Synchronization of noisy delayed feedback systems with delayed coupling

Nikola Burić,[1,*] Kristina Todorović, and Nebojša Vasović
[1]Institute of Physics, University of Beograd, PO Box 68, 11080 Beograd-Zemun, Serbia
[2]Department of Physics and Mathematics, Faculty of Pharmacy, University of Belgrade, Vojvode Stepe 450, Belgrade, Serbia
[3]Department of Applied Mathematics, Faculty of Mining and Geology, University of Belgrade, PO Box 162, Belgrade, Serbia.
(Received 16 August 2006; revised manuscript received 26 November 2006; published 16 February 2007)

Synchronization of delayed coupled and stochastically perturbed systems with delayed nonlinear feedback is studied, using as an example circular chains of three and four delayed coupled Ikeda oscillators. It is proved that in the case of multiplicative noise the exact synchronization in the mean occurs for sufficiently large coupling, and an analytic estimate of the sufficient coupling is given. The sufficiency condition is compared with numerical computations, and typical effects of noise on the exact and some generalized types of synchronization are illustrated.

DOI: 10.1103/PhysRevE.75.026209 PACS number(s): 05.45.Xt, 02.30.Ks

I. INTRODUCTION

Dynamical synchronization has been an important research topic for a long time (an overview and an extensive list of references can be found in Ref. [1]). Our goal in this paper is to study stability of synchronization with respect to random perturbations in an important class of models that correspond to a system of coupled feedback loops with some special properties. In order to have models of such systems which are relevant in real applications it is often necessary to take into account time-lags in the feedback and a finite duration of the transmission of information between each feedback unit. It is also natural to assume that many features of real systems have been neglected in making the model, but that they can be retaken into the consideration as different types of random perturbations of the deterministic model. We thus arrive at a model given by a stochastic dynamical system with two different characteristic time delays.

Two properties of delay-differential systems make the study of the influence of noise on such systems interesting from the theoretical point of view and nontrivial. First, a deterministic delay-differential system has a nonzero memory, i.e., it does not satisfy the Markov property, and addition of stochastic perturbations is not going to turn the dynamics into that of a Markov process. Consequently, some of the well-established methods which are valid for Markov processes, like Foker-Planck equations, cannot be used [2]. Second, a single nonlinear scalar deterministic delay-differential equation (DDE) with a single fixed time-lag τ gives an infinite dimensional dynamical system on the phase space $C(-\tau,0)$ of continuous functions on the interval $(-\tau,0)$ [3]. Large τ usually implies high-dimensional chaotic attractor, first studied in Ref. [4]. In fact, dynamical systems generated by a scalar DDE are hyperchaotic, i.e., they possess a chaotic attractor, such that there are more than one positive Lyapunov exponents for the restriction of the system on the attractor. Thus deterministic DDE can have quite chaotic dynamics, and it is not intuitively clear if a small noise will have any significant effect on such hyperchaotic evolution.

In this paper we shall report results of our study of the influence of different types of noise on a particularly important property of collective dynamics in a collection of bidirectionally coupled hyperchaotic feedback systems, i.e., on the stability of exact synchronization in such systems. Our aim is to use a well-known example of a hyperchaotic system generated by DDE in order to illustrate some analytical methods that can be used to study, and to present numerical indications of some interesting phenomena related to, the influence of noise on the synchronization. Thus, the choice of multiplicative or additive noise, and bidirectional or unidirectional coupling is not motivated and justified by a particular physical application of the model but by abstract interest in different possibilities.

DDE's with hyperchaotic dynamics often appear in applications, for example in biology [5], dynamics of lasers [6], and/or secure communication [7,8]. In this paper, we shall use, as the single deterministic feedback unit, the well-known example of scalar DDE's, the Ikeda model [9,10], for such values of the parameters and the feedback time lag that the single system is hyperchaotic. The Ikeda model

$$\dot{x}(t) = -x(t) + \mu \sin[x(t-\tau_1)], \tag{1}$$

is one of the first systems used to study the multidimensional chaos. It was originally introduced as a model of nonlinear optical resonators, but has also been used to model semiconductor laser with electro-optical feedback [8,11]. Our choice of the Ikeda model as the representative of hyperchaotic feedback units is motivated by the fact that various aspects of synchronization between Ikeda systems have been thoroughly studied. The total system is given by a system of Itô stochastic delay differential equations of the following form:

$$dx^i = f_i(x^i,(x^i)^{\tau_1})dt + g_i(x^i)dW_i + c[-2x^i + (x^{i+1})^{\tau_2} + (x^{i-1})^{\tau_2}]dt, \quad i=1,2,\ldots,N, \quad x^0 \equiv x^N, \quad x^{N+1} \equiv x^1, \tag{2}$$

where

$$x^{\tau}(t) \equiv x(t-\tau), \tag{3}$$

and f_i is the Ikeda system (1). We shall always assume that dW_i are stochastic increments of Wiener processes $\xi_i(t)$ with

*Electronic address: buric@phy.bg.ac.yu

zero mean and correlations $\langle \xi_i(t)\xi_j(t')\rangle = \delta_{i,j}\delta(t-t')$, which are added to the deterministic system either additively, in which case g_i are constants, or multiplicatively in which case we shall suppose that g_i are of the form $g_i = \text{const}\times x_i$.

Synchronization of two instantaneously coupled deterministic hyperchaotic systems in a master-slave configuration has been studied for the first time in Ref. [12]. In this paper, the hypechaotic dynamics of each unit is produced by a delayed feedback loop. Since then, the synchronization of hyperchaotic Ikeda systems has been studied by many. For example, Voss [13] has observed that in the master-slave configuration there is a type of synchronization such that the slaved system anticipates the state of the driver. Later, Shahverdiev [14], applied the same methods to two Ikeda systems in the master-slave configuration with a particular form of the unidirectional delayed coupling, and studied lag and anticipating synchronization. Anticipating synchronization was also studied numerically in (master-slave) two Lang-Kobayashi laser in Refs. [15,16]. Synchronization of three hyperchaotic bidirectionally delayed coupled Ikeda systems was studied in Ref. [17]. Zero lag synchronization has recently been demonstrated experimentally and numerically in a chain of three semiconductor lasers with long coupling delays [18]. Patterns of exact synchronization in a chain of arbitrary number of bidirectionally delayed coupled Ikeda systems were studied in Ref. [19]. Physically motivated sources of noise in the Ikeda system as a model of nonlinear optical media was discussed in Ref. [20]. The influence of additive or multiplicative noise on the bifurcation sequence leading to the hyperchaotic behavior of a single DDE system was also studied in Ref. [20]. The phenomenon of coherence resonance in optical feedback systems modelled by DDE, was studied for example in Refs. [21,22]. Stability of anticipating synchronization on the additive noise in unidirectionally coupled DDE models of semiconductor lasers was demonstrated numerically, for example, in Ref. [15].

