\tfrac{1}{2}). \tag{2.14}$$

Note that the angular momentum of the ground state is calculated *at the same number of electrons*, and is $M_0 = N[q(N-1)-1]/2$. Such a ground state only exists for N even, but we use the formula as an interpolation for N odd also, to yield (2.14). We interpret the states as having fermions created in orbitals of angular momentum $\Delta M = n+1/2$, $n=0,1,\ldots$, which is exactly the description of the ground state (antiperiodic) sector for Majorana-Weyl fermions on a circle. We should note, however, that if we choose a certain even number N_0 of electrons and use the ground state for $N=N_0$ as a reference ground state, then the odd-fermion-number sectors occur only when some charge has been either added or removed from the ground state. More precisely, odd-fermion numbers can occur when N is odd, so an odd amount of charge has been added to the edge, and similarly for even numbers. Thus, the parity of the fermion number is equal to the parity of the (integral) charge added. This seemingly trivial observation indicates that the fermion and charge edge degrees of freedom are not completely decoupled. This is analogous to the spin-charge separation at the edge of the Halperin states, where although the spin and charge form separate excitations that can be moved along the edge independently, there are global selection rules that relate the total spin and charge added at the edge, in a similar way to here.[23] This is closely related to the absence of any spin-charge separation in the quasiparticles in the bulk, which can carry spin 1/2, only if they also carry nonzero charge. These ''projection rules'' will be discussed more extensively later, including the hierarchy states, to one of which the Halperin state is isomorphic.

In addition to these states, we can also take any one of them multiplied by a symmetric polynomial in all the z_is, which is again a zero-energy eigenstate. These polynomials represent the ubiquitous chiral bosons associated with charge excitations and need not be considered further at the moment. To ensure that all these states represent linearly independent edge excitations, we must certainly take the limit $N \to \infty$ when studying each space of angular momentum eigenstates at ΔM fixed and finite. As we will see below, the states without symmetric polynomial factors appear to be linearly independent, but a full proof of this, and of independence of the symmetric polynomials, appears difficult. In Appendix A, we prove that all zero-energy states can be written as linear combinations of the forms (2.12) times symmetric polynomials. In Appendix B, we indicate how we showed, for ΔM up to 8, that all these states are linearly independent. This provides rather convincing evidence for our simple form (2.12).

For completeness, we also give expressions for the edge states in the other, ''twisted,'' sector, where the Majorana-Weyl field obeys periodic boundary conditions. These states occur when an odd number of quasiparticles are present far inside the edge. For simplicity, we consider a single quasihole at the center of the disk. The ground state in this sector can be produced by taking the two-quasihole state above, dividing by $w_2^{N/2}$ and letting $w_1 \to 0$, $w_2 \to \infty$. On including unpaired electrons as above, we obtain

$$\Psi_{n_1,\ldots,n_F}(z_1,\ldots,z_N)=\frac{1}{2^{(N-F)/2}(N-F)/2!}\sum_{\sigma\in S_N} \operatorname{sgn}\sigma \frac{\prod_{k=1}^{F} z_{\sigma(k)}^{n_k}\prod_{l=1}^{(N-F)/2}(z_{\sigma(F+2l-1)}+z_{\sigma(F+2l)})}{(z_{\sigma(F+1)}-z_{\sigma(F+2)})\cdots(z_{\sigma(N-1)}-z_{\sigma(N)})} \times\prod_{i<j} (z_i-z_j)^q \exp\left[-\tfrac{1}{4}\sum |z_i|^2\right]. \tag{2.15}$$

These states have angular momentum

$$M=\sum_{k=1}^{F} n_k+\frac{1}{2}qN(N-1). \tag{2.16}$$

Hence the ground state, in which $F=0$, has angular momentum $M_0 = qN(N-1)/2$ and the angular momentum relative to the ground state, $\Delta M = M - M_0$, is

$$\Delta M=\sum_{k=1}^{F} n_k . \tag{2.17}$$

Similar remarks to those above about the F odd cases apply here. These quantum numbers are exactly those expected for Majorana-Weyl fermions obeying periodic boundary conditions, in which fermions can be added in orbitals with angular momentum $n=0,1,\ldots$. In particular, note that for $F=1$, we can take $n_1=0$ and obtain a state with zero increased angular momentum. This state is entirely meaningful; since it has an odd number of electrons, it is not the same as the ground state (with the added quasihole), which has an even number of electrons. That all states in this sector have the form shown is also proved in Appendix A; proof of linear independence along the lines of Appendix B has been done at low ΔM.

Other numbers of quasiholes, either even or odd, are obtained by taking states in the untwisted or twisted sectors, respectively, and multiplying in Laughlin quasihole factors $\Pi_i z_i^m$. These states are similar to those in the two sectors above, but with additional charge at the edge, to which it has been pushed by the quasiholes at the center.

All these states with quasiholes at the center of the disk obey projection rules similar to those for the states without the quasiholes. In this case, as N varies, the charge at the edge runs over values which are integers plus a fixed fraction (defined mod 1, and equal to a multiple of $1/2q$). If the charge added at the edge is defined relative to a reference ground state at even $N=N_0$, for each number of quasiholes at the center, then the projection rules are unchanged. This avoids problems of definition in the case where q Laughlin quasiholes have been added at the center. A more satisfactory approach will be given for the case of two edges, as on the cylinder, in Sec. IV. The states with quasiholes added at the center of the disk can be viewed as a special case of this. One could argue that for the disk, it is more natural to exclude any bulk excitations, in which case there are no twisted or quasihole sectors. The other sectors arise only when both edges are present, or in the presence of bulk quasiparticles.

For completeness, we include a table of the dimensions of the spaces of fermion states for low ΔM in the untwisted sector:

ΔM	1	2	3	4	5	6	7	8
dim	0	1	1	2	2	3	3	5

The entries in the table are the ones that have been verified in Appendix B. The dimensions of the full space of edge excitations in the untwisted, even particle number sector are found by convoluting these numbers with those for the U(1) chiral boson system given earlier. Thus, all states in this sector have been verified to be linearly independent up to $\Delta M=8$. In a numerical calculation in Ref. 18, only the states for $\Delta M<5$, which, in fact, contain a maximum of two excited fermions, were found. The first state with four fermions excited appears at $\Delta M=8$ (and is included in the table).

C. Edge states of the Haldane-Rezayi state

The HR state[10] can be written in terms of the coordinates of $N/2$ up-spin electrons at $z_1^\uparrow,\dots,$ and $N/2$ down-spin electrons at $z_1^\downarrow,\dots$ as

$$\Psi_{\rm HR}(z_1^\uparrow,\dots,z_{N/2}^\uparrow,z_1^\downarrow,\dots,z_{N/2}^\downarrow)=\sum_{\sigma\in S_{N/2}}\operatorname{sgn}\sigma\frac{1}{(z_1^\uparrow-z_{\sigma(1)}^\downarrow)^2\cdots(z_{N/2}^\uparrow-z_{\sigma(N/2)}^\downarrow)^2}\prod_{i<j}(z_i-z_j)^q\exp\left[-\tfrac{1}{4}\sum|z_i|^2\right]. \tag{2.18}$$

Here q is even to describe fermionic electrons, and the filling factor is $1/q$. The first factor is of course just a determinant. The product over z_i's with no spin labels attached is over all particles. The fact that this describes a singlet is discussed carefully in Ref. 10. Strictly speaking, the form given is an abuse of notation. The correct way to write the functions is as a function of N electron coordinates, numbered from 1 to N, half of which have up spin and half down, and the permutations are over the subset of electrons of each spin. On including the proper sign factors, the spatial wave functions can be combined with the spinor wave functions of the N electrons, and then summed over all ways of choosing which electrons have which spin. In this way, wave functions that are totally antisymmetric under exchange of both the space and spin labels of particles are constructed. This procedure is standard and has been described in the literature;[10,24,25] it can be used to produce states of definite total spin. Since the construction of such states from the functions given below is straightforward, if tedious, it will be omitted, and we will continue to use the abused notation as in (2.18). In Ref. 7 it was pointed out that this state can be regarded as a BCS-type condensate of spin-singlet pairs of spin-1/2 neutral fermions that consist of an electron and q vortices, from which the spin-singlet property can be more easily understood. The HR state is the unique zero-energy state at $N_\phi=q(N-1)-2$ flux of a ''hollow-core'' pseudopotential Hamiltonian that gives any two particles a nonzero energy when their relative angular momentum is exactly $q-1$.[10]

As for the Pfaffian state, excitations of $hc/2e$ flux are expected and by flux quantization they should occur in pairs. In exact analogy with the Pfaffian state, the wave function for two quasiholes is

$$\Psi_{\rm HR}(z_1^\uparrow,\dots,z_{N/2}^\downarrow;w_1,w_2)=\sum_{\sigma\in S_{N/2}}\operatorname{sgn}\sigma\frac{\Pi_{k=1}^{N/2}[(z_k^\uparrow-w_1)(z_{\sigma(k)}^\downarrow-w_2)+(w_1\leftrightarrow w_2)]}{(z_1^\uparrow-z_{\sigma(1)}^\downarrow)^2\cdots(z_{N/2}^\uparrow-z_{\sigma(N/2)}^\downarrow)^2}\prod_{i<j}(z_i-z_j)^q\exp\left[-\tfrac{1}{4}\sum|z_i|^2\right]. \tag{2.19}$$

Due to the spin independence of the newly inserted factors acting on each pair inside the sum over permutations, the state is still a spin singlet and this suggests that the quasiholes carry no spin. We will see further evidence of this later. The two quasihole state is again a zero-energy eigenstate of the ''hollow-core'' Hamiltonian, where all pseudopotentials (including V_0) are zero, except V_{q-1}, i.e., for two particles to interact, their relative angular momentum should be $q-1$. To see this fact, expand the inserted factors for each pair in terms of powers of $z_k^\uparrow\pm z_{\sigma(k)}^\downarrow$. Due to the symmetry between $z_k^\uparrow$ and $z_{\sigma(k)}^\downarrow$ in each factor, it is easy to see that $z_k^\uparrow-z_{\sigma(k)}^\downarrow$ must occur to an even power. Thus, in the complete wave function, the absence of $(z_k^\uparrow-z_{\sigma(l)}^\downarrow)^{q-1}$ for any k, l, and hence the zero-energy property of the ground state, is preserved in the quasihole states.

It is possible to write down directly the forms of all the zero-energy states of the hollow core Hamiltonian, in analogy with those for the Pfaffian. In the untwisted sector, in terms of the coordinates of $N_\uparrow$ up electrons, $N_\downarrow$ down electrons, the wave functions are linear combinations of (omitting symmetric polynomial prefactors)

$$\frac{1}{(N_\uparrow - F_\uparrow)!}\sum_{\substack{\sigma\in S_{N_\uparrow}\\ \rho\in S_{N_\downarrow}}} \mathrm{sgn}\sigma\ \mathrm{sgn}\rho \times \frac{\prod_{k=1}^{F_\uparrow}(z^\uparrow_{\sigma(k)})^{n_k}\prod_{l=1}^{F_\downarrow}(z^\downarrow_{\rho(l)})^{m_l}}{(z^\uparrow_{\sigma(F_\uparrow+1)}-z^\downarrow_{\rho(F_\downarrow+1)})^2\cdots(z^\uparrow_{\sigma(N_\uparrow)}-z^\downarrow_{\rho(N_\downarrow)})^2} \times\prod_{i<j}(z_i-z_j)^q \exp\left[-\tfrac{1}{4}\sum|z_i|^2\right]. \tag{2.20}$$

Here, $N_\uparrow - F_\uparrow = N_\downarrow - F_\downarrow$ is the number of unbroken pairs, and we may assume the n_k's, m_k's are strictly increasing as for those in the Pfaffian edge states. These functions have a structure similar to the real space wave functions for a BCS state with some broken pairs, that is with BCS quasiparticles added; the latter would have a similar form for the sum over permutations, but the factors $z_i^{\uparrow n_k}$ would be replaced by plane waves. Here, of course, they represent edge states, in which the fermions do behave as if they occupied plane waves running along the one-dimensional edge. As written, these states do not have definite spin, but eigenstates of $\mathbf{S}^2$ and of S_z can be constructed as indicated above. Since the paired electrons form singlets, the spin is determined by the spin-1/2 unpaired fermions in the sums over σ and ρ, which behave identically to ordinary spin-1/2 fermions. Hence, the possible spin states are determined by adding the spins of electrons in different orbitals (labeled by n_k or m_k), with the only constraint that an orbital occupied with both an up and a down fermion must form a singlet.

The angular momentum of the wave functions given is

$$M=\sum_{k=1}^{F_\uparrow} n_k + \sum_{k=1}^{F_\downarrow} m_k + \frac{1}{2}[qN(N-1)-2(N-F_\uparrow-F_\downarrow)]. \tag{2.21}$$

Hence, the angular momentum relative to the ground state, $\Delta M = M - M_0$, is

$$\Delta M=\sum_{k=1}^{F_\uparrow}(n_k+1)+\sum_{k=1}^{F_\downarrow}(m_k+1). \tag{2.22}$$

Conformal invariance ideas suggest that this implies that the edge excitations are fermions of conformal weight 1, not conformal weight 1/2, as for the Pfaffian state (Sec. II B) and the 331 state (Sec. II D). This will be discussed further in Sec. III. We note that the projection rule arising from our states is the same as for the Pfaffian, i.e., the parities of the fermion and charge numbers are the same.

From these wave functions, we can obtain the total numbers of the low-lying edge excitations (excluding symmetric polynomials) of the HR state in the untwisted sector at fixed even N:

ΔM	1	2	3	4	5	6
dim	0	1	4	5	8	10

Again, though the count of states based on the wave functions given could of course be continued, the table has been terminated at the largest ΔM, where we were able to verify linear independence directly (see Appendix B). Note that spins larger than 1 do not occur at these low ΔM. Our numbers for spin 0 and 1 excitations agree, when convoluted with the U(1) numbers, with those calculated in Wen, Wu, and Hatsugai,[19] though our results for N large extend to higher ΔM than they can attain in a small system without encountering finite size effects.

The twisted sector is again obtained by including a spin-independent quasihole factor, this time $\prod_{l=1}^{N_\uparrow-F_\uparrow}(z^\uparrow_{\sigma(F_\uparrow+l)}+z^\downarrow_{\rho(F_\downarrow+l)})$, in the sum on permutations. The angular momentum of the excited states relative to the ground state in this sector is

$$\Delta M=\sum_{k=1}^{F_\uparrow}(n_k+1/2)+\sum_{k=1}^{F_\downarrow}(m_k+1/2). \tag{2.23}$$

Finally, it is once again possible to multiply in factors $\prod_i z_i^m$ that add charge to the edge, which are spin independent and identical to those for the Laughlin states.

