Dipole representation of half-filled Landau level

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We introduce a variant of a dipole representation for composite fermions in a half-filled Landau level, taking into account the symmetry under an exchange of particles and holes. This is implemented by a special constraint on a composite fermion and a composite hole degree of freedom (of an enlarged space), which makes the resulting composite particle (dipole) a symmetric object. We study an effective Hamiltonian that commutes with the constraint on the physical space and fulfills the requirement for boost invariance on the Fermi level. The calculated Fermi liquid parameter F_2 is in good agreement with numerical investigations in Phys. Rev. Lett. **121**, 147601 (2018).

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I. INTRODUCTION

The fractional quantum Hall effect (FQHE) is a phenomenon of strongly correlated electrons that is amenable to quasiparticle pictures and modeling. The presence of a strong magnetic field very often leads to dominance of the physics inside a Landau level (LL) (a subspace of the Hilbert space of the half-filled LL problem), and it justifies approaches that assume that the description of the problem can be confined to an isolated LL. The half-filled LL problem (relevant for systems at half-filling factors) is very interesting because it contains an additional symmetry that is not present in experiments or any other fillings of LLs, namely the symmetry of the exchange of particles (electrons) and holes, i.e., particle-hole (PH) symmetry. On the other hand, systems at half-filling contain physics that are specific for the FQHE, including the formation of a Fermi-liquid state of composite quasiparticles (fermions) that reside in the lowest LL (LLL) [1,2], and the incompressible half-integer filling factor, $5/2$, for electrons in the second LL [3]. Moreover, it is widely believed that the physics at $5/2$ is connected with a Cooper pairing of underlying quasiparticles that form so-called Pfaffian states [4]. The LL mixing (the influence of other LLs on the effective physics in the base LL) is very important for the physics of Pfaffian states because it selects and stabilizes a unique Pfaffian state among three possibilities: Pfaffian, anti-Pfaffian, and PH Pfaffian. On the other hand, the LL mixing is not that important for the physics of electrons in the LLL. Moreover, we are interested to know what will happen to the PH symmetry that is present if we assume that the Hilbert space of the system is an isolated LL. The state in the experiments may correspond to the spontaneous PH symmetry breaking state of a Hamiltonian that contains PH symmetry.

Even if we confine our description to the isolated LL, the problem at half-filling is still a strongly correlated one. Our hope is that by selecting appropriate quasiparticles, and by applying approximate methods (usually mean-field methods), we can arrive at an effective description of the system that is consistent with numerical and real experiments.

The Fermi liquid (FL) concept for the physics of composite quasiparticles, i.e., composite fermions, was introduced in the seminal work of Halperin, Lee, and Read (HLR) [1] on the basis of the Chern-Simons field-theoretical description, which does not include a projection to the LLL. To achieve a detailed description and understanding of the Fermi-liquid state inside the LLL, Pasquier and Haldane [5], and later Read [6], analyzed a related system of bosons at filling factor 1, with an exact representation of the composite fermion (CF) quasiparticles in an enlarged space of the half-filled LL problem. On the other hand, Shankar and Murthy [7] advanced a field-theoretical description of the dipole-CF representation of the problem [8]. In recent years, the concept of a Dirac, i.e., a two-component quasiparticle, was introduced by Son [9] for the description of the half-filled LL, i.e., a system that features PH symmetry. In Ref. [10], a microscopic derivation of such a theory was given in which the two components were connected to the two possibilities for quasiparticles, i.e., CFs and CHs (composite holes).

Here we propose an extension of the quasiparticle view of the physics inside an isolated LL based on a one-component fermion, namely a dipole. Our description does not distinguish between CFs and CHs. We employ the enlarged-space formalism [5–7], but with a special constraint that incorporates PH symmetry, in the system of electrons that fill half of an isolated LL. The special constraint and demand for the boost invariance defines an effective Hamiltonian and FL description, in agreement with numerical experiments [11].

The paper is organized as follows. In Sec. II, the enlarged-space formalism for the system of bosons at filling factor 1 is reviewed before the exposition of our proposal in the same section. At the end of the section, we discuss the FL description based on that proposal. In Sec. III, in order to further understand the nature of the introduced quasiparticles, the problem of the bilayer, i.e., two half-filled LLLs, is analyzed in the new representation. In Sec. IV, we discuss an additional implementation of our approach, i.e., a quantum Boltzmann equation based on the dipole picture that incorporates the

boost invariance and PH symmetry, and we end with conclusions. In Appendix A, the invariance under the change of basis, i.e., the SU(N) invariance for the proposed formalism, is described, and in Appendix B we provide details concerning the derivation of the quantum Boltzmann equation.