Stability of synchronization in systems with noise involving DDE which are not necessarily hyperchaotic was studied analytically, for example, in the context of coupled realistic and formal neural networks. Liao and Mao [23] have initiated the study of stability in stochastic neural networks, and this was extended to stochastic neural networks with discrete time delays in Refs. [24,25]. Some analytical techniques relevant for delayed systems with noise have also been used in the study of coupled bistable systems with delays [2], and in noisy oscillators with delayed feedback [26].

The paper is organized as follows. In the next section we illustrate an analytical technique that can be applied to prove possibility of global asymptotic stability in the mean of the exact synchronization. Using a generalization of the Lyapunov-Krasovskii method to the SDDE's we prove that the exact synchronization in the system (4) is asymptotically stable in the mean if the coupling is sufficiently strong, and we provide a criterion for the sufficient value of the coupling. In Sec. III we present a selection of numerical results that illustrate the effects of multiplicative and additive noise on the synchronization properties for the chains with $N=3$ and $N=4$ units. We first numerically analyze and comment on the degree of overestimation of the sufficiency criterion for the global asymptotic stability in the mean of the exact synchronization. Then we present an analyses of the influence of noise in the situation that occurs for smaller values of the coupling, when the deterministic system with $N=3$ is bistable and has two attractors, one corresponding to the exact and the other to some generalized type of synchronization. Finally we shall present some numerical results concerning the influence of noise in the case of the chain with four units. The deterministic system with $N=4$ can have, for some fixed values of the parameters, the globally stable exact synchronization between ith and $i+2$ units, and some generalized type of synchronization between i and $i+1$. We shall illustrate the fact that in this case the exact of noise on the different types of synchronization between the nearest and next to the nearest neighbors is quite different. Finally, in Sec. IV we summarize and discuss the presented results, and indicate some directions for future research.

II. MEAN EXPONENTIAL STABILITY OF EXACT SYNCHRONIZATION

In this section we show that the exact synchronization in the mean between stochastically perturbed Ikeda systems with delayed diffusive interaction is possible, and globally asymptotically stable, for sufficiently large coupling constant. As relevant but sufficiently simple example, we shall consider a system consisting of three Ikeda units.

In this section we consider the case when each unit is influenced by the same multiplicative noise. We shall first analyze the systems with bidirectional coupling, and then briefly comment on the system in a master-slave configuration. Thus, the system is described by the following set of stochastic delay differential equations (SDDE):

$$\begin{aligned} dx^i(t) = \{&-x^i(t) + \mu\sin[x^i(t-\tau_1)] \\ &+ c[x^{i-1}(t-\tau_2) + x^{i+1}(t-\tau_2) - 2x^i(t)]\}dt \\ &+ x^i(t)\sqrt{2D}dW, \quad i=1,2,3, \quad x^0\equiv x^3, \quad x^4\equiv x^1, \end{aligned} \tag{4}$$

where dW, formally written as $dW=\xi(t)dt$, is the stochastic increment dW of the Wiener process $\xi(t)$ for which

$$E(\xi)=0,$$

$$E[\xi(t)\xi(t')] = \delta(t-t'), \tag{5}$$

where $E(\ )$ denotes the mean with respect to the stochastic process. The increments satisfy

$$E(dW)=0, \quad dWdW=dt. \tag{6}$$

To study the stability of the exact synchronization between the ith and the jth unit $(i,j=1,2,3)$ it is convenient to analyze the dynamics of the difference

$$\Delta^{i,j}(t) = x^i(t) - x^j(t). \tag{7}$$

In fact only $\Delta^{1,2}$ and $\Delta^{2,3}$ are important, and the dynamics of each of these two functions is given by a scalar SDDE of the same form

$$d\Delta = \{-(1+2c)\Delta(t) - c\Delta(t-\tau_2) + \sigma(t)\sin[\Delta(t-\tau_1)/2]\}dt + \Delta(t)\sqrt{2D}dW, \tag{8}$$

where $\Delta = \Delta^{1,2}, \Delta^{\tau_1} = \Delta^{1,2}(t-\tau_1), \Delta^{\tau_2} = \Delta^{1,2}(t-\tau_2)$ or $\Delta = \Delta^{2,3}, \Delta^{\tau_1} = \Delta^{2,3}(t-\tau_1), \Delta^{\tau_2} = \Delta^{2,3}(t-\tau_2)$. In the two cases the time dependent parameter $\sigma(t)$ is given by

$$\sigma(t) = 2\mu \cos\frac{(x^1)^{\tau_1} + (x^2)^{\tau_1}}{2} \quad \text{or } \sigma(t) = 2\mu \cos\frac{(x^2)^{\tau_1} + (x^3)^{\tau_1}}{2}. \tag{9}$$

Although, the time dependence of $\sigma(t)$ could be quite complicated, its absolute value is always bounded by 2μ.

In the case of the deterministic DDE global asymptotic stability of $\Delta(t)=0$ implies that the global attractor of the deterministic part of (4) satisfies $x^1 = x^2 = x^3$. Sufficient condition for the global asymptotic stability of $\Delta = 0$ in the case of DDE's can be found by applying a generalization of the Lyapunov first method, on the phase space given by continuous functions Δ defined on the interval $[-\tau, 0]$, where $\tau = \max\{\tau_1, \tau_2\}$, with the norm $||\Delta||^2 = \int_{-\tau}^{0}\Delta^2(\theta)d\theta$. Krasovskii [27] has found the general form of the Lyapunov functional for systems of deterministic DDE's with multiple delays, and Pyragas [12] was the first to applied the Krasovskii-Lyapunov functional to study the stability of synchronization. He studied a pair of Ikeda systems with unidirectional instantaneous coupling. Since then, the Lyapunov-Krasovskii functional has been applied by many to prove the possibility of synchronization in various systems of deterministic DDE's. A slight improvement of the sufficient condition usually obtained by the Lyapunov-Krasovskii functional, but a significant simplification of its derivation, is provided by using Razumikhin-type theorems and a Lyapunov functional of a simpler form [3]. In the case of SDDE's the global exponential stability is replaced by the analogous stability in the mean value with respect to the distribution given by the stochastic process. In what follows we shall first provide an argument, based on heuristic generalization of the Razumikhin-type theorems on a stochastically perturbed DDE, which suggests the sufficient condition that can be used to prove the exponential stability in the mean of $\Delta^2(t)$ for Eq. (8). Then we prove that this condition indeed guarantees the exponential stability in the mean of $\Delta^2(t)$.