D. Edge states of the 331 state

The 331 state is just one of a family of two-component states, the so-called $mm'n$ states, first introduced by Halperin.[26] Using the notation $\uparrow$, $\downarrow$ for the two components, even though they need not represent spin, and bearing in mind that similar remarks to those at the beginning of Sec. II C about constructing totally antisymmetric wave functions apply here also, these states can be written

$$\Psi_{mm'n}(z_1^\uparrow,\ldots,z_{N/2}^\downarrow)=\prod_{i<j}(z_i^\uparrow-z_j^\uparrow)^m\prod_{k<l}(z_k^\downarrow-z_l^\downarrow)^{m'}\prod_{rs}(z_r^\uparrow-z_s^\downarrow)^n \times\exp\left[-\tfrac{1}{4}\sum_i|z_i|^2\right]. \tag{2.24}$$

The general $mm'n$ state is the unique lowest total-angular-momentum ground state of a spin-dependent pseudopotential Hamiltonian, that generalizes (2.1) to the two-component case, which gives positive energy to any state in which two $\uparrow$ or $\downarrow$ particles have relative angular momentum less than m or m', respectively, or in which an $\uparrow$ and a $\downarrow$ particle have relative angular momentum less than n.

For the case when the exponents in these states are of the form $m=m'=q+1$, $n=q-1$, $q\geqslant 1$ (which give filling factor $\nu=1/q$, and the partial filling factors for $\uparrow$, $\downarrow$ are both $1/2q$; for brevity, we will continue to refer to this class of states with general q as the 331 state), then use of the Cauchy determinant identity

$$\prod_{i<j}(z_i^{\uparrow}-z_j^{\uparrow})\prod_{k<l}(z_k^{\downarrow}-z_l^{\downarrow})\prod_{rs}(z_r^{\uparrow}-z_s^{\downarrow})^{-1}=\det\left(\frac{1}{z_i^{\uparrow}-z_j^{\downarrow}}\right) \tag{2.25}$$

allows the ground states to be written in a paired form, similar to the Pfaffian and HR states.[16,17] This identity can be understood physically, in terms of the description of bulk fractional quantum Hall effect wave functions as conformal field theory correlators,[7] as expressing bosonization of correlators of a chiral Dirac (or Weyl) field (on the right hand side), in terms of correlators in a Coulomb gas (or exponentials of a chiral scalar Bose field) (on the left hand side). In terms of BCS-type pairing, this function describes p-type spin-triplet pairing, with each pair in the $S_z=0$ state of a spin triplet.[16,17]

We will extend the fermionized description immediately to include the edge excitations in the untwisted sector, omitting symmetric polynomial prefactors:

$$\frac{1}{(N_{\uparrow}-F_{\uparrow})!}\sum_{\substack{\sigma\in S_{N_{\uparrow}}\\ \rho\in S_{N_{\downarrow}}}}\operatorname{sgn}\sigma\ \operatorname{sgn}\rho$$
$$\times\frac{\Pi_{k=1}^{F_{\uparrow}}(z_{\sigma(k)}^{\uparrow})^{n_k}\Pi_{l=1}^{F_{\downarrow}}(z_{\rho(l)}^{\downarrow})^{m_l}}{(z_{\sigma(F_{\uparrow}+1)}^{\uparrow}-z_{\rho(F_{\downarrow}+1)}^{\downarrow})\cdots(z_{\sigma(N_{\uparrow})}^{\uparrow}-z_{\rho(N_{\downarrow})}^{\downarrow})}$$
$$\times\prod_{i<j}(z_i-z_j)^q\exp\left[-\tfrac{1}{4}\sum|z_i|^2\right], \tag{2.26}$$

which is particularly similar to the HR case. For the angular momentum, we obtain

$$M=\sum_{k=1}^{F_{\uparrow}}n_k+\sum_{k=1}^{F_{\downarrow}}m_k+\frac{1}{2}[qN(N-1)-(N-F_{\uparrow}-F_{\downarrow})]. \tag{2.27}$$

Hence, the angular momentum relative to the ground state, $\Delta M=M-M_0$, is

$$\Delta M=\sum_{k=1}^{F_{\uparrow}}(n_k+1/2)+\sum_{k=1}^{F_{\downarrow}}(m_k+1/2). \tag{2.28}$$

This is the correct behavior for the states of a chiral Dirac (or Weyl) field, where the two types ↑ and ↓ denote particles and antiparticles. This is as expected from the general arguments based on the form of the bulk ground-state wave function,[7] which, as we have mentioned above, includes a correlator of this same type of fields. The projection rule is once again the same as for the Pfaffian.

The edge states can be reexpressed in bosonic form as

$$\mathcal{F}^{\uparrow}\mathcal{F}^{\downarrow}\Psi_{q+1,q+1,q-1}, \tag{2.29}$$

in which $\mathcal{F}^{\uparrow}$ ($\mathcal{F}^{\downarrow}$) are symmetric polynomials in the ↑ (↓) coordinates only, and the numbers $N_{\uparrow}$, $N_{\downarrow}$ of ↑ and ↓ particles need not be equal.

This system also has a twisted sector obtained in a similar way as in the other examples, by multiplying by factors $\Pi z_i^{\uparrow}$ or $\Pi z_i^{\downarrow}$ that represent the elementary quasiholes located at the center of the drop. For a single such factor, this leads to a formula for ΔM like (2.28), but in which the 1/2's in the expression are dropped.

In the bosonic form, it is easy to see that all these states, both twisted and untwisted, are zero-energy eigenstates for the above-mentioned pseudopotential Hamiltonian, as an extension of the arguments for the one-component Laughlin states, and that they span the space of such states. The equivalence of the bosonic and fermionic forms of edge state wave functions involves generalizations of the Cauchy determinant identity (2.25). We will return to the bosonized description in Sec. IV D.

III. FIELD THEORY OF THE HR STATE

A. Field theory of the edge states of the HR state

We have seen in the previous section that, apart from the charge-fluctuation excitations, the edge excitations in the states that we have studied are free fermions (for most of this section, we ignore the projection rules exhibited in Sec. II; they will be reincorporated in Sec. IV). For the Pfaffian and 331 states, these can clearly be described by relativistic Fermi fields in $1+1$ dimensions (i.e., distance along the edge and time) of scaling dimension 1/2, which we will denote ψ for the Majorana field in the Pfaffian case, and $\psi_{\uparrow}$, $\psi_{\downarrow}$ for the Dirac field and its adjoint in the 331 case. These standard field theories need not be described here. For the HR state, a natural candidate might have been the Dirac theory, with up and down excitations described by particle and antiparticle. However, in the Dirac field theory, there is no SU(2) symmetry that can be generated by local expressions for the spin density and current, and, in fact, it correctly describes the edge of the 331 state. Moreover the angular momentum quantum numbers show that the field for the HR state does not have scaling dimension 1/2, but instead dimension 1. This puzzle will be addressed in this section. Another attempt at its resolution has been made by Wen and Wu,[20] which described the bulk wave functions, but did not exhibit the simple Lagrangian description shown here.

First, we write a Hamiltonian that reproduces the angular momentum eigenvalues already found. Introducing a velocity v_s for the spin excitations when a term λM is present, the Hamiltonian for an edge of circumference L would be

$$H=v_s\sum_{n=1}^{\infty}k(a_{k\uparrow}^{\dagger}a_{k\uparrow}+a_{k\downarrow}^{\dagger}a_{k\downarrow}). \tag{3.1}$$

Here, the operators $a_{k\sigma}$, $a_{k\sigma}^{\dagger}$, with $k=2\pi n/L$, obey the canonical anticommutation relations $\{a_{k\sigma}^{\dagger},a_{k'\sigma'}\}=\delta_{kk'}\delta_{\sigma\sigma'}$, $\{a_{k\sigma},a_{k'\sigma'}\}=\{a_{k\sigma}^{\dagger},a_{k'\sigma'}^{\dagger}\}=0$. Comparing with the result for the Pfaffian state, where there is a real (Majorana), right-moving (Weyl) fermion (see, for example, Ref. 2), we see that apart from the extra S^z quantum number, the boundary condition is periodic in the ground-state (untwisted) sector here, while it was antiperiodic in the Majorana-Weyl system.

Therefore, we propose a new $1+1$-dimensional fermion theory for the neutral part of the HR edge, with a doublet of complex Fermi fields $\Psi_{\sigma}(x,t)$, $\sigma=\uparrow$ or $\downarrow$, and a chiral Lagrangian density (inspired by that for chiral scalar bosons[3]) of the explicitly SU(2) invariant form:

$$\mathscr{L}=\tfrac{1}{4}\,\varepsilon^{\sigma\sigma'}(\partial_t-v_s\partial_x)\Psi_\sigma\partial_x\Psi_{\sigma'}+\text{H.c.}\tag{3.2}$$

Here, $\varepsilon^{\sigma\sigma'}=-\varepsilon^{\sigma'\sigma}$, $\varepsilon^{\uparrow\downarrow}=1$. The canonical procedure leads to the following, simpler looking Hamiltonian:

$$H=\frac{1}{2}\int dx\ v_s(\partial_x\Psi_\uparrow\partial_x\Psi_\downarrow+\partial_x\Psi_\downarrow^\dagger\partial_x\Psi_\uparrow^\dagger),\tag{3.3}$$

together with canonical momenta $\Pi_\uparrow=\partial_x\Psi_\downarrow$, $\Pi_\downarrow=-\partial_x\Psi_\uparrow$. Using periodic boundary conditions, and going to Fourier modes, we see that, for the zero wave vector modes, we obtain the first-class constraints that the corresponding momenta vanish: $\Pi_\sigma(k=0)\equiv0$. The constraints can be included by simply omitting the zero modes henceforth in this chiral theory. Quantization using canonical anticommutation relations then leads to quantized fields:

$$\Psi_\uparrow=\sum_{k>0}\frac{a_{k\uparrow}}{\sqrt{kL}}\exp-ik(x+v_st)+\sum_{k>0}\frac{a_{k\downarrow}^\dagger}{\sqrt{kL}}\exp ik(x+v_st),\tag{3.4}$$

$$\Psi_\downarrow=\sum_{k>0}\frac{a_{k\downarrow}}{\sqrt{kL}}\exp-ik(x+v_st)-\sum_{k>0}\frac{a_{k\uparrow}^\dagger}{\sqrt{kL}}\exp ik(x+v_st),\tag{3.5}$$

and their adjoints where $\{a_{k\sigma}^\dagger,a_{k'\sigma'}\}=\delta_{kk'}\delta_{\sigma\sigma'}$, $\{a_{k\sigma},a_{k'\sigma'}\}=\{a_{k\sigma}^\dagger,a_{k'\sigma'}^\dagger\}=0$, and $k=2\pi n/L$, with $n=1,2,\ldots$ for a system of circumference L. It is assumed that the vacuum $|0\rangle$ obeys $a_{k\sigma}|0\rangle=0$. The normal ordered version of (3.3) then yields (3.1).

Because of the Fermi statistics chosen for the field, and the positive-definite norm imposed, as usual, on the Hilbert space, this field theory is not Lorentz invariant, in spite of the gapless linear spectrum. Consequently, it is not conformally invariant either. This may be surprising, since we have become used to the edge theories being some chiral conformal system, but in fact, since we started with a nonrelativistic system of electrons in a high magnetic field, nothing guarantees that the edge must be Lorentz invariant, even when there is a linear dispersion relation for the excitations. Nonetheless, we will see that there is a closely related conformal field theory.

Returning to the chiral theory, the SU(2) currents can be found using the standard Noether procedure:

$$\mathscr{J}_0^a=\frac{1}{2}\sum_{\sigma,\sigma'}\frac{\partial\mathscr{L}}{\partial(\partial_t\Psi_\sigma)}i\tau^a_{\sigma\sigma'}\Psi_{\sigma'}+\text{H.c.},\tag{3.6}$$

where we specify the Grassmann derivative $\delta\mathscr{L}/\delta(\partial_t\Psi_i)$ to be taken from the left, τ^a $(a=1,2,3)$ are Pauli matrices, and normal ordering is assumed. Then the total spin operators for the edge are $S^a=\int dx\ \mathscr{J}_0^a(x)$, and

$$S^z=\sum_k\frac{1}{2}(a_{k\uparrow}^\dagger a_{k\uparrow}-a_{k\downarrow}^\dagger a_{k\downarrow}),\tag{3.7}$$

$$S^+=\sum_k a_{k\uparrow}^\dagger a_{k\downarrow},\tag{3.8}$$

$$S^-=\sum_k a_{k\downarrow}^\dagger a_{k\uparrow},\tag{3.9}$$

which are easily seen to satisfy the SU(2) commutation relations. However, unlike other FQHE systems, such as the Halperin state, where the edge theory is not only conformally invariant, but also has a Kac-Moody current algebra as a spectrum-generating algebra, here the currents do not form a Kac-Moody algebra, and their correlation functions contain logarithmic factors.

We are ready to identify the operators describing the addition or removal of electrons at the edge in our conjectured edge field theory. The operators $\partial\Psi_\uparrow e^{-i\sqrt{q}\varphi}$ and $\partial\Psi_\downarrow e^{-i\sqrt{q}\varphi}$ represent the electron annihilation field operators $\Psi_{\uparrow\text{el}}$ and $\Psi_{\downarrow\text{el}}$. In these expressions, $\varphi(x,t)$ is the usual chiral boson field representing the density fluctuations at the edge, with propagator

$$\langle\varphi(x,t)\varphi(0,0)\rangle=-\ln(x-vt);\tag{3.10}$$

it is related to the electronic charge density by $\rho=-i\partial\varphi/\sqrt{q}$.[5,7] The exponential of the boson operator creates a bosonic object whenever q is even, so the scalar fermion, like the Majorana fermion in the Pfaffian case, field makes the whole thing into a fermion, as the electron should be. Note that the gradient of the scalar fermion field appears here, not the field itself; this reproduces the spin 1 field found earlier. Taking the charge excitations to propagate with velocity v_c, we find that the electron propagator is

$$\langle0|\Psi_{\uparrow\text{el}}^\dagger(x,t)\Psi_{\uparrow\text{el}}(0,0)|0\rangle\propto\frac{1}{(x+v_st)^2(x+v_ct)^q},\tag{3.11}$$

where the space-time separation of the two fields should be small compared with the circumference of the disk. The total exponent is thus $q+2$, in contrast to that for the Pfaffian and 331 state at $\nu=1/q$, which give $q+1$, whereas the Laughlin states give q. Consequently, the expectation $n(k)$ of the occupation number of the kth single-*electron* state, which can be obtained by Fourier transforming (in one dimension, along the edge) the equal-time electron Green's function, has a power-law singularity $n(k)\sim|k-k_{\max}|^{q+1}$ for the HR state, while the exponent is q for the Pfaffian and 331 states, $q-1$ for the Laughlin states [for the full Landau level, $q=1$, and there is a discontinuity in $n(k)$]. Numerical simulations have been performed for both the Haldane-Rezayi and Pfaffian states,[27] but no conclusion about the exponents in occupation number $n(k)$ versus k relevant to the edge field theory is drawn in the published work.