Applying the Itô derivative on

$$L[\Delta(t)] = \Delta^2(t), \tag{10}$$

and using the formal notation dL/dt for convenience, the derivative of (10) along an orbit for which $|\Delta(t)| \geq |\Delta(t-\tau_k)|$, $k=1,2$ satisfies

$$\begin{aligned}
\frac{dL}{dt} &= 2\Delta(t)\left[-\Delta(t) + 2\mu\cos\left(\frac{x^1(t-\tau_1)+x^2(t-\tau_1)}{2}\right)\sin[\Delta(t-\tau_1)/2]\right] + 2c\Delta(t)[-\Delta(t-\tau_2) - 2\Delta(t)] + 2\Delta^2(t)\sqrt{2D}\xi(t) + \Delta^2(t)2D \\
&\leq -2\Delta^2(t) + 2\mu\Delta(t)\Delta(t-\tau_1) - 2c\Delta(t)\Delta(t-\tau_2) - 4c\Delta^2(t) + 2\Delta^2(t)\sqrt{2D}\xi(t) + \Delta^2(t)2D \\
&\leq 2[-1-2c+D+\sqrt{2D}\xi(t)]\Delta^2(t) + 2\mu|\Delta(t)||\Delta(t-\tau_1) + 2c|\Delta(t)||\Delta(t-\tau_2)| \\
&\leq -2[1+2c-D-\sqrt{2D}\xi(t)-\mu-c]\Delta^2(t),
\end{aligned} \tag{11}$$

where we used $|\Delta(t)| \geq |\Delta(t-\tau_k)|$, $k=1,2$.

Because of (6) the mean of the expression in the last line of (11) is equal to $2\mu + 2D - 2 - 2c$ and the previous argument suggests that if this expression is negative the system (8) should be stable in the mean. Indeed we now prove the following theorem.

Theorem 1. If the constants μ, D, and c satisfy

$$\mu + D - 1 - 1c < 0 \tag{12}$$

then the system (8) is exponentially stable in the mean value of $\Delta^2(t)$ (i.e., exponentially stable in mean square). Proof: See the Appendix.

Notice that the exponential stability in mean square implies exponential stability in mean (i.e., $E[\Delta(t)]$ [28]).

Finally, let us point out that the same method could be applied to SDDE that describe the model studied by Pyragas [12] with added multiplicative noise,

$$dx^1(t) = \{-x^1(t) + \mu\sin[x^1(t-\tau_1)]\}dt + x^1(t)\sqrt{2D}dW,$$

$$dx^2(t) = \{-x^2(t) + \mu\sin[x^2(t-\tau_1)] + c[x^1(t) - x^2(t)]\}dt + x^2(t)\sqrt{2D}dW, \tag{13}$$

or to the system with the term $c[x^1(t) - x^2(t)]$ replaced by $c[x^1(t-\tau) - x^2(t)]$.

The equations (13) describe two stochastically perturbed Ikeda systems with instantaneous (or delayed) coupling in the master-slave configuration. It is easily checked that the application of Razumikhin theorem with the functional (10) in this case gives the same estimate (12) as for the system (2).

A few remarks concerning Theorem 1, and especially the estimate (12), are in order.

First, the method of the proof can be applied to prove the possibility of exact synchronization in the mean whenever the equation for the differences $\Delta(t)$ can be approximately expressed in a closed form. Furthermore, the method could be extended to the cases when the equation for the differences $\Delta(t)$ contains terms that depend on $x_i(t)$ and for which there is no explicate bound. In this case, to prove the exitances of the exact synchronization for sufficiently large coupling, one must prove that $x_i(t)$ remain bounded all the time. Of course, in this case one does not get an explicit sufficiency condition.

Second, it is well known even in the case of deterministic DDE, that although the exitances of the Lyapunov-Krasovskii functional proves the possibility of synchronization, the estimate of the sufficiency condition that one gets by applying the method largely overestimates the numerical results. As we shall see in the next section, the situation appears to be the same with the estimate for the SDDE. In this sense, Theorem 1 should be understood more like a claim of possibility of exact synchronization in the mean, rather than an accurate estimate of the sufficiency conditions. We shall say more about this in the next section.

Third, the Theorem 1 does not say anything about a single realization or a path of the stochastic process given by (8). It claims that on the average the square of the difference Δ converges to zero exponentially fast. This implies exponential convergence to zero on the average of Δ but possibly with slower rate. Related to this is a question of possible extensions of the method, applied in the proof, to the study of possible synchronization between $x_i(t)$ and $x_i(t')$ for different moments of time t and t'. It is an interesting phenomenon that in the case of master-slave deterministic DDE, the slave $x_2(t)$ can synchronize with the master at an advanced time $x_1(t+\tau)$. Then, the relevant difference $\Delta(t)=x_2(t)-x_1(t+\tau)$ is not a nonanticipating function, and it is not clear to us how to apply the Ito calculus in this case.

III. EFFECTS OF NOISE ON THE LOCAL STABILITY OF SYNCHRONOUS SOLUTIONS

Local stability of the synchronous solutions for bidirectionally delayed coupled deterministic systems was studied numerically in Refs. [17,19]. In this section we report the major effects of the noise on the typical properties of synchronization in the deterministic case. In Ref. [17], only the system with $N=3$ Ikeda equations was analyzed, and in Ref. [19] the case with arbitrary N distributed on a discrete chain was studied. In the first case different types of synchronization have been analyzed, and in the latter case the main interest was in the possible spatial patterns of exact synchronization.

In order to discuss the influence of the noise we need to recapitulate briefly the typical properties of synchronization in the deterministic case. Numerical calculations showed that the exactly synchronous solutions could be locally stable for the values of the coupling constant c smaller than that which implies global stability. For such c there could be several coexisting local low-dimensional attractors that describe various types of generalized synchronization. The dynamics on this locally stable synchronization manifold could be low-dimensional chaotic, quasiperiodic or periodic. In the case of arbitrary odd N it was observed that the only spatial pattern of exact synchronization was that between all units, i.e., between the nearest neighbors. For arbitrary even N, and moderate values of c, it is possible to have exact synchronization between the next to the nearest units with some more complicated type of synchronization between the nearest neighbors. However, sufficiently large c implies the exact synchronization between all units also in the case of even N. Dependence of the local synchronization threshold value of c on the number of units N was also studied, and in general this threshold increases with N. The synchronization was detected using simple observations, largest transverse Lyapunov exponent [12,17,19] and the statistical correlations between units given by the so-called lag function [29,17,19].

In what follows we shall concentrate on the numerical study of the bidirectionally coupled system with $N=3$ and the multiplicative noise. For the system with additive noise and/or with more units we shall only briefly indicate some interesting phenomena that we have observed and that deserve further study. In our numerical computations we have used the package XPPAUT for numerical integration and analyses of dynamical systems [30]. The calculated mean values of the differences Δ for different integration time steps Δt in the interval 0.005–0.05 are qualitatively the same, and all presented results have been obtained with $\Delta t=0.01$.