One might also expect that, just as the bulk system has pairing of the composite fermions,[7] similar algebraic BCS-type expectations should appear at the edge. Indeed, because of the form of the operators (3.5), we have, for example,

$$\langle0|\partial\Psi_\uparrow(x,t)\partial\Psi_\downarrow(0,0)|0\rangle\propto\frac{1}{(x+v_st)^2},\tag{3.12}$$

which can be viewed as a pairing function. However, this correlator omits the exponentials of φ needed to represent the electron operators; if these were included, the correlator would decay rapidly, since the fields carry the same, not opposite, charge. This could be taken to illustrate, for the edge theory, how pairing occurs for the composite fermions, not for electrons. Similar phenomena can be found in the Pfaffian and 331 states. However, while the correlator shown is legitimate as it stands for the scalar fermion field theory, it is not a legitimate correlator for the HR edge theory, because the required intermediate states, where only a single fermion has been added or removed from the ground state (or excited states, in the finite temperature case), do not obey the projection rule found in Sec. II C. Since fermions can be created from the ground state only by breaking pairs, states with an odd number of fermions occur only when an odd number of charges have also been added at the edge (assuming there are no changes in the interior of the system). Only operators that respect this rule can be constructed in the edge field theory. Thus, in the theory of the HR edge, this correlator can be constructed only for equal times, and must then be viewed as the expectation of a single, nonlocal, operator.

B. Conformal field theory of the bulk HR state

In Ref. 7, a mathematical connection between FQH wave functions and correlators in CFT was presented, namely, the elegant wave functions of some important FQHE states are actually correlators (or conformal blocks) in a chiral two-dimensional conformal field theory. However, the question of what theory this would be for the HR state was left unresolved. The difficulty was to understand the SU(2) symmetry (in fact, the singlet nature of the state) in terms of CFT. We will present a solution to this problem here. The natural choice for the CFT in the HR case is a nonunitary theory with the Euclidean action (containing at this stage both right- and left-moving degrees of freedom for convenience):

$$S=\int \frac{d^2x}{8\pi}\varepsilon^{\sigma\sigma'}\partial_\mu\Psi_\sigma\partial_\mu\Psi_{\sigma'} \tag{3.13}$$

and the Grassman field Ψ_σ is regarded as real. Thus, Ψ_σ is a relativistic scalar fermion and this model is conformally invariant, but its states do not all have positive self-overlaps, because of the violation of the spin-statistics connection, so we say that the Virasoro representations are nonunitary; the central charge is $c=-2$. [We note that the spin-statistics theorem relates the statistics of the fields in a positive-definite, Lorentz-invariant field theory to the ''spin'' defined by rotations of the Euclidean two-dimensional space-time, which in a conformal theory is the difference of the right- and left-moving conformal weights, and not to what we have been calling spin, which describes the transformation properties under SU(2) rotations that leave the spatial coordinates unchanged.] This system is an anticommuting counterpart to the system of a pair of scalar boson fields, which is unitary and has $c=2$. $\partial\Psi_\sigma$ is a field of conformal weight 1, the correlators

$$\langle\partial\Psi_\uparrow(z_1^\uparrow)\cdots\partial\Psi_\downarrow(z_{N/2}^\downarrow)\rangle \tag{3.14}$$

of which reproduce the determinant in the HR state. The action S is manifestly invariant under the group of real symplectic transformations Sp(2,**R**) [which preserves the reality property of the fields, and has the same complexified Lie algebra as SU(2)] and thus the correlator produces a singlet (since the vacuum is invariant). However, the Noether currents are of the form $\Psi_\sigma\partial\Psi_{\sigma'}-(\sigma\leftrightarrow\sigma')$, which are not ''good'' conformal fields, since their correlators contain logarithms, so there is no Kac-Moody symmetry. We note that Wen and Wu[20] arrived at an equivalent description of this $c=-2$ CFT system in terms of OPE's, but did not give the simple lagrangian description above.

It is possible to construct ''twist'' fields[29] for the field Ψ_σ, which play a role similar to the spin field of the Ising (or Majorana) field theory in the construction of the bulk quasihole wave functions.[7] These fields obey identical relations, as those defined in the next subsection, so we postpone discussion until then.

C. Relation of bulk and edge field theories

Next we will explain briefly how the relation of the bulk and edge field theories can be used in order to define quasihole operators at the edge. The field theories as defined in Sec. III A, III B appear very similar. The difference is that, in Sec. III A the fields were not required to satisfy a reality condition, they and their adjoints both appeared in the action, and the Hilbert space was found to have positive norms (but no Lorentz invariance), while in Sec. III B fields were real, Lorentz invariance was maintained and the self-overlaps of some states were negative. Here we will consider the positive-definite theory of Sec. III A, and exhibit a conformal structure in this system. This does not contradict the earlier statements, because the stress-energy tensor involved is not self-adjoint (with respect to this inner product).

The correlators of the (gradients of the) fields in this system, as already exhibited in (3.12), are clearly conformally invariant. If we work in imaginary time τ, and use the space-time coordinate $z=x+iv_s\tau$, then the fields obey the operator product expansion (ope),

$$\partial\Psi_\sigma(z)\partial\Psi'_\sigma(0)\sim\varepsilon_{\sigma\sigma'}/z^2, \tag{3.15}$$

up to the usual less singular terms (these have the same form as those for the real fields in the theory in Sec. III B). Then, *if we consider only correlators of* $\partial\Psi_\sigma$, *not of* $\partial\Psi_\sigma^\dagger$, these correlations are conformally invariant. The stress-energy tensor that generates these transformations is $T(z)=-\varepsilon_{\sigma\sigma'}:\partial\Psi_\sigma\partial\Psi_{\sigma'}:/2$, which can be verified to obey the ope's of a stress-energy tensor, using only the ope (3.15). This operator is not self-adjoint, so its Fourier components L_n, in general, do not obey $L_n^\dagger=L_{-n}$. However, the Hamiltonian L_0 derived from T coincides with that found above (3.1). Naturally, if we consider instead $\partial\Psi_\sigma^\dagger$, there will be another non-Hermitian stress tensor generating conformal transformations of those correlators. The vacuum is annihilated by the modes L_n, $n\geqslant-1$ of either of these stress tensors, as required in a conformal field theory.

We may now consider twist operators in this theory, which quite generally are operators that twist the boundary conditions on the fields, in the manner already described in Sec. II. We will require these to be also Virasoro primary conformal fields for the stress tensor T above, as they are in the nonpositive-definite theory. They are introduced by the operator product expansion,

$$\partial\Psi_{\sigma}(z)\mathscr{S}(w)\sim\frac{1}{\sqrt{z-w}}\hat{\mathscr{S}}_{-\sigma}(w), \tag{3.16}$$

where $\mathscr{S}$ is a twist field and $\hat{\mathscr{S}}_{\sigma}$ is an excited twist field of spin σ. In the presence of the $\mathscr{S}$ fields, the correlator for $\partial\Psi_{\uparrow}$ and $\partial\Psi_{\downarrow}$ which, in the untwisted sector, is given by the leading term in the ope (3.15)

$$\langle\partial\Psi_{\uparrow}(z)\partial\Psi_{\downarrow}(w)\rangle=\frac{1}{(z-w)^2}, \tag{3.17}$$

becomes

$$\langle\mathscr{S}(\infty)\Psi_{\uparrow}(z)\Psi_{\downarrow}(w)\mathscr{S}(0)\rangle=\frac{\frac{1}{2}\left(\sqrt{\frac{w}{z}}+\sqrt{\frac{z}{w}}\right)}{(z-w)^2}, \tag{3.18}$$

and by a standard calculation,[29] we come to the equation

$$\langle\mathscr{S}(\infty)T(z)\mathscr{S}(0)\rangle=-\frac{1}{8}\frac{1}{z^2}, \tag{3.19}$$

which means that the conformal weight for the primary field $\mathscr{S}$ is $-1/8$, and that the correlator

$$\langle\mathscr{S}(z)\mathscr{S}(0)\rangle=z^{1/4} \tag{3.20}$$

increases with separation. Negative scaling dimensions cannot appear in a conformal field theory described by unitary representations of the Virasoro algebra. Since, in theories obeying the BPZ axioms, operators correspond one-to-one with states, they indicate the existence of states with energy below that of the ground state. Since our Virasoro generators L_n generate a nonunitary representation, this is not a problem here. Moreover, L_0 coincides with the physical Hamiltonian in both the untwisted and twisted sectors, at least up to an overall constant in the twisted case. (We note that, in order to calculate this from ΔM of the zero-energy states, a term related to the contribution of the bulk to the angular momentum must be subtracted; see the next section.) We now propose that this constant is $-2\pi v_s/8L$, as predicted by the conformal considerations above, since this appears to contradict no principles. There is no real inconsistency in asserting that the energy of the supposedly ''excited'' twisted ground state lies below that of the untwisted ''true'' one. We could take the twisted sector (where Ψ_{σ} obeys antiperiodic boundary conditions) as the ground-state sector. We do not do so, because (i) the theory for the disk (i.e., the chiral theory) clearly identifies the periodic sector as the ground state, which has no quasiholes in the bulk; (ii) the ground state in the antiperiodic sector is not invariant under SL(2,**C**) generated by L_0, $L_{\pm 1}$, as required in a conformal theory, whereas that in the periodic sector is.

For the edge states of the HR quantum Hall state (as in all the paired theories considered in this paper), the twisted sector of the fermions occurs only when the charge added at the edge is $1/2q$ (modulo $1/q$). Making use of conformal arguments for the states with added charge, we expect the contribution to the energy from the charge sector to be $(2\pi v_c/L)Q^2/2q$, where Q/q is the charge added to the edge, as will be discussed in the next section. Thus, depending on the ratio v_c/v_s, the net energy of these states in the HR case will be positive in most cases, except for the lowest added charge sectors. In particular, for $\nu=1/2$, the orginal case of interest for HR, there will be a sector with ground-state energy $-2\pi v/16L$, if $v_s=v_c$. These sectors correspond to operators of the form $\mathscr{S}e^{i\varphi/2\sqrt{q}}$, which are also the operators used in the bulk conformal field theory to generate quasihole states that are single valued, with respect to the electron operators.[7]

IV. EDGE STATES AND THEIR FIELD THEORIES ON A CYLINDER

A. General results and the Laughlin states

In this section, we consider zero-energy states on a cylinder. For the Laughlin states, considered in the present subsection, the structure of the edge states is well known (see especially Ref. 2), but will be reviewed here to ensure that the ideas are clear, and so as to introduce the partition function for two oppositely moving edges.

On a right cylinder of circumference L, we work in the Landau gauge. In terms of a complex coordinate z, the single-particle wave functions in the lowest Landau level are $e^{2\pi inz/L}e^{-(1/2)y^2}$, where n is an integer and $y=\text{Im}z$; it has been assumed that the boundary condition is that wave functions are periodic under $z\rightarrow z+L$. A more general boundary condition is that the wave function changes by $e^{i\phi}$ under such a transformation; in that case, the wave functions become $e^{i(2\pi n+\phi)z/L}e^{-(1/2)y^2}$. Returning to the case $e^{i\phi}=1$ from here on, we can write many-particle wave functions in terms of $Z_j=e^{2\pi iz_j/L}$, for example, the Laughlin state:[28]

$$\Psi_L=\prod_{i<j}(Z_i-Z_j)^q\exp\left[-\tfrac{1}{2}\sum_i y_i^2\right]. \tag{4.1}$$

As any z_i approaches any z_j, this function clearly retains the properties of the Laughlin state in the plane, namely, it vanishes as the qth power.

The Landau gauge has explicit symmetry under translation around the cylinder, and the corresponding conserved quantum number, M, is the sum of the powers M_i's of the Z_i's in the many-particle wave functions. M is the angular momentum; alternatively, we could use the linear momentum equal to $2\pi M/L$. For $e^{i\phi}=1$, M is integral. In the Laughlin state above, $M=qN(N-1)/2$, and the M_i's of individual particles are in the range $0,\ldots q(N-1)$. The single-particle wave function with $M_i=n$ is peaked at $y=-2\pi n/L$, so the Laughlin state occupies a corresponding range in the y direction. Due to translational symmetry in the y direction, there are, however, also an infinite number of other Laughlin states obtained by shifts, which are produced by acting repeatedly on the wave function with $\Pi_i Z_i$ (or its inverse). This operator shifts the M_i by 1, so since the filling factor is $1/q$, it corresponds to shifting a charge $1/q$ from one edge to the other. Fractional shifts change the boundary condition on the wave function, so they are not allowed in the Hilbert space at fixed $e^{i\phi}$.

The infinite set of Laughlin states (''ground states'') are the most dense or compact zero-energy states, in the sense that the M_i's lie in a range of minimum possible width (note that the pseudopotential Hamiltonian in arbitrary geometry is defined in terms of the order of vanishing of the wave functions as two-particles approach each other, which in the plane is equivalent to relative angular momentum). The total angular momentum of the Laughlin states is $M = \frac{1}{2}qN(N-1) - pN$, for the state where we have applied ΠZ_i^{-p}, p integral, to the Laughlin state (4.1). The range of M_i values found in this state is then $-p \leqslant M_i \leqslant q(N-1) - p$. [Note that for $q(N-1)$ even, we could choose p, such that $M=0$ and obtain a state symmetrical about $M_i=0$.] All other zero-energy states have a broader range of M_i's, and are obtained as edge excitations of the two edges. In the $N\to\infty$ limit, the different Laughlin states become infinitely far apart in angular momentum, and the assignment of edge excitations as belonging to a particular ground state (from which their angular momentum differs by a finite amount) becomes unambiguous. There are then two sets of elementary edge excitations, out of which these excited states can be built, and as for the disk, these are linearly independent in the $N\to\infty$ limit. The elementary bosons that create them are the operators $s_n = \Sigma Z_i^n$ and $\overline{s_n} = \Sigma Z_i^{-n}$ $(n>0)$. We refer to the excitations created by the action of the s_n's, the contribution of which to ΔM_{tot}, the change in M relative to the corresponding Laughlin state, is positive, as right moving. The other operators, $\overline{s_n}$'s, create excitations at the other edge, have $\Delta M_{\mathrm{tot}} < 0$, and are viewed as left moving. (Here right and left refer to the two directions parallel to the edge.) Thus, if we can split ΔM_{tot} into $\Delta M - \overline{\Delta M}$, where the two terms are the contributions at the two edges, then we would like to view $\Delta M + \overline{\Delta M}$ as the "pseudoenergy" (within a scale factor), where both terms are defined as nonnegative for the case in the present subsection. The direction of motion then follows from the group velocity, the change in pseudoenergy with momentum of an elementary excitation.

A more realistic, and well-defined, method would be to introduce a Hamiltonian that breaks the degeneracy of the zero-energy ground states and edge excitations; for example, a parabolic confinement potential $\Sigma_i M_i^2$, would suffice, but unfortunately the wave functions that can be easily written down are not eigenstates of such a form, except for $q=1$. We assume that as such a term is turned on and the eigenstates evolve, they stay in one-one correspondence with those found here. We expect on physical grounds that excitations that move electrons further from the minimum of this potential, as the edge excitations do, have higher energy, and so for small $|\Delta M|$ the bosons have a dispersion relation $E \sim |\Delta M|$ for either edge, and form the modes of a (nonchiral) scalar Bose field. The effect of the Hamiltonian on the low-lying states of the system can then be determined through a renormalization group analysis, as has been done for the present case already.[9] In fact, the analysis of the effective field theory of low-lying excitations, and its operator content, given here is the basis for such an analysis for the paired states.

If the electron number in the Laughlin ground state is changed by 1, the width changes by q units. We can view this as an operation that adds charge to a single edge, without disturbing the bulk ground state. The full set of possible ground-state systems can then be parametrized by the positions of the two edges, which are almost independent. Each edge can be shifted by q units without affecting the other. Shifts by $m=1,\ldots,q-1$ must be performed on both edges together. A shift by q units at both edges is equivalent to removing an electron from one edge and inserting it in the ground state at the other. If we extend the idea of a charge sector to include in a single sector all those that differ by integral charges, then there are only q sectors, of charge 0, $1/q,\ldots,1-1/q$ (modulo integers), where the charge is shifted by the stated amount from one edge to the other. For the pseudopotential Hamiltonian without the confining potential, these q sectors are on an equal footing and the choice of zero is arbitrary. Above these ground states, bosonic excitations can be created at either edge, and have the same character in all sectors. Thus, the Hilbert space of the edge excitations can be written in the form

$$\mathbf{V} = \oplus_{r=0}^{q-1} \mathbf{V}_{r/q} \otimes \overline{\mathbf{V}}_{r/q}. \tag{4.2}$$

Each Hilbert space $\mathbf{V}_{r/q}$ $(\overline{\mathbf{V}}_{r/q})$, $r=0,1,\ldots q-1$, is the span of the full set of states in the extended charge sectors at the right- (left-) moving edge. The conformal field theory of the edge states that includes the operators of charge 1 in the chiral algebra is known as the "rational torus" (see, e.g., Ref. 7). Loosely, the chiral algebra is the algebra of operators that affect only a single edge. There is such an algebra for both edges, the two algebras are isomorphic, and operators in one commute or anticommute with those in the other. Each Hilbert space $\mathbf{V}_{r/q}$ $(\overline{\mathbf{V}}_{r/q})$, $r=0,1,\ldots q-1$, is an irreducible representation of the fully extended right- (left-) moving chiral algebra.

To extend our definition of the pseudoenergy, which was given for the edge excitations of a single Laughlin state at fixed (but large) particle number, to the full set of sectors just described, we introduce an arbitrary reference ground state with $N=N_0$ particles and shift $p=p_0$, so that the angular momentum $M_0 = \frac{1}{2}qN_0(N_0-1) - p_0N_0$. We calculate ΔM_{tot} for Laughlin states, where an integral amount of charge has been added to a single edge, by changing N and adjusting the shift p from p_0, such that either the maximum or minimum occupied M_i is unchanged compared with the reference state in the case of charge added to the left- or right-moving edge, respectively. After subtracting a quantity related to the bulk of the system, a step analogous to measuring momentum from the Fermi wave vector in a Fermi gas, this gives a formula for ΔM or $\overline{\Delta M}$, valid in these special cases. This will then be used in all the sectors. The occupied states, in general, lie in the interval $-p \leqslant M_i \leqslant qN(N-1)/2 - p$. To add charge at the right-moving edge, we let $N=N_0+\Delta N$, and $p=p_0$. Then we calculate

$$\Delta M_{\mathrm{tot}} = q\Delta N^2/2 + \Delta N[q(N_0-1/2)-p_0]. \tag{4.3}$$

Therefore, if we define $E \equiv \Delta M_{\mathrm{tot}} - [q(N_0-1/2)-p_0]\Delta N$ for the excitation pseudoenergy at the right edge, we obtain $E = q\Delta N^2/2$, for the Laughlin states, and this is consistent with result for the charge-fluctuation bosons, which do not change the charge at the edge. Similarly, for the left edge, where we must use the adjusted shift $p=p_0+q\Delta N$, so that the right edge is unmoved, we obtain

$$\Delta M_{\rm tot} = -q\Delta N^2/2 + \Delta N[-q/2 - p_0]. \quad (4.4)$$

Then for the excitation pseudoenergy $\overline{E}$ at this edge, we must use $\overline{E} = -\Delta M - [q/2 + p_0]\Delta N$. For each edge, the coefficient of N is the mean of the angular momenta of the highest (respectively, lowest) occupied single-particle states in the reference state with N_0 and with N_0+1 electrons, so it resembles the Fermi wave vector in a Fermi sea, on two sides of the Fermi sphere. We now use these formulas also for the ground (and excited) states in any charge sector, having a combination of electrons added to N_0, shifts of charge from one edge to the other, and charge fluctuations; for such states we can always calculate, for each edge separately, the amount of charge effectively added, which may now be fractional, but is always a multiple of $1/q$.

All information about the number and quantum numbers of the edge excitations in the thermodynamic limit (taken with L^2/N fixed) can be conveniently summarized in a partition function analogous to that in Sec. II A. The partition function is a double series in a complex parameter x and its complex conjugate $\overline{x}$, which contains information about right and left movers, respectively. In fact, if H is the Hamiltonian the eigenvalues of which are $2\pi v/L$ times the sum of an E and an $\overline{E}$ found above, where v is the speed of propagation of the edge excitations, and taking $x = \overline{x} = e^{-2\pi\beta v/L}$, then this partition function is the Gibbs grand canonical partition function of statistical mechanics, $\mathrm{Tr}e^{-\beta H}$. (This is consistent with the assignments in Sec. II and Sec. III.) The E's of the charge sectors that we have found agree with the conformal weights that are found in the conformal field theory.[2] The result is expected to be the same for a more realistic Hamiltonian with a confining potential, as discussed above. The partition function is defined as

$$\mathscr{Z}(x,\overline{x}) = \mathrm{Tr}x^{E}\overline{x}^{\overline{E}}. \quad (4.5)$$

The structure of $\mathbf{V}$ given above now allows us to express $\mathscr{Z}$ in terms of the trace over each space $\mathbf{V}_{r/q}$, and we define [using the Euler partition sum $p(n)$ for the bosons from Sec. II A]

$$\chi^{\pm}_{r/q}(x) \equiv \sum_{m=-\infty}^{\infty} (\pm 1)^m x^{(mq+r)^2/2q} \prod_{n=1}^{\infty} (1-x^n)^{-1} \quad (4.6)$$

and the complex conjugate for $\overline{\mathbf{V}}_{r/q}$. The traces, such as the $\chi^{+}_{r/q}$'s, over the right- and left-moving spaces are known as characters, since they are essentially the characters, in the algebraic sense, of the irreducible representations $\mathbf{V}_{r/q}$ of the chiral algebra. (The greater generality afforded by the insertion of ± 1, and by allowing r to be an arbitrary real number, will be useful later.) The partition function can then be written as

$$\mathscr{Z}(x,\overline{x}) = \sum_{r=0}^{q-1} |\chi^{+}_{r/q}(x)|^2. \quad (4.7)$$

Similar structures to those found here for the Laughlin states on a cylinder will be found for the other states in the following subsections; unfortunately, for the paired states, while the method for calculating E and $\overline{E}$ produces a similar contribution for the different charge sectors, it is too crude to produce the analogous energies that we expect to originate in the fermion sectors, such as the $-1/8$ discussed in the preceding section, though we believe that they could, in principle, be obtained in a refined calculation.

B. Pfaffian state

As for the Laughlin states, the zero-energy states on the cylinder can be obtained from those in Sec. II B by replacing z_i by Z_i, the Gaussian factor by $\exp[-\frac{1}{2}\Sigma_i y_i^2]$, and recalling that the exponents of the Z_i's run over all the integers, positive and negative. The formula for ΔM of the edge excitations in the untwisted sector containing fermions only still applies,

$$\Delta M_{\rm tot} = \sum_{k=1}^{F} (n_k + 1/2), \quad (4.8)$$

but now $n_k \geqslant 0$ describes right-moving fermions, $n_k < 0$ left moving. For the latter, we can define $\overline{\Delta M} = -\Delta M_{\rm tot}$.

By inspection of the resulting states, we deduce that, in the untwisted sector, if N even is fixed at N_0, then the total number of fermions excited is even, and the parity of the number at each edge (i.e., whether it is even or odd) must be the same. But if we increase N by 1, we must create or destroy a fermion at one edge, as well as increase the charge by 1, which can be done at the same edge without affecting the other. So the chiral algebra includes the operator $\psi e^{i\sqrt{q}\varphi}$ which does this.[7] Then the parity of the fermion numbers at the two edges can be opposite, provided the parity of the integral amount of charge added relative to the reference state is also opposite. Applying the operation, or its adjoint, once more to the same edge, we find states where the charge has changed by 0 or 2 relative to the reference state, but the number of fermions has the same parity, still without affecting the other edge. Thus, all these states lie in the same extended charge sector, and similar results hold in the other untwisted sectors, where a fractional charge m/q has been shifted from one edge to the other. A total of $2q$ sectors results from these considerations.

The twisted sector is obtained from the untwisted by inserting a factor

$$\prod_{l=1}^{(N-F)/2} (Z_{\sigma(F+2l-1)} + Z_{\sigma(F+2l)})$$

in the states with F unpaired fermions (in the notation of Sec. II B) to produce the effective shift by a half unit that transfers charge $1/2q$ from one edge to the other. The angular momentum of the excited fermions is then

$$\Delta M_{\rm tot} = \sum_{k=1}^{F} n_k, \quad (4.9)$$

where once again n_k can run over the integers. This time $n_k > 0$ represents right movers, $n_k < 0$ left movers. $n_k = 0$ is the zero mode, which cannot be assigned to the right- or left-moving sectors. On the other hand, the requirement that the *total* number of fermions must be even whenever the charge added to the reference state is even, and odd when it is odd, still applies. Thus, we have the general projection rule that applies in all the sectors, twisted and untwisted, that the total number of fermions created in right moving, left moving, or zero modes together must be equal to the parity of the total charge added to the system at the two edges together. In the twisted sectors, it can be satisfied by allowing either parity of right- and of left-moving fermions in all charge sectors, then choosing the occupation number (either 0 or 1) of the zero mode to obey the condition. Consequently, the distinction between ''even'' and ''odd'' sectors, that existed among the untwisted sectors and was responsible for the factor 2 in the $2q$ sectors, no longer applies, and there are just q distinct sectors (or irreducible representations of the chiral algebra). The total number of sectors is therefore $3q$, which, in line with the general connection between bulk and edge states made in Ref. 7, is the same as the number of zero-energy ground states found in the toroidal geometry.[12] Note that in this case the description of the chiral algebra as affecting only a single edge is not quite correct, because the right-moving operator ψ, and its left-moving analog $\overline{\psi}$, each contain a term that changes the occupation number of the zero mode. Nonetheless, in the Majorana field theory, these operators anticommute; similarly, the algebra of operators assigned to one edge does (anti-)commute with those assigned to the other, even in the twisted sector.

The calculation of the partition function, which formalizes the above remarks, is conveniently performed in terms of characters. The basic objects are characters for the states at one edge that differ in charge only by integers, and, in the untwisted sector, the parity of the charge difference from that in the lowest energy state is equal to the parity of the change in fermion number. Characters for the fermions alone will be useful; these are, for untwisted (antiperiodic) boundary conditions,

$$\chi_0^{\mathrm{MW}}(x)=\frac{1}{2}\left[\prod_{n=0}^{\infty}(1+x^{n+1/2})+\prod_{n=0}^{\infty}(1-x^{n+1/2})\right],$$

$$\chi_{1/2}^{\mathrm{MW}}(x)=\frac{1}{2}\left[\prod_{n=0}^{\infty}(1+x^{n+1/2})-\prod_{n=0}^{\infty}(1-x^{n+1/2})\right], \tag{4.10}$$

which are, respectively, for even and odd numbers of Majorana-Weyl (MW) fermions. The subscripts are the conformal weights of the corresponding primary fields, or the energies of the ground state in each sector. In the twisted sector (periodic boundary conditions), there is only a single nonvanishing character:

$$\chi_{1/16}^{\mathrm{MW}}(x)=x^{1/16}\prod_{n=1}^{\infty}(1+x^{n}). \tag{4.11}$$

(The zero mode is omitted here, as it will be accounted for separately, as already explained.) The constant 1/16 in E for the twisted ground state here is the analog of those in the different charge sectors as derived in Sec. IV A, where it was mentioned that we cannot at present derive this one directly from our zero-energy wave functions. 1/16 is the conformal weight of the corresponding operator σ, the spin field, which twists the boundary condition on the Majorana fermion, like the twist field $\mathscr{S}$ discussed in Sec. III. These three expressions are well known as the Virasoro characters of the critical two-dimensional Ising model, as well as in other contexts. The characters of the chiral algebra relevant to the edge states of the Pfaffian are

$$\chi_{r/q,\mathrm{even,untwisted}}^{\mathrm{Pf}}(x)=\tfrac{1}{2}\chi_0^{\mathrm{MW}}(x)[\chi_{r/q}^{+}(x)+\chi_{r/q}^{-}(x)] + \tfrac{1}{2}\chi_{1/2}^{\mathrm{MW}}(x)[\chi_{r/q}^{+}(x)-\chi_{r/q}^{-}(x)],$$

$$\chi_{r/q,\mathrm{odd,untwisted}}^{\mathrm{Pf}}(x)=\tfrac{1}{2}\chi_{1/2}^{\mathrm{MW}}(x)[\chi_{r/q}^{+}(x)+\chi_{r/q}^{-}(x)] + \tfrac{1}{2}\chi_0^{\mathrm{MW}}(x)[\chi_{r/q}^{+}(x)-\chi_{r/q}^{-}(x)],$$

$$\chi_{(r+1/2)/q,\mathrm{twisted}}^{\mathrm{Pf}}(x)=\chi_{1/16}^{\mathrm{MW}}(x)\chi_{(r+1/2)/q}^{+}(x). \tag{4.12}$$

It can easily be seen that these expressions are sums over states with the necessary constraints on the combinations of fermion and charge states included, apart from those that enter on combining right and left movers and zero modes. The partition function is, finally,

$$\mathscr{Z}^{\mathrm{Pf}}(x,\overline{x})=\sum_{r=0}^{q-1}\left[|\chi_{r/q,\mathrm{even,untwisted}}^{\mathrm{Pf}}(x)|^2+|\chi_{r/q,\mathrm{odd,untwisted}}^{\mathrm{Pf}}(x)|^2 + |\chi_{(r+1/2)/q,\mathrm{twisted}}^{\mathrm{Pf}}(x)|^2\right]. \tag{4.13}$$

C. HR state

As for the Pfaffian state, the edge states of the HR state on a cylinder can be deduced almost immediately from the results for the disk. In the HR case, the untwisted sector is found now to contain zero modes, while the twisted sector does not. The zero mode, like the nonzero, right- and left-moving modes, can be occupied by a spin up or a spin down fermion, or both. From these states we can deduce the projection rule. It has the same form as for the Pfaffian, in all sectors. The projection rule requires even fermion number when no charge has been added, and that a fermion is created or destroyed whenever a unit of charge is added to a single edge, so that the chiral algebra includes an operator $\partial\Psi_{\sigma}e^{i\sqrt{q}\varphi}$, very similarly to the Pfaffian. We can, therefore, write down the characters without further ado. The characters for the fermions are

$$\chi_0^{\Psi}(x)=\frac{1}{2}\left[\prod_{n=1}^{\infty}(1+x^{n})^2+\prod_{n=1}^{\infty}(1-x^{n})^2\right],$$

$$\chi_1^{\Psi}(x)=\frac{1}{2}\left[\prod_{n=1}^{\infty}(1+x^{n})^2-\prod_{n=1}^{\infty}(1-x^{n})^2\right],$$

$$\chi_{-1/8}^{\Psi}(x)=\frac{1}{2}x^{-1/8}\left[\prod_{n=0}^{\infty}(1+x^{n+1/2})^2 + \prod_{n=0}^{\infty}(1-x^{n+1/2})^2\right],$$

$$\chi_{3/8}^{\Psi}(x)=\frac{1}{2}x^{-1/8}\left[\prod_{n=0}^{\infty}(1+x^{n+1/2})^2-\prod_{n=0}^{\infty}(1-x^{n+1/2})^2\right]. \tag{4.14}$$