A. Numerical illustration of the sufficiency criterion

We shall first compare the predictions of the sufficiency criterion (12) for global exponential stability in the mean of the exact synchronization with numerical calculations. After that, we shall consider the situation when the coupling is not large enough to imply the global stability of the exact synchronization, but, on the other hand, is large enough to render local stability of exact synchronization for the deterministic system. In this case, the deterministic system has two attractors, one corresponding to the exact synchronization and the other corresponding to some generalized synchronization. We shall analyze the influence of multiplicative noise on these two attractors.

Figure 1 illustrates the application of the sufficiency criterion (12). The value of $c=0.31$ is such that the deterministic system has globally stable attractor corresponding to the exact synchronization. It is also just above the critical value $c_0\approx 0.3$, below which there are two stable attractors of the deterministic system, so that the exact synchronization is then only locally stable. In Figs. 1(a) and 1(b) we show $E[\Delta_{1,2}(t)]$ [Fig. 1(a)] and $E[\Delta_{1,2}^2(t)]$ [Fig. 1(b)] over some sufficiently late interval of time t, where the expectations are calculated as averages over 100 paths with the same initial condition. The paths from the same initial condition far away from $\Delta=0$ are used. The expectations $E(\Delta_{1,2})$ and $E(\Delta_{1,2}^2)$ are shown for the value of the stochasticity parameter $D=0.31=c$ and for two values of the coupling $c=0.31$ and $c=2.36$. The second one satisfies the criterion (12) and, as is clear from the figures, leads to exact synchronization in the mean. The same results have been obtained using other initial conditions, other values of D, and other values of the time lags.

Numerical computations show that samples paths of the process converge into a small neighborhood of the exact synchronization manifold for much smaller values of c than those required by the condition (12). Like in the determinis-

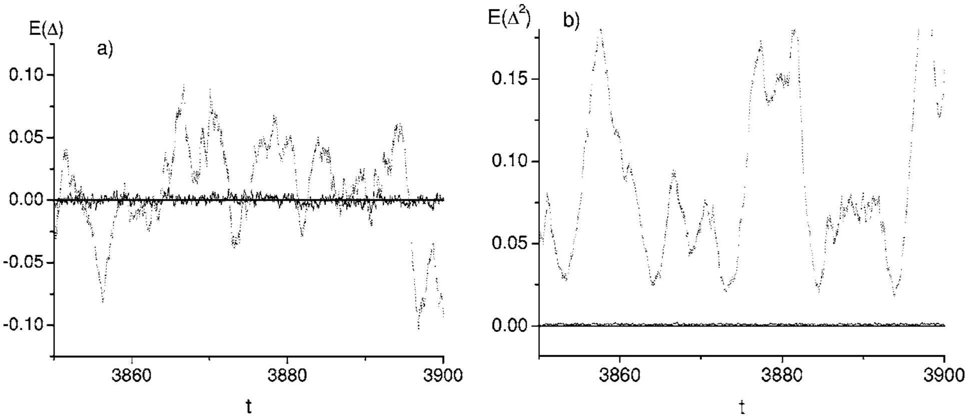

FIG. 1. The expectations $E(\Delta)(t)$ (a) and $E(\Delta^2)(t)$ (b) averaged over 100 realizations of the process from the same initial conditions. The parameters are $\mu=3, \tau_1=30, \tau_2=20$ and in both figures $D=0.31$; the thick line is for $c=2.32$ and the condition (12) is satisfied, and the dotted line corresponds to $c=0.31$.

tic case, it is expected that the condition (12) grossly overestimates the necessary values of c that imply synchronization in the mean. Because of a large parameter space $c, D, \tau_1, \tau_2, \mu$, and because the space of initial conditions is in fact infinite, it is difficult to give a complete quantitative comparison of the condition (12) and numerical calculations. However, in the view of the fact that parameter D enters in (12) in a simple way, one is tempted to use (12) as a guidance to make a heuristic prediction of the sufficient c which could agree much better with the numerics. Namely, let us denote numerical estimate of the sufficient value of c in the noiseless case, i.e., for $D=0$, by $c_0(\mu, \tau_1, \tau_2)$. For, example the formula (12) predicts for $D=0$ and $\mu=3$ the sufficient value of $c=2$. However, numerically obtained c_0 depends on $\tau_{1,2}$ and is much smaller than 2 for any $\tau_{1,2}$. For example, for $\tau_1=30, \tau_2=20, \mu=3$ the numerical c_0 is $c_0 \approx 0.3$. Substituting the numerical estimate c_0 in the noiseless case, instead of $\mu-1$ in the formula (12) gives a simple heuristic estimate for the case $D \neq 0$,

$$c \geq c_0 + D, \tag{14}$$

of the value of c which guarantees the exact synchronization of the noisy system. Because of the mentioned difficulties and because the results seem to be negative as far as asymptotic stability in the mean is concerned, we shall concentrate on numerical illustrations of the condition (14) for just one combination of parameter values and initial conditions. The results for $c=0.64$ and $D=0.31$ are illustrated in Fig. 2. The numerically calculated expectations $E(\Delta^2)$ [Fig. 2(b)] and $E(\Delta)$ [Fig. 2(a)] converge as the number of averaged sample paths is increased, which is not the case for $c=0.31; D=0.31$ (Fig. 1). However, the averages are close to but different from zero. The distance from zero remain as illustrated for averages taken along much later segments of the paths. The same behavior of the expectations $E(\Delta^2)$ and $E(\Delta)$ are obtained using different initial conditions, some quite far away from the exact synchronization manifold. Thus, numerical evidence suggests that, there is some type of stability in the mean of the exact synchronization, but the exponential stability in the mean is not satisfied for $c=0.64$ and $D=0.31$. We should say that the same qualitative conclusion is suggested by numerical computations for other initial conditions and different values of c close to $D+c_0$, but we cannot make any more precise and quantitative conclusions.

B. Influence of noise on different types of synchronization

Theorem 1, and the numerical computations, show that, for any value of the multiplicative noise there is sufficiently large c such that the exact synchronization is asymptotically globally stable on the average. In this sense the exact synchronization is more stable than any other type of locally stable synchronization that occurs for some values of c and D and some τ_1, τ_2. However, it is interesting to study the case when the coupling c is such that there are two locally stable types of synchronization in the deterministic system. Thus, when $D=0$ and c is smaller than c_0 it is possible to have two attractors (for certain τ_1, τ_2), one corresponds to the

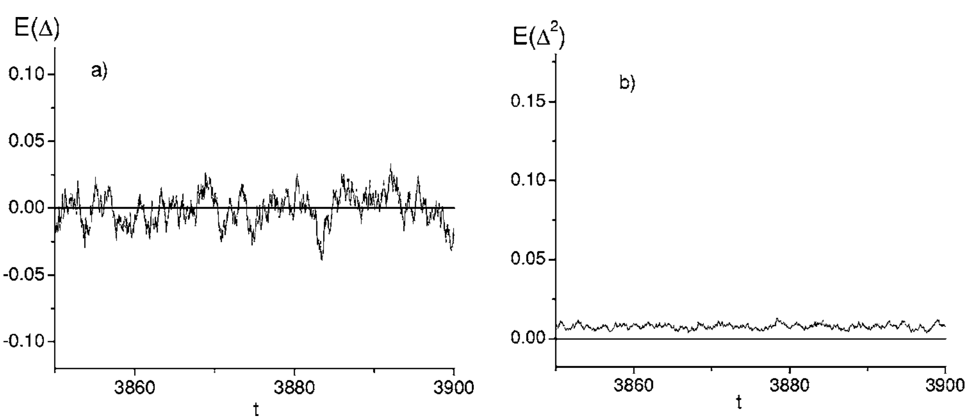

FIG. 2. The expectations $E(\Delta)(t)$ (a) and $E(\Delta^2)(t)$ (b) averaged over 100 realizations of the process from the same initial conditions. The parameters are $D=0.31$, $c=0.63>D+c_0$, $\mu=3$, $\tau_1=30$, $\tau_2=20$.

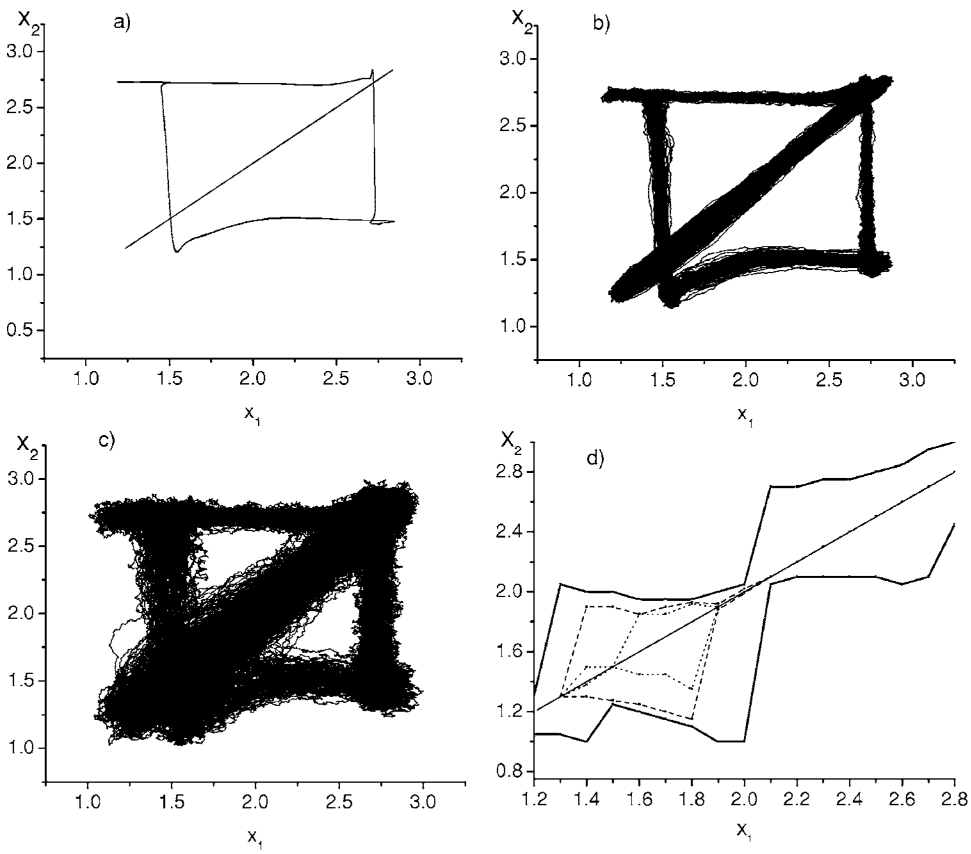

FIG. 3. Illustrates possible realizations of the process with increasing the multiplicative noise in the case when the deterministic system has two coexisting attractors. In all figures $\mu=3, \tau_1=30$, $\tau_2=15$, and $c=0.2$. In (a) $D=0$; (b) $D=0.05$, and (c) $D=0.1$. In (d) the full line represents a part of the boundary of the exact synchrony domain for $D=0$, the dashed line for $D=0.005$ and the dotted line for $D=0.01$.

exact and the other to some type of the generalized synchronization [Fig. 3(a)]. We have studied the influence of noise in this case, and in particular have tried to estimate numerically how the stability domains change as the noise is increased for a fixed c. The results are illustrated in Figs. 3. Figure 3(a) represents the two attractors in the stochastic case. Figures 3(b) and 3(c) show quite long segments of two typical paths that could occur when noise is $D=0.05$ [Fig. 3(b)] and $D=0.1$ [Fig. 3(c)]. Thus, different realizations of the stochastic process from the same initial conditions could end up in a neighborhood of either of the two attractors. When the noise is small the average over many paths, and the attractor of each path, largely depend on the initial condition, but as the noise is increased this dependence is lost. Inother words, a path that started from an initial condition close to one of the attractors almost certainly remains in a neighborhood of this attractor if the noise is small, but as the noise is increased the path can spend most of the time in a neighborhood of either of the attractors. Nevertheless, all calculated paths spent most of the time close to one or the other of the attractors for values of noise up to $D=0.2=c$. We were led to the same conclusions by calculations for many initial conditions and different values of c or τ_1, τ_2 for which there are two attractors in the deterministic system.

Finally, in Fig. 3(d), we illustrate, for different noise intensities, the domain of points that with large probability remain forever in a neighborhood of the exact synchronization. The points (x_1,x_2) within the full line correspond to $D=0$. They represent orbits that at $t=0$ go through $(x_1,x_2,0)$ and are attracted to the attractor of exact synchronization. The points (x_1,x_2) within the dashed line correspond to $D=0.005$. They represent the points such that 90 out of 100 paths with the same initial condition $(x_1,x_2,0)$ remain close to the attractor of exact synchronization. The dotted line corresponds to paths chosen by the same criterion but for $D=0.01$. Thus, as the noise is increased the probability that an initial state will converge to the almost exact synchronization is diminishing, however as numerical calculations represented in Figs. 3(b) and 3(c) indicate, it never gets small or zero. This qualitative conclusion summarizes results of our study in the case of two coexisting attractors, and with this we finish the description of the effects of multiplicative noise in the $N=3$ case.