Note that the first two are the even- and odd-fermion-number states in the untwisted sector, omitting the zero modes, while the last two are the same for the twisted sector, and we have used the negative conformal weight of the twist fields inwriting the latter. In this case, we have maintained the distinction between even- and odd-fermion numbers in the sec-tor that contains the zero modes, so as to exhibit its fate explicitly. (If desired, the following approach can also be taken for the Pfaffian state, and the expressions already given for the partition function can be derived in this manner, verifying the argument, given in words in the preceding text, that there are only $3q$ sectors.) The characters for the chiral algebra of the HR state are

$$\begin{aligned}
\chi^{\rm HR}_{r/q,\rm ev,untw}(x) &= \tfrac{1}{2}\,\chi^{\Psi}_{0}(x)[\chi^{+}_{r/q}(x)+\chi^{-}_{r/q}(x)] \\
&\quad + \tfrac{1}{2}\,\chi^{\Psi}_{1}(x)[\chi^{+}_{r/q}(x)-\chi^{-}_{r/q}(x)], \\
\chi^{\rm HR}_{r/q,\rm od,untw}(x) &= \tfrac{1}{2}\,\chi^{\Psi}_{1}(x)[\chi^{+}_{r/q}(x)+\chi^{-}_{r/q}(x)] \\
&\quad + \tfrac{1}{2}\,\chi^{\Psi}_{0}(x)[\chi^{+}_{r/q}(x)-\chi^{-}_{r/q}(x)], \\
\chi^{\rm HR}_{(r+1/2)/q,\rm ev,tw}(x) &= \tfrac{1}{2}\chi^{\Psi}_{-1/8}(x)[\chi^{+}_{(r+1/2)/q}(x)+\chi^{-}_{(r+1/2)/q}(x)] \\
&\quad + \tfrac{1}{2}\,\chi^{\Psi}_{3/8}(x)[\chi^{+}_{(r+1/2)/q}(x) \\
&\quad - \chi^{-}_{(r+1/2)/q}(x)], \\
\chi^{\rm HR}_{(r+1/2)/q,\rm od,tw}(x) &= \tfrac{1}{2}\,\chi^{\Psi}_{3/8}(x)[\chi^{+}_{(r+1/2)/q}(x)+\chi^{-}_{(r+1/2)/q}(x)] \\
&\quad + \tfrac{1}{2}\,\chi^{\Psi}_{-1/8}(x)[\chi^{+}_{(r+1/2)/q}(x) \\
&\quad - \chi^{-}_{(r+1/2)/q}(x)].
\end{aligned} \tag{4.15}$$

We may now form the partition function, by combining the sectors subject to the rules already mentioned. In particular, in the untwisted sector where the zero mode occurs, we may combine right- and left-moving sectors of the same parity, in which case the zero mode may be either unoccupied or doubly occupied, or we may combine sectors of opposite parity if the zero mode is occupied once, which may be with either spin. Thus, we find for the partition function:

$$\begin{aligned}
\mathscr{Z}^{\rm HR}(x,\overline{x}) &= \sum_{r=0}^{q-1} \{2|\chi^{\rm HR}_{r/q,\rm ev,untw}(x)|^2+2|\chi^{\rm HR}_{r/q,\rm od,untw}(x)|^2 \\
&\quad + 2[\chi^{\rm HR}_{r/q,\rm ev,untw}(x)\overline{\chi^{\rm HR}_{r/q,\rm od,untw}(x)} \\
&\quad + \chi^{\rm HR}_{r/q,\rm od,\ untw}(x)\overline{\chi^{\rm HR}_{r/q,\rm ev,untw}(x)}] \\
&\quad + |\chi^{\rm HR}_{(r+1/2)/q,\rm ev,tw}(x)|^2+|\chi^{\rm HR}_{(r+1/2)/q,\rm od,tw}(x)|^2\} \\
&= \sum_{r=0}^{q-1} [2|\chi^{\rm HR}_{r/q,\rm ev,untw}(x)+\chi^{\rm HR}_{r/q,\rm od,untw}(x)|^2 \\
&\quad + |\chi^{\rm HR}_{(r+1/2)/q,\rm ev,tw}(x)|^2+|\chi^{\rm HR}_{(r+1/2)/q,\rm od,tw}(x)|^2],
\end{aligned} \tag{4.16}$$

which shows that there are in fact $4q$ sectors. The untwisted characters have combined into simpler ones, similarly to the twisted Pfaffian state characters:

$$\chi^{\rm HR}_{r/q,\rm untw}(x)=\prod_{n=1}^{\infty}(1+x^n)^2\chi^{+}_{r/q}(x), \tag{4.17}$$

which, however, appear twice in $\mathscr{Z}$. The $4q$ sectors show that there are $4q$ primary fields of the chiral algebra in the system, which (in the notation of Sec. III, except that the fields Ψ_σ, $\mathscr{S}$ are now the nonchiral fields that act on both right and left movers) are 1 (the identity operator), $\Psi_\uparrow\Psi_\downarrow$, $\mathscr{S}e^{i(\varphi+\overline{\varphi})/2\sqrt{q}}$, $\hat{\mathscr{S}}_{\sigma\overline{\sigma}}e^{i(\varphi+\overline{\varphi})/2\sqrt{q}}$ ($\hat{\mathscr{S}}_{\sigma\overline{\sigma}}$ is the twist field excited in both left and right sectors, and $\sigma,\overline{\sigma}=\uparrow,\downarrow$), and these operators times additional factors $e^{i(\varphi+\overline{\varphi})/\sqrt{q}}$, which shift charge $1/q$ from one edge to the other. All of these fields are spin singlets, except $\hat{\mathscr{S}}_{\sigma\overline{\sigma}}$, which transforms as spin $\frac{1}{2}\otimes\frac{1}{2}=0\oplus 1$. The field $\Psi_\uparrow\Psi_\downarrow$ is not strictly a primary field, since it has weight zero, but this is not an important distinction here. All other fields are descendants of these, that is they can be obtained by acting with operators in the chiral algebras; as a particular example, the state created from the untwisted ground state with unoccupied zero modes by the zero mode of $\Psi_\uparrow$ times a unit charge at the right-moving edge is obtained from $\Psi_\uparrow\Psi_\downarrow$, by acting with $\partial\Psi_\uparrow e^{i\sqrt{q}\varphi}$. Acting again, with a similar operator, leads us to the identity, so the representations are not irreducible, which is a peculiarity of this system.

D. 331 state and the hierarchy and its generalizations

The edge states and the partition function for the 331 state are, by now, easily obtained. There are zero modes in the twisted sector, as for the Pfaffian, but there are two types of fermions (particles and antiparticles), as in the HR state. Similar selection rules governing even- and odd-fermion numbers apply as in the other cases. Accordingly, the partition function can be written using the Weyl (or chiral Dirac) characters:

$$\begin{aligned}
\chi^{\rm Weyl}_{0}(x) &= \frac{1}{2}\left[\prod_{n=0}^{\infty}(1+x^{n+1/2})^2+\prod_{n=0}^{\infty}(1-x^{n+1/2})^2\right], \\
\chi^{\rm Weyl}_{1/2}(x) &= \frac{1}{2}\left[\prod_{n=0}^{\infty}(1+x^{n+1/2})^2-\prod_{n=0}^{\infty}(1-x^{n+1/2})^2\right], \\
\chi^{\rm Weyl}_{1/8}(x) &= \frac{1}{2}x^{1/8}\left[\prod_{n=1}^{\infty}(1+x^{n})^2+\prod_{n=1}^{\infty}(1-x^{n})^2\right], \\
\chi^{\rm Weyl}_{9/8}(x) &= \frac{1}{2}x^{1/8}\left[\prod_{n=1}^{\infty}(1+x^{n})^2-\prod_{n=1}^{\infty}(1-x^{n})^2\right].
\end{aligned} \tag{4.18}$$

The characters entering the partition function are

$$\begin{aligned}
\chi^{331}_{r/q,\rm ev,untw}(x) &= \tfrac{1}{2}\,\chi^{\rm Weyl}_{0}(x)[\chi^{+}_{r/q}(x)+\chi^{-}_{r/q}(x)] \\
&\quad + \tfrac{1}{2}\,\chi^{\rm Weyl}_{1/2}(x)[\chi^{+}_{r/q}(x)-\chi^{-}_{r/q}(x)], \\
\chi^{331}_{r/q,\rm od,untw}(x) &= \tfrac{1}{2}\,\chi^{\rm Weyl}_{1/2}(x)[\chi^{+}_{r/q}(x)+\chi^{-}_{r/q}(x)]+\tfrac{1}{2}\,\chi^{\rm Weyl}_{0}(x) \\
&\quad \times[\chi^{+}_{r/q}(x)-\chi^{-}_{r/q}(x)],
\end{aligned}$$

$$\chi^{331}_{(r+1/2)/q,\mathrm{ev,tw}}(x)=\tfrac{1}{2}\chi^{\mathrm{Weyl}}_{1/8}(x)[\chi^{+}_{(r+1/2)/q}(x)+\chi^{-}_{(r+1/2)/q}(x)]+\tfrac{1}{2}\chi^{\mathrm{Weyl}}_{9/8}(x)[\chi^{+}_{(r+1/2)/q}(x)-\chi^{-}_{(r+1/2)/q}(x)],$$

$$\chi^{331}_{(r+1/2)/q,\mathrm{od,tw}}(x)=\tfrac{1}{2}\chi^{\mathrm{Weyl}}_{9/8}(x)[\chi^{+}_{(r+1/2)/q}(x)+\chi^{-}_{(r+1/2)/q}(x)]+\frac{1}{2}\chi^{\mathrm{Weyl}}_{1/8}(x)[\chi^{+}_{(r+1/2)/q}(x)-\chi^{-}_{(r+1/2)/q}(x)]. \tag{4.19}$$

The partition function is

$$\mathscr{Z}^{331}(x,\overline{x})=\sum_{r=0}^{q-1}[|\chi^{331}_{r/q,\mathrm{ev,\ untw}}(x)|^2+|\chi^{331}_{r/q,\mathrm{od,untw}}(x)|^2+2|\chi^{331}_{(r+1/2)/q,\mathrm{ev,tw}}(x)+\chi^{331}_{(r+1/2)/q,\mathrm{od,tw}}(x)|^2]. \tag{4.20}$$

Again the twisted terms have combined to form a simpler character,

$$\chi^{331}_{(r+1/2)/q,\mathrm{tw}}(x)=(\chi^{\mathrm{MW}}_{1/16}(x))^2\chi^{+}_{(r+1/2)/q}(x), \tag{4.21}$$

and there are two distinct sectors with this character. The equality of some characters of distinct sectors may also happen with the rational torus or Laughlin state characters, for which the characters obey $\chi^{\pm}_{r/q}(x)=\pm\chi^{\pm}_{(q-r)/q}(x)$.

By bosonization,[3] the Dirac (or Weyl) characters can be written, using the Jacobi triple product formula, in terms of characters for a chiral boson with $q=1$ (summed over charge sectors):

$$\chi^{\mathrm{Weyl}}_{0}(x)\pm\chi^{\mathrm{Weyl}}_{1/2}(x)\equiv\chi^{\pm}_{0/1}(x), \tag{4.22}$$

$$2(\chi^{\mathrm{MW}}_{1/16}(x))^2\equiv\chi^{+}_{(1/2)/1}(x), \tag{4.23}$$

and so the 331 partition functions can be written in the form of sums for two boson fields, which is described in detail below. As mentioned in Sec. II, the bosonized description of the field theories is closely related to the description of the bulk wave functions as two-component generalizations of the Laughlin states. The E's for the various sectors can be obtained in that description by an argument similar to that given in Sec. IV A for the Laughlin state. Since this includes a contribution from the fermions, as well as from the charge degrees of freedom, this holds out some hope that a derivation in the ''pairing'' representation of wave functions, which might also be applicable to the Pfaffian and HR states, should exist.

Here we will give without proof the general results for the hierarchy and its generalizations, restricting ourselves, for simplicity, to the case where the matrix G below is positive definite, with the 331 state as a special case. Physically, this is the case where all modes at the same edge propagate in the same direction. The other case has been discussed recently in Refs. 9 and 30. The bosonized field theory for the right-moving edge[2] can be formulated in terms of chiral boson fields φ_α, which have correlators (in imaginary time)

$$\langle\varphi_\alpha(z)\varphi_\beta(0)\rangle=-\delta_{\alpha\beta}\ln z, \tag{4.24}$$

where α,β run from 1 to n, and here, in the hierarchy n, is the number of levels. The $\alpha=1$ component could be taken to be the density fluctuation field that we have used up to now. The others represent the internal, neutral degrees of freedom. In the composite fermion approach, n is the number of Landau levels for the fermions,[14] and the density mode is usually taken to be the sum of the φ_α. The chiral operators that are allowed to be used at this edge without affecting the other (which generate the chiral algebra) are of the form $e^{iv_\alpha\varphi_\alpha(z)}$ (we use the summation convention). The vectors $\mathbf{v}$, the components of which in the basis labeled by α are v_α, lie on a (Bravais) lattice Λ in n-dimensional Euclidean space, that is they take the form of integral linear combinations of n linearly independent vectors $\mathbf{e}_a$. Thus, the chiral algebra can be generated by $e^{\pm ie_{a\alpha}\varphi_\alpha(z)}$, $a=1,\ldots,n$. The scalar products of the $\mathbf{e}_a$ are $G_{ab}=\mathbf{e}_a\cdot\mathbf{e}_b$, which defines the Gram matrix of Λ; G is positive definite here, because we assumed Euclidean space. In the (generalized) hierarchy theory, G is a matrix of integers, and so Λ is an integral lattice. If $\mathbf{v}=v_a\mathbf{e}_a$, then we have $\mathbf{v}\cdot\mathbf{v}'=v_aG_{ab}v'_b$. Since the basis labeled by α (which is not an integral basis) is orthonormal, we find that the conformal weight of the chiral operators is $\mathbf{v}^2/2$, and so is either integral or half integral.