We would like now to report briefly a couple of observations that resulted from a preliminary numerical study of the influence of additive noise in the case $N=3$, and of the multiplicative or additive noise in larger chains. The first remark concerns an interesting phenomenon, which can be thought of as an instance of generalized stochastic coherence, that occurs in the situation treated in the preceding paragraph but in the case of additive noise with different intensities at different sites. We have observed that there are such initial conditions that small intensities of inhomogeneous additive noise imply that a majority of sample paths (in our computation about 70 out of 100) are attracted to a neighborhood of the manifold of generalized synchronization. On the other hand, larger additive noise implied that majority (60 out of 100) paths converged into a small neighborhood of the manifold of exact synchronization. Qualitatively the same effect was observed with initial points that are, in the deterministic system, attracted to one or the other of the two attractors.

Our final observation concerns the influence of noise on different types of synchronization that occur for chains with even $N=2n$. It has been reported before [19], that in such chains the state of full exact synchronization $x_1(t) = x_2(t) \cdots x_{2n}(t)$ is preceded, for weaker coupling c, by a state in which the next to the nearest neighbors are exactly synchronized $x_i(t)=x_{i+2}(t)$ but the nearest neighbors only satisfy some condition of generalized synchronization $x_i(t) = f[x_{i+1}(t)]$. Our numerical computations show that the influence of multiplicative versus additive noise on these two types of synchronization is quite clearly qualitatively different. In Fig. 4, we illustrate what happens using the chain with $N=4$ units. Figures 4(a) and 4(b) correspond to deterministic system and illustrate the attractors of generalized and exact synchronization between x_1 and x_2 and between x_1 and x_3, respectively. Both types of synchronization are globally stable for the shown values of $c=0.5, \tau_1=30, \tau_2 = 15, \nu=3$. Figures 4(c) and 4(d) show the influence of small multiplicative noise $D=0.005$. Figures 4(e) and 4(f) illustrate what happens with the additive noise with noise parameters at each site equal to $D_1=0.01$, $D_2=D_3=0.004$, and $D_4 = 0.006$. Notice that we illustrate the case when x_2 and x_3 are directly influenced by the same amount of noise $D_2=D_3$, wile the noise at x_1 and x_3 is of different intensity $D_1 \neq D_3 \neq D_4$, but the observed effects occurs also with all D_i different.

Figures 4(c)–4(f) clearly illustrate that the small multiplicative noise has the same effect on the generalized and on the exact synchronization. On the other hand, a small inhomogeneous additive noise destroys the generalized synchronization between x_1 and x_2 while the dynamics of x_1 and x_3 is changed but is still exactly synchronous. Let us point out that the same behavior has been obtained with all sample paths from the same initial condition, because the coupling c is sufficiently large compared to D or D_i, and with different initial conditions, because the attractors are globally stable in the deterministic system. Also the same figures have been obtained with different values of c and D or D_1, D_2, D_3, D_4. However, due to large parameter space we cannot, at this moment, make any quantitative estimate of c, D_1, D_2, D_3, D_4 that for some μ, τ_1, τ_2 lead to the described situation.

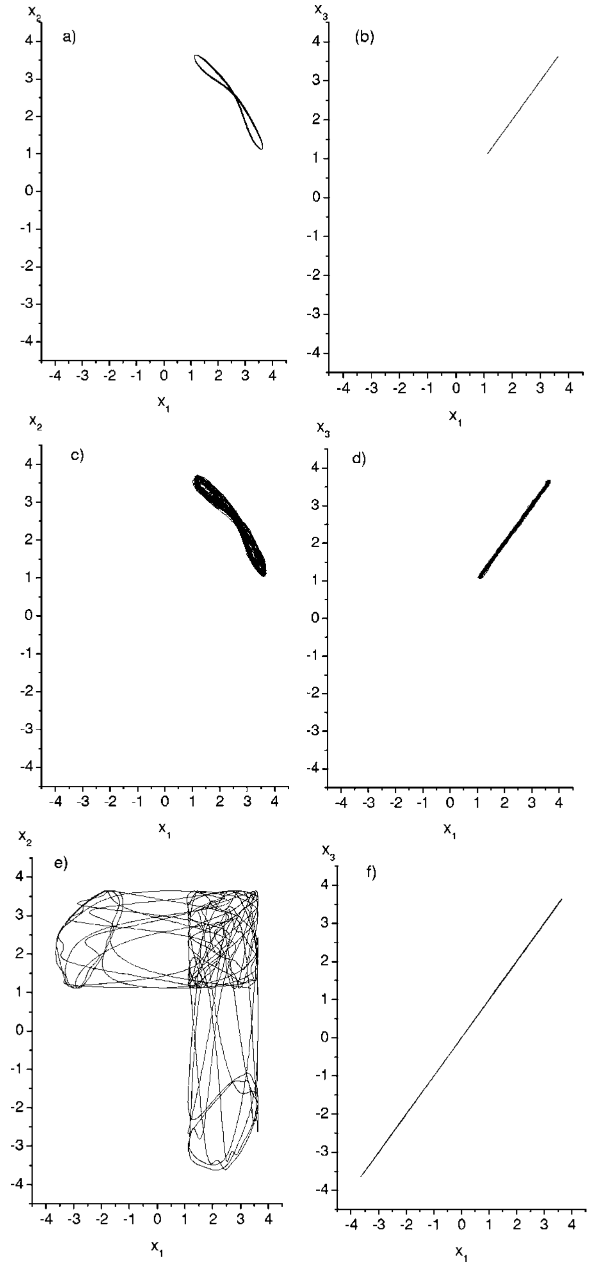

FIG. 4. The chain with $N=4$ units. For $c=0.5; D=0$ there is generalized synchronization between x_1 and x_2 (a), and exact synchronization between x_1 and x_3 (b). Multiplicative noise $D=0.03$ slightly perturbs the generalized (b) as well as the exact (d) synchronization. Additive noise $D_1=0.01, D_2=0.004=D_3, D_4=0.006$ completely destroys the generalized synchronization (e), but x_1 and x_3 remain almost exactly synchronous (f). In all figures the fixed parameters are $\mu=3$, $\tau_1=30$, $\tau_2=15$, $c=0.5$.

IV. SUMMARY AND PERSPECTIVES

We have studied influence of noise on the synchronization between delayed coupled Ikeda systems. The Ikeda system was chosen as a typical example of the delayed feedback system which can have hyperchaotic dynamics, so our interest was in synchronization of noisy hyperchaotic delayed feedback systems with delayed interaction. In the analytical investigation of the stability of synchronization we have concentrated on the system with multiplicative noise and bidirectional coupling, but the exact synchronization of unidirectionally coupled system in the master-slave configuration was also briefly considered. Numerical computation was used to illustrate interesting examples of the influence of multiplicative and additive noise on the synchronization.

In the case of multiplicative noise we have proved the exponential stability in the mean of the exact synchronization for sufficiently strong coupling. Thus, in the case of multiplicative noise, the exact synchronization can be made globally asymptotically stable in the mean for any value of the time delays or noise parameter by using the sufficiently strong coupling. This is the main result of our study. The estimate of the sufficient condition is based on an application of Lyapunov-Krasovskii functional or Razumikhin theorems for DDE's and stochastic calculus, and could be applied in similar situations, whenever the problem of the stability of exact synchronization can be restated as the problem of stability of a stationary solution of the equations with coefficients known within a bounded error. However the method cannot be used to study the possibility of anticipating synchronization in delayed master-slave configuration, due to inherent conditions on type of functions considered in the Itô calculus.

The estimated sufficient condition for the stability of the exact synchronization is illustrated by numerical computations. The condition grossly overestimates the numerically suggested values. This is no surprise, since the estimates of the sufficient coupling provided by the Lyapunov-Krasovskii functional or Razumikhin theorems are known to overestimate the numerically obtained values also in the deterministic case. The major reason for this in the case of the deterministic equations is in the majorization of the time-dependent coefficients in the deterministic part of the equation for the difference. This is probably the main reason for the error in the sufficiency estimate also in the case of the noise system.

Further numerical computation are used to illustrate and compare the influence of multiplicative and additive noise on the exact and more general types of synchronization in the system with $N=3$ units, and in the system with $N=4$ units. Here we have not performed and presented a completed study, but have just illustrated some interesting observations. In particular, in the case of $N=3$, we have illustrated what can happen when the coupling is such that the deterministic system has two coexisting attractors, one corresponding to the exact and the other to a generalized synchronization. In the case of $N=4$ we have shown that the generalized synchronization between the nearest neighbors is much less stable on the additive noisy perturbations than the exact synchronization between the next to the nearest neighbors. On the other hand, the multiplicative noise produces qualitatively the same effects on both types of synchronization.

In this paper we have concentrated on the effects of noise on the properties of exact synchronization. Effects of noise on more general types of synchronization should be studied and compared with the case of the exact synchronization. Also, the analytical methods applied here, could be used to study synchronization between delayed coupled noisy phase oscillators, and phase synchronization between noisy (chaotic) oscillators. We have illustrated the analytical methods using the system with the simplest spatial distribution of coupled units. However the methods could be applied for analyses of synchronization in more complicated noisy feedback systems, distributed on a lattice, with more complicated coupling matrix.

ACKNOWLEDGMENTS

One of the authors (N.B.) to acknowledges the support and warm hospitality of the Condensed matter group at ICTP-Trieste. This work is partly supported by the Serbian Ministry of Science Contract No. 141003.

APPENDIX

We must prove that

$$E[\Delta^2(t)] \leq CE[\Delta^2(0)]\exp(-\lambda t), \quad t \geq 0, \tag{A1}$$

where $\lambda > 0$ is a constant and C is independent of t, but could depend on c, μ, τ, and λ.

Applying the Itô formula to $\Delta(t)^2$, we have

$$d\Delta^2(t) = 2\Delta(t)\left[-\Delta(t) + 2\mu\cos\left(\frac{[X^1(t-\tau_1)+X^2(t-\tau_1)]}{2}\right)\times\sin\frac{\Delta(t-\tau_1)}{2} + c[-\Delta(t-\tau_2) - 2\Delta(t)]\right]dt + \Delta^2(t)2Ddt + 2\Delta^2(t)\sqrt{2D}dW(s).$$

For $U(\Delta^2(s),s)=e^{-(2+4c)(t-s)}\Delta^2(s)$, and integrating with respect to s, we get

$$\int_0^t d[e^{-(2+4c)(t-s)}\Delta^2(s)] = \int_0^t [(\partial U/\partial s)ds + (\partial U/\partial x)dX + (1/2)(\partial^2 U/\partial x^2)G^2 ds],$$

where $X=\Delta^2(s)$ and $dX=d[\Delta^2(s)]=Fdt+GdW$. Thus,

$$e^{-(2+4c)(t-t)}\Delta^2(t) - e^{-(2+4c)(t-0)}\Delta^2(0) = \int_0^t (2+4c)\Delta^2(s)e^{-(2+4c)(t-s)}ds + \int_0^t e^{-(2+4c)(t-s)} \times\left(-2\Delta^2(s) + 2\Delta(s)2\mu\cos\frac{[X^1(s-\tau_1)+X^2(s-\tau_1)]}{2}\times\sin\frac{\Delta(s-\tau_1)}{2} - 2c\Delta(s)\Delta(s-\tau_2) - 4c\Delta(s)\Delta(s)ds + \Delta^2(s)2Dds + 2\Delta^2(s)\sqrt{2D}dW(s)\right). \tag{A2}$$

It follows that

$$\Delta^2(t) - e^{-(2+4c)t}\Delta^2(0) \leq \int_0^t (2+4c)\Delta^2(s)e^{-(2+4c)(t-s)}ds + \int_0^t e^{-(2+4c)(t-s)}\{[-2\Delta^2(s) + 2\mu\Delta(s)\Delta(s-\tau_1) - 2c\Delta(s)\Delta(s-\tau_2) - 4c\Delta^2(s)ds + \Delta^2(s)2Dds + 2\Delta^2(s)\sqrt{2D}dW(s)]\}.$$

From the preceding expression we have

$$\Delta^2(t) \leq \Delta^2(0)e^{-(2+4c)t} + \int_0^t e^{[-(2+4c)(t-s)]}\{[(2+4c)\Delta^2(s) - (2+4c)\Delta^2(s) + 2\mu|\Delta(s)||\Delta(s-\tau_1)| + 2c|\Delta(s)|| \times\Delta(s-\tau_2)| + 2D\Delta^2(s)]ds + 2\Delta^2(s)\sqrt{2D}dW(s)\}.$$

From (12), there exists some sufficiently small positive constant λ,

$$1+2c>\lambda>0$$

such that

$$(2+4c)-2\lambda-2\mu e^{(\lambda\tau)}-2ce^{(\lambda\tau)}-2D>0,\quad \tau=\max(\tau_1,\tau_2). \tag{A3}$$

Denote by

$$G(t)=\sup_{(-\tau\le\theta\le t,-\tau\le\psi\le t)}E[|\Delta(\theta)||\Delta(\psi)|]e^{\lambda\theta}e^{\lambda\psi}.$$