The possible shifts of charge or the other U(1) quantum numbers related to the components of φ_α from one edge to the other are described by similar operators that act on both edges simultaneously, as in the states considered earlier. The right-moving part of such an operator is of the form $e^{iw_\alpha\varphi_\alpha(z)}$, where $\mathbf{w}$ is a vector in the dual (or reciprocal) lattice Λ^* of Λ; the dual lattice is defined as the set of all vectors $\mathbf{w}$, such that $\mathbf{w}\cdot\mathbf{v}=$ an integer for all $\mathbf{v}\in\Lambda$. Clearly Λ is a sublattice of Λ^*. The right-moving conformal weight (or ground-state energy in the corresponding charge sector) of the operators $e^{iw_\alpha\varphi_\alpha(z)}$ is again $\mathbf{w}^2/2$, which is the same (modulo 1) as the statistical parameter $\theta/2\pi$ of the quasiparticles in the bulk of the same state, which are also labeled by vectors in Λ^* (the ''excitation lattice'' in the terminology of Ref. 14). The lattice Λ (the ''condensate lattice'') labels the combinations of quasiparticles that make up the possible order parameters. The relation of these to changes in the charges at one edge was mentioned in Ref. 14, and forms the basis for the results quoted here. If we view the lattices as additive groups, we find that Λ has index $\det G$ in Λ^*, and so the quotient group Λ^*/Λ is a finite group of $\det G$ elements. The extended charge sectors are labeled by the possible shifts modulo fields in the chiral algebra, that is, by the elements of Λ^*/Λ. These can be described by a set of vectors $\mathbf{w}_A$, $A=1,\ldots,\det G$, in Λ^*, one in each coset of Λ. The simplest case is the Laughlin states, where $n=1$, $G=(q)$, $\Lambda=\sqrt{q}\mathbf{Z}$, $\Lambda^*=\mathbf{Z}/\sqrt{q}$, and $\Lambda^*/\Lambda=\mathbf{Z}_q$, which is equivalent to the description in Sec. IV A. Another simple case is the integral quantum Hall effect, in which for n Landau levels, G is (in a convenient basis) the $n\times n$ identity matrix, so Λ is the n-dimensional simple (hyper-) cubic lattice, $\det G=1$, and $\Lambda^*=\Lambda$. Thus, in this case, all edge excitations are just electrons in the various Landau levels.

For the 331 states, the matrix G in the basis natural for the ground-state wave function in the form (2.24) is

$$G^{331}=\begin{pmatrix} q+1, & q-1\\ q-1, & q+1\end{pmatrix}. \tag{4.25}$$

The resulting lattice Λ embodies the projection rules by being distinct from an orthogonal direct sum of one-dimensional lattices. We note that there are $\det G = 4q$ extended charge sectors, as found earlier. Moreover, unlike the lattices for the hierarchy states, for the generalized hierarchy states, and the 331 state, in particular, the sublattice $(\Lambda^*)^\perp$ of operators that are neutral is not the same as the sublattice $\Lambda^\perp$ of Λ. The lowest-weight neutral operator that is in Λ^*, but not in Λ, represents the Fermi field ψ_σ of weight 1/2. Since it does not appear in Λ, it cannot be applied to one edge, but must be combined with a similar operator for the other edge, or with another Fermi field or a charged field at the same edge. This is a consequence of the projection rules that we saw using the fermionic form of wave functions; states differing only by one fermion at one edge do not both exist in the Hilbert space. In contrast, for the hierarchy, the projection rules place no restriction on the neutral operators that can be applied to one edge, since all vectors in $(\Lambda^*)^\perp$ are also in $\Lambda^\perp$.

We may now describe the partition function for the (generalized) hierarchy states, in the case where G is positive definite. The sectors are labeled by the $\det G$ cosets of Λ, and in each sector the chiral characters are sums over vectors in the coset, together with fluctuations in the φ_α. That is, define

$$\chi_A(x) \equiv \sum_{\mathbf{v}\in\Lambda} x^{(\mathbf{w}_A+\mathbf{v})^2/2} \prod_{m=1}^{\infty} (1-x^m)^{-n}. \tag{4.26}$$

The partition function is simply

$$\mathscr{Z}^\Lambda(x,\overline{x}) = \sum_{A=1}^{\det G} |\chi_A(x)|^2. \tag{4.27}$$

For the 331 states, this agrees with that derived by bosonization from $\mathscr{Z}^{331}$. It is interesting that, in the theory of the edge states, the relation of composite boson and composite fermion approaches maps exactly onto the usual $1+1$-dimensional bosonization (or its inverse, fermionization).

E. Orbifolds, chiral superalgebras, and modular transformations

For readers familiar with, or ready to learn about, CFT, we mention that the theories for the cylinder described in this section are examples of the construction known as "orbifolding." Definitions, results, and examples of orbifolds can be found in Refs. 29, 31, and 32. In brief, the general algebraic definition of an orbifold involves starting with a rational CFT with a chiral algebra $\mathscr{A}$ on which some finite group $\mathscr{G}$ acts as a symmetry. One then takes the subalgebra $\mathscr{A}_0$ that is invariant under $\mathscr{G}$ as the new chiral algebra. The representations of $\mathscr{A}$ will be representations of $\mathscr{A}_0$ also, but will, in general, be reducible; each irreducible component transforms as an irreducible representation of $\mathscr{G}$. In addition, there will be "twisted" representations of $\mathscr{A}_0$ that are not representations of $\mathscr{A}$. The same operations are applied to the left-moving chiral algebra $\overline{\mathscr{A}}$ and its representations. The (symmetric, diagonal) orbifold CFT then has a primary field for each representation of $\mathscr{A}_0$, which at the same time are primary for the isomorphic left-moving algebra $\overline{\mathscr{A}}_0$. The rule for combining left- and right-moving representations is that all fields must be invariant under the simultaneous action of $\mathscr{G}$ on left and right movers, and untwisted (twisted) fields must combine with untwisted (twisted).

The CFT's of all the paired states described in this section are examples of orbifolds with $\mathscr{G}=\mathbf{Z}_2$. The algebra $\mathscr{A}$ for the Pfaffian is generated by the fields $e^{\pm i\sqrt{q}\varphi}$, ψ, and contains the U(1) current algebra generated by $\partial\varphi$ and the Virasoro algebra for ψ as subalgebras. The primary fields are $e^{ir\varphi/\sqrt{q}}$, $r=0,1,\ldots,q-1$ (we suppress the left-moving operators for now). $\mathbf{Z}_2$ acts simultaneously on φ,ψ by $\varphi\rightarrow\varphi+\pi\sqrt{q}$ and $\psi\rightarrow-\psi$. The algebra $\mathscr{A}_0$ is generated by $\psi e^{\pm i\sqrt{q}\varphi}$, and the primary fields are $e^{ir\varphi/\sqrt{q}}$, $\psi e^{ir\varphi/\sqrt{q}}$, $r=0,1,\ldots,q-1$, which result from the splitting of the representations of $\mathscr{A}$ (the untwisted representations), together with the twisted representations $\sigma e^{i(r+1/2)\varphi/\sqrt{q}}$, $r=0,1,\ldots,q-1$, which include the spin field σ, which is the analog for the Majorana fermion of the twist field discussed in Sec. III. The states (or descendant fields) in these representations obey the "projection rules" found earlier, and the full description of the combination of left and right movers, and the resulting partition functions, can be done in agreement with the rules obtained from the wave functions in this section. Very similar descriptions work for the HR and 331 states. [For the 331 state, the orbifold that we find is that where $\psi_\uparrow$ transforms by $e^{i\pi}$, $\psi_\downarrow$ transforms by $e^{-i\pi}$. These factors are both equal to -1, but the point is that the factors written describe the way the phase of either field winds on taking it round the twist field; there is also an adjoint twist field around which they wind the reverse way. This behavior is required by the structure of the twisted states, which have definite pseudospin as well as charge quantum numbers. It is most easily understood in the bosonized representation $\varphi=\varphi_1$, $\psi_\uparrow=e^{i\varphi_2}$, in the same notation as in Sec. IV D. Then the symmetry is $\varphi_1\rightarrow\varphi_1+\pi\sqrt{q}$, $\varphi_2\rightarrow\varphi_2+\pi$. This orbifold leads back to the lattice described in Sec. IV D. In particular, the spin (or twist) field for ψ_σ is bosonized as $e^{i\varphi_2/2}$, which must appear in combination with some other charged fields, as for the other orbifolds and the generalized hierarchy theories in Sec. III D.] In all cases, the rationale for the structure is that the electron (or other fundamental charged particle) is represented in the edge theory by a field like $\psi e^{\pm i\sqrt{q}\varphi}$, which has fixed boundary conditions in all sectors, and all fields must be local with respect to it, just as in the bulk, all wave functions must be single valued functions of the electron coordinates.[7]

Our description of the orbifolds glossed over one aspect of the systems discussed here. The usual definition of chiral algebras assumes that all fields in the chiral algebras (both $\mathscr{A}$ and $\mathscr{A}_0$) have integral conformal weight. In our examples, ψ and ψ_σ that appear for the Pfaffian and 331 states have half-odd-integral weight, and for q odd, so does $e^{\pm i\sqrt{q}\varphi}$. Thus, $\mathscr{A}$ is actually a chiral superalgebra in these cases,[7] and so is $\mathscr{A}_0$ in some cases (and also for the algebra of the Laughlin state for q odd, and the generalized hierarchy states whenever applicable to electrons). We emphasize that, for our purposes, a superalgebra is one where some fields have half-odd-integral conformal weight, rather than one where some of the relations are anticommutators instead of commutators. In fact, to describe electrons, which are fermions, rather than the quantum Hall effect of charged bosons, the chiral algebra is always a superalgebra, except in the case of the HR states, due to the violation of the spin-statistics theorem there as discussed earlier.

The fact that the chiral algebra is sometimes a superalgebra has consequences for the modular transformation properties that we may expect for the partition functions calculated in this section. If $x=e^{2\pi i\tau}$, and $\mathrm{Im}\tau>0$ (τ should not be confused with earlier uses of the same symbol), then modular transformations act as

$$\tau\mapsto\frac{a\tau+b}{c\tau+d},\tag{4.28}$$

and the matrix

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\tag{4.29}$$

is a member of $SL(2,\mathbf{Z})$, the group of 2×2 integer matrices of determinant 1. [The group of modular transformations themselves is $SL(2,\mathbf{Z})/\{\pm I\}$.] The modular group is generated by the elements $T:\tau\rightarrow\tau+1$, represented by

$$T=\begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix},\tag{4.30}$$

and $S:\tau\rightarrow-1/\tau$, represented by

$$S=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}.\tag{4.31}$$

When the chiral algebra is strictly an algebra (i.e., not a superalgebra), then the partition functions will be modular invariant, if we modify the definition to include the factor $(x\bar{x})^{-c/24}$, where c is the central charge of the CFT (not the matrix element just above). Central charges are additive; the values of the central charge are $c=1$ for the Laughlin state, $1+1/2=3/2$ for the Pfaffian, $1-2=-1$ for the HR, $1+1=2$ for the 331 (all independent of the value of q), and n for the (generalized) hierarchy states. Modular invariance occurs for q even in the case of the Laughlin state, and q odd for the Pfaffian and 331 states, all of which describe the fractional quantum Hall effect of charged bosons, not electrons. When some of the fields that generate the chiral algebra have half-integral conformal weight, they will obey an antiperiodic boundary condition in the space direction, and the (modified) partition function cannot be invariant under the full modular group; the only boundary conditions that are invariant under the whole group are periodic around any cycle on the torus. In these cases, which as we have seen apply to all states considered here that can describe electrons, with the sole exception of the HR states with q even, we expect that our expressions are invariant only under the subgroup of the modular group that leaves the antiperiodic boundary condition on the electron field invariant. This subgroup is generated by the elements S and T^2 and can be shown to be isomorphic to $\Gamma_0(2)/\{\pm I\}$, where $\Gamma_0(2)$ is the subgroup of $SL(2,\mathbf{Z})$ consisting of matrices where the matrix element $c\equiv0 \pmod 2$. For the HR state, the situation is a little more subtle. Modular invariance may be expected when using the nonunitary CFT of Sec. IV B for the scalar fermions, in the cases with q even but requires modification of the partition function to include the factor $(-1)^F$ in the trace over states, where F is the total number of fermions. The same should apply for $\Gamma_0(2)/\{\pm I\}$ invariance for q odd.

We also mention here some isomorphisms of the chiral algebras of our systems to known algebras. (In this paragraph, c is the central charge, and N is not the number of particles.) For the Laughlin state (of bosons) at $\nu=1/2$, the fields $e^{\pm i\sqrt{2}\varphi}$, $\partial\varphi$ generate the SU(2) current (Kac-Moody) algebra of level 1. For the Laughlin state at $\nu=1/3$, we have[7] the $N=2$ superconformal algebra at $k=1$, generated by $e^{\pm i\sqrt{3}\varphi}$, $\partial\varphi$. For the Pfaffian state (of bosons) with $q=1$, the operators $e^{\pm i\sqrt{q}\varphi}$ are the bosonized representation of a Dirac field, or of a pair of Majorana fields $\psi_{\pm1}$, which together with the Majorana field $\psi=\psi_0$ forms a triplet of Majoranas. This $c=3/2$ theory contains an SU(2) current algebra of level 2, or equivalently an O(3) algebra of level 1, in which the currents are the bilinears $\psi_a\psi_b$, $a,b=\pm1,0$. This symmetry shows up, for example, in the degeneracies of the excited energy levels, as long as the velocities for φ and ψ are equal. The $3q=3$ sectors, even untwisted, odd untwisted, and twisted, correspond to primary fields that transform respectively as spins 0, 1, 1/2, under both the left- and right-moving SU(2). Moreover, the product $\psi_1\psi_0\psi_{-1}$ generates $N=1$ superconformal symmetry,[33] though this operator does not survive the projection to $\mathscr{A}_0$. Finally, the algebra $\mathscr{A}_0$ for the $\nu=1/2$ Pfaffian state is generated by $\psi e^{\pm i\sqrt{2}\varphi}$, which has weight 3/2, and the algebra can be recognized as superconformal $N=2$ at $k=2$.[33] In this case, the unprojected algebra $\mathscr{A}$ contains SU(2) level 1 and an SU(2) triplet of supercurrents $\psi e^{\pm i\sqrt{2}\varphi}$, $\psi\partial\varphi$, which generate an $N=3$ superconformal algebra.[33]

V. CONCLUSION

To conclude, we have found complete descriptions of the wave functions, the Hilbert spaces and the field theories of the edge states of the paired systems considered. The explicit wave functions are very appealing and make the enumeration of excited states in terms of elementary excitations straightforward. The combination of complete proofs of some results, and enumeration for low excited states in others, makes the correctness of those results not explicitly proven here almost certain. The results confirm the general prediction in Ref. 7 of a relation between bulk and edge properties. For the Pfaffian and HR states, this provides indirect evidence for the prediction of nonabelian statistics of the quasiparticles in the bulk of these states. The twist fields in the edge conformal field theories proposed here certainly have such properties when exchanged in space time at the edge (''monodromy''). For the 331 state, as for all generalized hierarchy states, the monodromy, and the statistics of the bulk quasiparticles, is abelian.

The explicit wave functions for the edge excitations, particularly for two edges on a cylinder, are reminiscent of results for integrable one-dimensional systems, especially those of the Calogero-Sutherland (CS) type[34] for which there are explicit, simple wave functions for the ground state and many excited energy eigenstates. Indeed, the similarity of the Laughlin state and the ground state of the CS model has often been remarked. In the limit $L\gg N$, the Laughlin state on the cylinder essentially becomes the Calogero-Sutherland ground state;[27] the coordinate transverse to the edges can be viewed as the canonical momentum. The edge excitations of the Laughlin state are in one-one correspondence with the excitations of the CS model, and the low-energy CFT of the latter is once again a Luttinger liquid. It is interesting to speculate that there might be some integrable one-dimensional Hamiltonians with long-range interactions, generalizing the CS model, for which the ground and excited states might be related in a similar way to the wave functions discussed in this paper. If so, then we expect the low-energy field theories of the one-dimensional models to be the $\mathbf{Z}_2$ orbifolds discussed in Sec. IV.