Then, using

$$E\left(\int_0^t e^{[2\lambda t-(2+4c)(t-s)]}\Delta^2(s)\sqrt{2D}dW(s)\right)=0,$$

we have

$$\begin{aligned}E[\Delta^2(t)]e^{2\lambda}t\le e^{[2\lambda-(2+4c)]t}E[\Delta^2(0)]+\int_0^t e^{2\lambda t-(2+4c)(t-s)}\\ \times\{2\mu E[|\Delta(s)||\Delta(s-\tau_1)|]e^{\lambda s}e^{s-\tau_1}e^{-\lambda s}e^{-\lambda(s-\tau_1)}\\ +2cE[|\Delta(s)||\Delta(s-\tau_2)|]e^{\lambda s}e^{(s-\tau_2)}e^{-\lambda s}e^{-\lambda(s-\tau_2)}\\ +2DE[\Delta^2(s)]e^{2\lambda s}e^{-2\lambda s}\}ds,\end{aligned} \tag{A4}$$

that is

$$\begin{aligned}E[\Delta^2(t)]e^{2\lambda}t\le e^{[2\lambda-(2+4c)]t}E[\Delta^2(0)]+\int_0^t e^{[2\lambda-(2+4c)](t-s)}\\ \times\{2\mu e^{\lambda\tau_1}E[|\Delta(s)||\Delta(s-\tau_1)|]e^{\lambda s}e^{s-\tau_1}\\ +2ce^{\lambda\tau_2}E[|\Delta(s)||\Delta(s-\tau_2)|]e^{\lambda s}e^{\lambda(s-\tau_2)}\\ +2DE[\Delta^2(s)]e^{2\lambda s}\}ds.\end{aligned} \tag{A5}$$

Thus, we obtain

$$\begin{aligned}E[\Delta^2(t)]e^{2\lambda}t\le e^{[2\lambda-(2+4c)]t}E[\Delta^2(0)]\\ +\int_0^t e^{[2\lambda-(2+4c)](t-s)}ds[2\mu e^{\lambda\tau}G(t)\\ +2ce^{\lambda\tau}G(t)+2DG(t)],\end{aligned}$$

and

$$\begin{aligned}E[\Delta^2(t)]e^{2\lambda}t\le E[\Delta^2(0)]+\frac{1}{[(2+4c)-2\lambda]}\\ \times(2\mu e^{(\lambda\tau)}+\,+2ce^{(\lambda\tau)}+2D)G(t).\end{aligned}$$

Now, we get

$$\begin{aligned}[(2+4c)-2\lambda]G(t)\le(2+4c-2\lambda)E[\Delta^2(0)]\\ +(2\mu e^{(\lambda\tau)}+2ce^{(\lambda\tau)}+2D)G(t)\end{aligned}$$

and

$$\begin{aligned}(2+4c-2\lambda-2\mu e^{\lambda\tau}-2ce^{\lambda\tau}-2D)G(t)\\ \le(2+4c-2\lambda)E[\Delta^2(0)].\end{aligned} \tag{A6}$$

Further, we can write

$$\begin{aligned}(2+4c-2\lambda-2\mu e^{\lambda\tau}-2ce^{\lambda\tau}-2D)E[\Delta^2(t)]e^{2\lambda t}\\ \le(2+4c-2\lambda-2\mu e^{\lambda\tau}-2ce^{\lambda\tau}-2D)G(t)\\ \le(2+4c-2\lambda)E[\Delta^2(0)]\le(2+4c)E[\Delta^2(0)],\end{aligned} \tag{A7}$$

which finally gives

$$E[\Delta^2(t)]\le\frac{(2+4c)E[\Delta^2(0)]}{(2+4c-2\lambda)-2\mu e^{\lambda\tau}-2ce^{\lambda\tau}-2D}e^{-2\lambda t}. \tag{A8}$$

The proof is completed.

[1] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization. A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, 2001).

[2] D. Goldobin, M. Rosenblum, and A. Pikovsky, Physica A **327**, 134 (2003).

[3] J. Hale and S. V. Lunel, *Introduction to Functional Differential Equations* (Springer-Verlag, New York, 1993).

[4] J. D. Farmer, Physica D **4**, 366 (1982).

[5] C. T. H. Baker, G. A. Bocharov, and F. A. Rihan, Numerical Analysis Report No. 343, University of Manchester, 1999 (available at www.ma.man.ac.uk./MCCM/MCCM.html).

[6] K. Green, B. Krauskopf, and K. Engelborghs, Physica D **173**, 114 (2002).

[7] G. D. VanWiggeren and R. Roy, Science **279**, 1198 (1998).

[8] J. P. Goedgebuer, L. Larger, and H. Porte, Phys. Rev. Lett. **80**, 2249 (1998).

[9] K. Ikeda and K. Matsumoto, Physica D **29**, 223 (1987).

[10] K. Ikeda and K. Matsumoto, J. Stat. Phys. **44**, 955 (1986).

[11] R. Vicente, J. Daudén, P. Colet, and R. Toral, IEEE J. Quantum Electron. **41**, 541 (2005).

[12] K. Pyragas, Phys. Rev. E **58**, 3067 (1998).

[13] H. U. Voss, Phys. Rev. E **61**, 5115 (2000).

[14] S. Sivaprakasam, E. M. Shahverdiev, P. S. Spencer, and K. A. Shore, Phys. Rev. Lett. **87**, 154101 (2001).

[15] C. Masoller, Phys. Rev. Lett. **86**, 2782 (2000).

[16] A. Locquet, C. Masoller, and C. R. Mirasso, Phys. Rev. E **65**, 056205 (2002).

[17] N. Burić and D. Todorović, Phys. Rev. E **67**, 066222 (2003).

[18] I. Fischer, R. Vicente, J. M. Buldu, M. Peil, C. R. Mirasso, M. C. Torrent, and J. Garcia-Ojalvo, Phys. Rev. Lett. **97**, 123902 (2006).

[19] N. Burić, K. Todorović, and N. Vasović, Int. J. Bifurcation Chaos Appl. Sci. Eng. **16**, 1569 (2006).

[20] J. Garcia-Ojalvo and R. Roy, Phys. Lett. A **224**, 51 (1996).

[21] G. Giacomelli, M. Giudici, S. Balle, and J. R. Tredicce, Phys. Rev. Lett. **84**, 3298 (2000).

[22] J. M. Buldu, J. Garcia-Ojalvo, C. R. Mirasso, M. C. Torrent, and J. M. Sancho, Phys. Rev. E **64**, 051109 (2001).

[23] X. Liao and X. Mao, Stoch. Anal. Appl. **14**, 165 (1996).
[24] S. Blythe, X. Mao, and X. Liao, J. Franklin Inst. **338**, 481 (2001).
[25] J. Sun and L. Wan, Phys. Lett. A **343**, 331 (2005).
[26] D. S. Goldobin and A. S. Pikovsky, Physica A **351**, 126 (2005).
[27] N. N. Krasovskii, *Stability of Motion* (Stanford University Press, Stanford, 1963).
[28] L. Arnold, *Stochastic Diffential Equations: Theory and Applications* (Krieger, Malabar, FL, 1992).
[29] G. Hu, Y. Zhang, H. A. Cerdeira, and S. Chen, Phys. Rev. Lett. **85**, 3377 (2000).
[30] The package XPPAUT for numerical analyzes of dynamical systems and in particular systems given by SDDE's is available from: www.pitt.edu/ phase or from www.math.pitt.edu/bard/xpp/xpp.html