Finally, we note that the approach used here can be applied to other states for which the ground state is the zero-energy eigenstate of a suitable local Hamiltonian, as here. An example is another paired state, the permanent state,[7] which is a spin singlet, and is the densest zero-energy eigenstate of a certain three-body Hamiltonian.[35] The resulting theory is a $\mathbf{Z}_2$ orbifold containing spin-1/2 bosons of conformal weight 1/2 at the edge. Such a system, like the HR state, violates the spin-statistics connection, so that the edge field theory is not conformal. The corresponding nonpositive conformal field theory, the correlators of which reproduce the bulk wave functions, is the β-γ ghost system,[7] so the relation of bulk and edge theories is maintained.

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APPENDIX A: ZERO-ENERGY STATES FOR THE THREE-BODY AND HOLLOW-CORE HAMILTONIANS

In this Appendix, we will justify directly the general form of the zero-energy eigenstates of (2.10), and by extension its analog for $q>1$, and show, in particular, that they lead to the forms for the edge states in (2.12), (2.15). We then briefly address similar questions for the hollow-core Hamiltonian for which the HR state is the unique ground state, and corresponding issues on the cylinder.

The Hamiltonian (2.10), taken with Bose statistics for the particles so that the Pfaffian state with $q=1$ is a possible ground state, implies that the wave function of a zero-energy state vanishes whenever any three (or more) particles coincide. This implies that zero-energy states can be written in the form of the Vandermonde determinant $\Pi_{i<j}(z_i-z_j)$, times an antisymmetric function that, as a function of any two coordinates z_i,z_j, may have a simple pole at $z_i=z_j$, times the usual Gaussian factors. Such a state will have zero energy provided the antisymmetric function involved does not have a triple pole of the form

$$[(z_i-z_j)(z_j-z_k)(z_k-z_i)]^{-1},$$

as i,j,k approach one another, for any i,j,k. A form like

$$[(z_i-z_j)(z_i-z_k)]^{-1}$$

cannot appear either, because it is symmetric in j and k, while a form

$$(z_j-z_k)(z_i-z_j)^{-1}(z_i-z_k)^{-1}$$

could, but this can be rewritten as a difference of simple poles,

$$(z_i-z_j)^{-1}-(z_i-z_k)^{-1}.$$

So all possible functions can be written as linear combinations of the forms already given, where the singularities involve disjoint pairs of particles.

Without loss of generality, the general state can be taken to be a linear combination of states written by antisymmetrizing a function obtained by dividing the particles into pairs, writing an odd factor for each pair and symmetrizing over exchange of pairs. These conditions are of course sufficient, but not necessary for the final antisymmetrization over all particles to be nonvanishing. That is, neglecting the omnipresent factor $\Pi(z_i-z_j)\exp(-\frac{1}{4}\Sigma|z_i|^2)$, we must have

$$\sum_{\sigma\in S_N}\operatorname{sgn}\sigma\frac{\Sigma_{\tau\in S_{N/2}}\Pi_{k=1}^{N/2}f_k(z_{\sigma[2\tau(k)-1]},z_{\sigma[2\tau(k)]})}{(z_{\sigma(1)}-z_{\sigma(2)})\cdots(z_{\sigma(N-1)}-z_{\sigma(N)})},\tag{A1}$$

where the f_k are symmetric polynomials in two variables. For N odd, we can write a similar form with $k=1,\ldots,(N-1)/2$, and include for the unpaired particle an arbitrary polynomial factor $f_0(z_{\sigma(N)})$.

A convenient way to describe the symmetric functions in two variables z_1,z_2, is the following. We know that symmetric functions can be written as sums of products of the sums of powers $s_n^{(2)}=z_1^n+z_2^n$. We will separate the symmetric functions that vanish at $z_1=z_2$ by writing the disjoint sets of functions,

$$A_m=\{(z_1-z_2)^{2m}s_n^{(2)}:n=0,1,2,\ldots\},\tag{A2}$$

for $m=0,1,2,\ldots$. We claim that the full set of symmetric functions in two variables is spanned by linear combinations of the polynomials in the set

$$\cup_{m=0}^{\infty}A_m\tag{A3}$$

(there is no need to take products of these functions). This can be shown by induction from the fact that products of sums of powers span the symmetric polynomials, together with the identities

$$s_{n_1}^{(2)}s_{n_2}^{(2)}=2s_{n_1+n_2}^{(2)}-(z_1^{n_1}-z_2^{n_1})(z_1^{n_2}-z_2^{n_2}),\tag{A4}$$

$$z_1^n-z_2^n=(z_1-z_2)(z_1^{n-1}+z_1^{n-2}z_2+\cdots+z_2^{n-1}),\tag{A5}$$

which (by induction on the order) express a product of elements of A_0 as a linear combination of elements of A_0, A_1, ...

Now each f_k in (1.1) can be chosen to be an element of $\cup_{m=0}^{\infty}A_m$. If f_k is an element $s_n^{(2)}$ of A_0, we will try to pull outside the sum on permutations σ the corresponding sum of powers in all N coordinates, s_n [see Eq. (2.6)]; this will leave behind terms with fewer $f_k\in A_0$. Repeating this procedure, eventually all f_k's remaining inside the sum will be in $\cup_{m=1}^{\infty}A_m$ and we will have finished. f_k that are in A_m $(m\geqslant 1)$ contain $(z_{\sigma[2\tau(k)-1]}-z_{\sigma(2\tau(k))})^2$, which cancels a factor in the denominator, so these particles are unpaired in this term and the wave function will be a linear combination of the forms (2.12).

For the basic (untwisted) sector of edge states, we can consider N large and most $f_k = 1$, though this is not necessary and the results below are valid for all wave functions of the stated form. Then we observe that if $f_1 = s_n^{(2)} \in A_0$, then

$$\sum_{\tau \in S_{N/2}} \prod_{k=1}^{N/2} f_k \propto s_n \sum_{\tau \in S_{N/2}} \prod_{k=2}^{N/2} f_k - \sum_{\tau \in S_{N/2}} \sum_{k=2}^{N/2} \hat{f}_k \prod_{k'=2, k' \neq k}^{N/2} f_{k'}, \tag{A6}$$

where for $k = 2,3, \ldots,$

$$\hat{f}_k = \begin{cases} 0 & \text{if } f_k = 1 \\ f_k s_n^{(2)} & \text{if } f_k \neq 1, \end{cases} \tag{A7}$$

and s_n can be taken outside the sum on permutations σ. The functions $f_k s_n^{(2)}$ can then be reduced using the identity (A4), and all the terms in the many-particle state are now of the form of symmetric polynomials in N variables times antisymmetric functions with fewer f_k that are members of A_0 and $\neq 1$. Eventually, all f_k are either 1 or are $\in A_m$ $(m \geq 1)$, and these states are linear combinations of the states in the text. Similar methods work for N odd. Thus, we have shown that all zero-energy states are linear combinations of symmetric polynomials times the form in (2.12).

To obtain the twisted sector, we can replace f_k by $f_k s_1^{(2)}$ in the above proof, leaving the $s_1^{(2)}$ factors intact inside the sum on σ and τ at each step. Of course, in a finite system, our proof shows that these states can be expressed as combinations of the others, but to study the Hilbert spaces of edge states, we take $N \to \infty$ before the number of $f_k \neq 1$ becomes large, and thus we obtain two different sectors in this limit. Similar arguments apply if it is desired to include any other factor in every f_k in the state.

The hollow-core Hamiltonian[10] requires that zero-energy states have no pairs of particles with relative angular momentum $q-1$. (Another way to say this, which is useful in other geometries, is in terms of the order of vanishing of the functions.) We recall that the relative angular momentum of a pair, say 1,2, is defined by expressing the wave function in the form (neglecting the Gaussian factor, and the spin labels if any)

$$\Psi(z_1, z_2, z_3, \ldots, z_N) = \sum_{m,n=0}^{\infty} (z_1 - z_2)^m (z_1 + z_2)^n \times \Psi_{mn}(z_3, \ldots, z_N), \tag{A8}$$

in which each term in the sum is an eigenstate of relative angular momentum of 1 and 2 of eigenvalue m. The densest zero-energy state of the hollow-core Hamiltonian occurs at filling factor $\nu = 1/q$. The largest q for which the pairing in which we are interested can occur is $q = 2$. For $q > 2$, zero-energy states can be obtained from those for $q = 2$ by multiplying by the Vandermonde determinant. For $q = 2$, the wave functions are required to be totally antisymmetric when the spin states are included (see Sec. II C). For fixed spins of the N particles, the wave functions can be written as $\Pi_{i<j}(z_i - z_j)^2$ times a meromorphic function, and the meromorphic function must be antisymmetric among particles of the same spin. For zero-energy states this function must have, as any two particles come to the same point, either a double pole, with zero residue, or be analytic. Because of antisymmetry, double poles can appear only for opposite spin particles. All antisymmetric functions can be obtained by antisymmetrization of functions of indefinite symmetry, though we may as well omit functions that would vanish on antisymmetrization. If a double pole is present for a pair i,j, then it cannot be present for any other pair of the form i,k or j,k. This is because the Vandermonde squared contains the factors

$$(z_i - z_j)^2 (z_i - z_k)^2 (z_j - z_k)^2 = (z_i - z_j)^2 \{[\tfrac{1}{2}(z_i + z_j) - z_k]^2 - \tfrac{1}{4}(z_i - z_j)^2\}^2, \tag{A9}$$

the expansion of which contributes only even powers to the relative angular momentum of i and j. In view of the double pole in $z_i - z_j$, the whole wave function is a zero-energy eigenstate, provided there is not a single or double pole in $z_i - z_k$ or $z_j - z_k$, for any k. Therefore, all pairing factors $(z_i - z_j)^{-2}$ must contain distinct pairs. The unantisymmetrized function can thus be written as a product of pair factors for as many opposite spin pairs as possible, times functions $f_k(z_i^{\uparrow}, z_j^{\downarrow})$ of the paired coordinates that can be taken either symmetric or antisymmetric, and must be either nonvanishing at $z_i^{\uparrow} = z_j^{\downarrow}$, or vanish at least as fast as $(z_i^{\uparrow} - z_j^{\downarrow})^2$, so as not to spoil the zero-energy property. The zero-energy wave functions thus have, without loss of generality, a form similar to (A1). We now try to pull any one of the f_k that does not vanish at $z_i^{\uparrow} = z_j^{\downarrow}$ outside the sum on permutations by the same procedure as for the Pfaffian, using (A6). Use of (A4),(A5) then shows that the resulting functions still obey the zero-energy property. The procedure can then be repeated until linear combinations of the form (2.20) are reached. We conclude that the wave functions in the form (2.20) span all the zero-energy states, in the untwisted sector. Similar arguments apply to the twisted sector, to $N_{\uparrow} \neq N_{\downarrow}$, and to combinations of these.

Finally, we comment on zero-energy states on the cylinder. On replacing z_i by Z_i (see Sec. IV), we see that in (A1) f_k must still be symmetric, but may contain negative powers of Z_i. [The pairing factors $(Z_i - Z_j)^{-1}$ can be left unchanged without loss of generality.] We extend the definition of the sets of symmetric functions (A2) by allowing the exponents n in the symmetric polynomials $s_n^{(2)}$ to be negative as well as positive or zero, while m is still non-negative; we claim that these span all symmetric holomorphic functions in two variables on the cylinder. (A4),(A5) apply unchanged to all integral values of n_1, n_2, although (A5) becomes an infinite series. The proof then works as before, by pulling sums of (positive or negative) powers outside the sum on permutations. The HR case works similarly.

APPENDIX B: LINEAR INDEPENDENCE FOR SMALL ΔM IN THE PFAFFIAN AND HR CASES

In this appendix, we construct and verify the linear independence of the states in the untwisted, even N sector for all $\Delta M \leq 8$ for the Pfaffian and $\Delta M \leq 6$ for the HR state. We use a different basis from that derived in Appendix A. We first return to the two-quasihole states for the Pfaffian. Due to the symmetry of exchanging w_1 and w_2, they may be expanded in the form

$$\Psi(z_1,\ldots,z_N;w_1,w_2)=\sum_{m=0}^{N/2}\sum_{n=0}^{m}\Psi_{mn}(z_1,\ldots,z_N)\times(w_1^n w_2^{m-n}+w_2^n w_1^{m-n}), \quad \text{(B1)}$$

where all the Ψ_{mn} are linearly independent. This may be interpreted as saying that the quasiholes behave as two bosons, which may each occupy any one of $N/2+1$ states. This does not, however, mean that the quasiholes are bosons in general, which would contradict the assertion that they obey nonabelian statistics.[35] To obtain the expansion, we first expand the numerator in (2.11) inside the Pfaffian, i.e., for a fixed choice of pairs, described by the permutation σ [each pairing is obtained from $2^{N/2}(N/2)!$ different σ's]. For each pair $\sigma(2k-1)$, $\sigma(2k)$ the factor $[(z_{\sigma(2k-1)}-w_1)(z_{\sigma(2k)}-w_2)+(w_1\leftrightarrow w_2)]$ will contribute $z_{\sigma(2k-1)}z_{\sigma(2k)}$, $(z_{\sigma(2k-1)}+z_{\sigma(2k)})$ or a constant to an expansion coefficient Ψ_{mn}. This observation suggests the use of an alternative basis for the space of edge states spanned by the Ψ_{mn}, defined by

$$\Phi_{\Delta M,s}(z_1,\ldots,z_N)=\frac{1}{2^{N/2}(N/2)!}\sum_{\sigma\in S_N}\frac{\mathrm{sgn}\sigma}{(z_{\sigma(1)}-z_{\sigma(2)})\cdots(z_{\sigma(N-1)}-z_{\sigma(N)})}\{[(z_{\sigma(1)}z_{\sigma(2)})^{m_1}\cdots(z_{\sigma(N-1)}z_{\sigma(N)})^{m_{N/2}}\times(z_{\sigma(1)}+z_{\sigma(2)})^{n_1}\cdots(z_{\sigma(N-1)}+z_{\sigma(N)})^{n_{N/2}}]\}\prod_{i<j}(z_i-z_j)^q\exp\left[-\tfrac{1}{4}\sum|z_i|^2\right], \quad \text{(B2)}$$

where $\Delta M=M-M_0$ is again the difference between the total angular momentum M of the edge state and the angular momentum of the ground state M_0. The expression in curly brackets is defined as the sum over permutations of $N/2$ pairs:

$$\{[(z_{\sigma(1)}z_{\sigma(2)})^{m_1}\cdots(z_{\sigma(N-1)}z_{\sigma(N)})^{m_{N/2}}(z_{\sigma(1)}+z_{\sigma(2)})^{n_1}\cdots(z_{\sigma(N-1)}+z_{\sigma(N)})^{n_{N/2}}]\}=\mathcal{N}^{-1}\sum_{\tau\in S_{N/2}}(z_{\sigma(1)}z_{\sigma(2)})^{m_{\tau(1)}}\cdots(z_{\sigma(N-1)}z_{\sigma(N)})^{m_{\tau(N/2)}}(z_{\sigma(1)}+z_{\sigma(2)})^{n_{\tau(1)}}\cdots(z_{\sigma(N-1)}+z_{\sigma(N)})^{n_{\tau(N/2)}}, \quad \text{(B3)}$$

which makes this expression invariant under permutations of the pairs, and under permutations $n_\alpha, m_\alpha \mapsto n_{\tau'(\alpha)}, m_{\tau'(\alpha)}$. $\mathcal{N}$ is the number of permutations in $S_{N/2}$ that leave the sequence of pairs n_α, m_α, $\alpha=1,\ldots,N/2$ invariant. In the states $\Phi_{\Delta M,s}$ the numbers n_α, m_α are defined to be 0 or 1, such that $n_\alpha+m_\alpha\leqslant 1$, and $\Sigma_{\alpha=1}^{N/2}m_\alpha=\Delta M-s$, $\Sigma_{\alpha=1}^{N/2}n_\alpha=2s-\Delta M$. With these restrictions, there is clearly just one distinct polynomial of the form (B3) for each s, which will be denoted $P_{\Delta M,s}$, and we see that $s\leqslant\Delta M\leqslant 2s$, $\Delta M-s\leqslant N/2$, $2s-\Delta M\leqslant N/2$, and $s\leqslant N/2$. Comparing $\Phi_{\Delta M,s}$ and Ψ_{mn}, we see that $\Delta M=N-m$.

From (B2), it is easy to calculate how many edge states of fixed ΔM the expansion of two quasiholes gives, as $N\to\infty$. There are $1+\Delta M/2$ linearly independent states for ΔM even and $(\Delta M+1)/2$ for ΔM odd. But for fixed ΔM,

$$\sum_{s\geqslant\Delta M/2}^{\Delta M}P_{\Delta M,s}=e_{\Delta M}, \quad \text{(B4)}$$

where $e_{\Delta M}$ is an elementary symmetric polynomial, independent of the permutation σ, which can, therefore, be brought outside the sum on permutations as a multiplicative factor. This arises because bringing the two quasiholes to the same position, $w_1=w_2$, produces a single Laughlin quasihole. The remaining edge states, which span spaces of dimensions $\Delta M/2$ for ΔM even and $(\Delta M-1)/2$ for ΔM odd, require nontrivial factors inside the sum over permutations and represent degrees of freedom that are not simply density fluctuations at the edge. At each ΔM, the number of such states coincides with the number of states with two fermions added to the ground state and the same ΔM in the Majorana field theory.

Further expansions of the states with more than two quasiholes should generate, besides further symmetric polynomial factors, all even-fermion-number excitations. Wen[18] has demonstrated numerically for (2.10) for up to $N=10$ particles that the number of low ΔM zero-energy states coincides with that in the Majorana field theory. We will go a little further analytically, for arbitrary N. States with $2n$ quasiholes, $n>1$, at positions $\{w_1,\ldots,w_{2n}\}$ will, in place of the factor

$$\prod_{k=1}^{N/2}[(z_{\sigma(2k-1)}-w_1)(z_{\sigma(2k)}-w_2)+(w_1\leftrightarrow w_2)] \quad \text{(B5)}$$

inside the sum on permutations σ in (2.11), have a *product* of such factors, each involving a distinct pair of w's. Thus, the degree of the wave function will be $N_\phi=q(N-1)-1+n$. There are $2^n n!$ distinct ways to associate the w's in pairs, but only 2^{n-1} of the resulting electron wave functions are linearly independent (as functions of the z_i's for fixed w's) for $n>1$.[35] Provided that they are degenerate in energy, which is true by inspection for the appropriate three-body Hamiltonian, the fact that this number is >1 is the basis for nonabelian statistics. When the quasiholes are exchanged adiabatically, the usual Berry phase is replaced by a matrix acting in this space of degenerate quasihole states; however, this has not yet been explicitly demonstrated in this or any other example (see Ref. 35). Here we are interested to see what edge excitations we can obtain by expanding these states. Evidently expanding in powers of $w_1, \ldots, w_{2n}$ will generate the general polynomials (B3) inside the sum on permutations, without restrictions on the m_α's and n_α's. We will then have to take into account the linear relations among some of these states, corresponding to those discussed in Ref. 35 for $n=2$.

The linear relations among some of the states are obtained from the following general identity. For any set of complex numbers a_i, $i=1,\ldots,P$, $P>2$ even,

$$\mathrm{Pf}(a_i-a_j)=0. \tag{B6}$$

This follows because the Pfaffian is the square root of a determinant in which any three rows or columns obey a linear relation. All cases with $P>4$ can be viewed as applications of the identity for $P=4$. One consequence of the identity is that when we insert the expression

$$\{[(z_{\sigma(1)}-z_{\sigma(2)})^2(z_{\sigma(3)}-z_{\sigma(4)})^2]\} \tag{B7}$$

[defined in analogy with (B3)], or similar expressions, into the Pfaffian the resulting expression vanishes. (B7) can be expanded in the basis (B3) and this gives a linear relation among the states obtained. Therefore, for $\Delta M=4$, we ''lose'' one state. For $\Delta M=5$, the expressions are

$$\{[(z_{\sigma(1)}-z_{\sigma(2)})^2(z_{\sigma(3)}-z_{\sigma(4)})^2(z_{\sigma(5)}+z_{\sigma(6)})]\}, \tag{B8}$$

$$\{[(z_{\sigma(1)}-z_{\sigma(2)})^2(z_{\sigma(3)}-z_{\sigma(4)})^2]\}e_1. \tag{B9}$$

Notice that in the first expression the distinct pair $\sigma(5)$, $\sigma(6)$ is introduced, which does not affect the vanishing, which is due to the summation over permutations of the other four particles. In the second, we have simply multiplied the insertion (B7) by an elementary symmetric polynomial, which is linearly independent of the other expression. From here on, we will omit the expressions which are products of the ones valid for lower ΔM and symmetric polynomials. Then, for $\Delta M=6$, the linearly independent expressions producing linear relations among the states are

$$\{[(z_{\sigma(1)}-z_{\sigma(2)})^2(z_{\sigma(3)}-z_{\sigma(4)})^2(z_{\sigma(5)}-z_{\sigma(6)})^2]\}, \tag{B10}$$

$$\{[(z_{\sigma(1)}-z_{\sigma(2)})^2(z_{\sigma(3)}-z_{\sigma(4)})^2(z_{\sigma(5)}+z_{\sigma(6)})^2]\}, \tag{B11}$$

$$\{[(z_{\sigma(1)}-z_{\sigma(2)})^2(z_{\sigma(3)}-z_{\sigma(4)})^2(z_{\sigma(5)}+z_{\sigma(6)})\times(z_{\sigma(7)}+z_{\sigma(8)})]\}, \tag{B12}$$

and

$$\{[(z_{\sigma(1)}-z_{\sigma(2)})^2(z_{\sigma(1)}+z_{\sigma(2)})(z_{\sigma(3)}-z_{\sigma(4)})^2\times(z_{\sigma(3)}+z_{\sigma(4)})]\}. \tag{B13}$$

The last expression uses (B6), with $a_i=z_i^2$. For $\Delta M=7$,

$$\{[(z_{\sigma(1)}-z_{\sigma(2)})^2(z_{\sigma(3)}-z_{\sigma(4)})^2(z_{\sigma(5)}-z_{\sigma(6)})^2\times(z_{\sigma(7)}+z_{\sigma(8)})]\}, \tag{B14}$$

$$\{[(z_{\sigma(1)}-z_{\sigma(2)})^2(z_{\sigma(3)}-z_{\sigma(4)})^2(z_{\sigma(5)}+z_{\sigma(6)})^3]\}, \tag{B15}$$

$$\{[(z_{\sigma(1)}-z_{\sigma(2)})^2(z_{\sigma(3)}-z_{\sigma(4)})^2(z_{\sigma(5)}+z_{\sigma(6)})^2\times(z_{\sigma(7)}+z_{\sigma(8)})]\}, \tag{B16}$$

$$\{[(z_{\sigma(1)}-z_{\sigma(2)})^2(z_{\sigma(3)}-z_{\sigma(4)})^2(z_{\sigma(5)}+z_{\sigma(6)})(z_{\sigma(7)}+z_{\sigma(8)})\times(z_{\sigma(9)}+z_{\sigma(10)})]\}, \tag{B17}$$

and

$$\{[(z_{\sigma(1)}-z_{\sigma(2)})^2(z_{\sigma(1)}+z_{\sigma(2)})(z_{\sigma(3)}-z_{\sigma(4)})^2(z_{\sigma(3)}+z_{\sigma(4)})\times(z_{\sigma(5)}+z_{\sigma(6)})]\}. \tag{B18}$$

All these expressions are linearly independent of each other.

Since symmetric polynomials can always be multiplied into zero-energy states to obtain another zero-energy state, it is convenient to decompose all states into a product of a symmetric polynomial and another part that is linearly independent of symmetric polynomials. The latter represents excitations that are not density fluctuations at the edge. The full Hilbert space of edge excitations thus can be written as a tensor product of a bosonic Fock space of density excitations, as described earlier, and another space of independent excitations. Since the ΔM's of the excitations add, the dimension of the full space at any ΔM can be obtained by convoluting those of the two factor spaces. It is easy to calculate the dimensions obtained for the latter space by building its states up from products of the $P_{\Delta M,s}$ (rendered linearly independent of $e_{\Delta M}$) and then subtracting the number of linear relations just obtained. For $\Delta M\leqslant 7$, we find that the linear relations eliminate all the states obtained from more than two quasiholes. Thus, we find that for $\Delta M\leqslant 7$ the edge excitations of the Pfaffian state exactly match those in the chiral boson times Majorana fermion system, in the fermion number zero or two sectors.

In principle, it is possible to find the number of the edge states at arbitrarily high ΔM, by deriving these expressions in a systematic way. First, we list all polynomials of degree ΔM of the form

$$\{[(z_{\sigma(1)}-z_{\sigma(2)})^2(z_{\sigma(3)}-z_{\sigma(4)})^2\cdots]\}, \tag{B19}$$

where dots denote additional squared differences or sums that multiply the first two terms. Then we take the space of expressions that vanish when inserted in the Pfaffian at lower ΔM, multiplied with all possible products of symmetric polynomials that make the degree of the expression ΔM. We expand these in the terms of the form (B19). Some terms in the expansion are obviously zero when inserted in the Pfaffian; then the rest must give zero too. Fortunately for low momenta ($\Delta M\leqslant 7$) each wave function of the form (B19) is zero and that leaves us to prove only that the rest of the wave functions are nonzero and linearly independent. This can be done by taking pairwise limits $z_1\to z_2$, etc., of particle coordinates in the Pfaffian alone (i.e., without Laughlin-Jastrow factor), whenever these are singular, and examining the linear independence of the resulting functions of the remaining variables, that are the residues of these poles.

We will just state that the number of the edge states that we found at $\Delta M=8$ implies that the four fermion state $(\tfrac{1}{2},1\tfrac{1}{2},2\tfrac{1}{2},3\tfrac{1}{2})$ is present in the spectrum. So the numbers found are those given in the table in Sec. II B.

We now turn to the HR state. States with any even number, $2n$, of quasiholes are zero-energy states and can serve as generating functions for edge states. As in the Pfaffian case, they suggest an overcomplete basis of the states in which polynomials of the (B3) type are inserted. For more than two quasiholes, linear dependences arise when more than one factor like $(z_i^{\uparrow}-z_j^{\downarrow})^2$ cancels a similar factor in the denominator; this time, the factors in the denominator are themselves squared, and the sum over permutations of the relevant particles gives a determinant, not a Pfaffian, so the identity that replaces (B6) is simply that

$$\det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & & \\ & \vdots & & \\ 1 & 1 & & 1 \end{pmatrix} = 0. \tag{B20}$$

The linear dependencies that we mentioned in the Pfaffian case with $(i\uparrow\sigma(i)\downarrow)$ pairs instead of $[\sigma(i)\sigma(i+1)]$ are then valid here, except the last one in the $\Delta M=6$ and $\Delta M=7$ cases. Namely, if we insert any of these in the HR state (2.18), the sum over permutations will produce zero. Therefore, the numbers of nontrivial edge states, that is, the states without symmetric polynomials that we found so far in the Haldane-Rezayi case are

ΔM	1	2	3	4	5	6	7
dim	0	1	1	2	2	4	4

All these states are singlets, because they consist of singlet pairs.

The complete spectrum of edge states of the HR system should contain also nonzero spin states. At present we have no quasiholelike generating functions for these, but we can still obtain the edge states by writing down suitable functions directly. Each such state is formed when some of spin-singlet pairs in the ground state are excited into triplet states of zero energy. The broadest class of edge states made in this way, with $S_z=0$, has this polynomial in the numerator of the term with a fixed arrangement of coordinates in pairs:

$$\{[(z_1^{\uparrow}-z_{\sigma(1)}^{\downarrow})^{n_1}\cdots(z_{N/2}^{\uparrow}-z_{\sigma(N/2)}^{\downarrow})^{n_{N/2}} \times(z_1^{\uparrow}+z_{\sigma(1)}^{\downarrow})^{m_1}\cdots(z_{N/2}^{\uparrow}+z_{\sigma(N/2)}^{\downarrow})^{m_{N/2}}]\}, \tag{B21}$$

where no $n_i=1$, and at least one of $n_i's$ is an odd number.

For $\Delta M=3$, we have only $\{[(z_i^{\uparrow}-z_{\sigma(i)}^{\downarrow})^3]\}$, which corresponds to $S=1,S_z=0$, i.e., belongs to the permutation group representation $(2^{N/2-1}1^2)$. Similarly the two remaining states of the triplet can be constructed. For example, the $S_z=-1$ state is

$$\Psi_{S_z=-1}(z_1^{\uparrow},\ldots,z_{N/2+1}^{\downarrow}) = \sum_{\sigma\in S_{N/2+1}} \mathrm{sgn}\,\sigma(z_{\sigma(1)}^{\downarrow}-z_{\sigma(2)}^{\downarrow}) \times\frac{1}{(z_1^{\uparrow}-z_{\sigma(3)}^{\downarrow})^2\cdots(z_{N/2-1}^{\uparrow}-z_{\sigma(N/2+1)}^{\downarrow})^2} \times\prod_{i<j}(z_i-z_j)^q\exp\left[-\tfrac{1}{4}\sum|z_i|^2\right]. \tag{B22}$$

Note that these functions are simply related to the general form of (2.20).

Then we proceed counting only linearly independent states that do not contain symmetric polynomial factors. For low angular momenta, we get these numbers:

ΔM	1	2	3	4	5	6
dim	0	0	1	1	2	2

where each state in the table is the $S_z=0$ element of a triplet, $S=1$. The total number of low-lying fermion edge states in the untwisted sector of the HR state is then as given in the table in Sec. II C. Finally, we note that the sets of linearly independent functions obtained here for each ΔM can be rearranged into the general form derived in Appendix A.

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