II. DIPOLE REPRESENTATION IN AN ISOLATED LANDAU LEVEL

We study an extension of the formalism for CFs that was developed in [5,6] for the case of bosons at filling factor 1 to the case of electrons in an isolated LL that is half-filled. Although an extension of the CF formalism to the case of half-filled LL of electrons has to incorporate *bosonic* correlation holes (i.e., artificial degrees of freedom in the setup), we will show that with a hard-core constraint in an enlarged space, we can delineate a physical subspace and reach a faithful description. In the following, we will review the basic formalism for the system of bosons at filling factor 1.

A. Bosons at filling factor $\nu = 1$

1. Review of the CF formalism for bosons at filling factor $\nu = 1$

In this section, we will briefly review the CF formalism that was introduced in [5] and further developed in [6] for bosons at filling factor $\nu = 1$. The CF is a composite quasiparticle of an underlying system of bosons (or electrons) that consists of a boson (an electron) and its (associated) correlation hole in the incompressible or weakly compressible (like the FL state of CFs) FQHE states. For the case of bosons at filling factor $\nu = 1$, we have one boson on average per state Ψ_n in an isolated LL, $n = 1, \ldots, N_\phi$. N_ϕ denotes the number of flux quanta through the system. Thus n enumerates states in a chosen basis of the relevant LL.

The Laughlin solution of the FQHE at $\nu = 1/3$ of electrons—an excellent description of the ground-state wave function, which incorporates basic correlations among particles that are solely interaction-driven in an isolated LL, motivates the introduction of the composite quasiparticles. In the regime of the FQHE, we can easily envision a structure of a neutral quasiparticle: an underlying boson (or an electron) and its associated correlation hole, which we can express and regard as (Laughlin) quasihole excitation (eigenstate) of the system with quantized charge and statistics [or nearly quantized and localized, almost an eigenstate in a weakly compressible systems (like the FL state of CFs)].

In the case of bosons at filling factor $\nu = 1$ of an isolated LL, and by following the Laughlin ansatz, we can easily find out that the most natural assignment for the statistics of the correlation hole is fermionic (so that the statistics of CFs is fermionic and counterbalances the Vandermonde determinant fermionic correlations, and produces overall bosonic correlations and wave functions), and that it represents a deficiency of a unit of charge (a hole). Thus we are inclined to consider a composite fermionic object (CF) and associate an annihilation operator with two indices m and n, c_{nm}, where n and m refer to two states of a chosen basis in a LL: the left index, n, describes the state of an (underlying, elementary) boson, and the right index, m, describes the state of its correlation hole—an artificial (additional) degree of freedom. In this way, we are enlarging the space that we associate with the system's description; we introduce also creation operators, $c^\dagger_{mn}$, so that

$$\{c_{nm}, c^\dagger_{m'n'}\} = \delta_{n,n'}\delta_{m,m'}, \tag{1}$$

and we consider states in the enlarged space,

$$c^\dagger_{mn} \cdots c^\dagger_{pq}|0\rangle. \tag{2}$$

The physical subspace of this enlarged space can be delineated by projecting out artificial, nonphysical degrees of freedom, i.e., "vortices" (correlation holes), that possess fermionic statistics:

$$|n_1, \ldots, n_N\rangle = \sum_{m_1,\ldots,m_N}^{N_\phi} \varepsilon^{m_1\cdots m_N} c^\dagger_{m_1 n_1} \cdots c^\dagger_{m_N n_N}|0\rangle, \tag{3}$$

where $\varepsilon^{m_1\cdots m_N}$ is the Levi-Civita symbol.

We may notice that this construction is invariant under an SU(N) transformation, i.e., a change of basis in the LL in the R sector only, i.e.,

$$c^\dagger_{mn} \to \sum_{m'}^{N_\phi} U_{mm'} c^\dagger_{m'n}, \tag{4}$$

where $U_{mm'}$ is an SU(N) matrix. (The physical states are spin-singlets under this transformation.)

In the CF representation, one can consider $\rho^L_{mm'}$ and $\rho^R_{mm'}$, density operators for physical (L) and unphysical (holelike, R) degrees of freedom:

$$\rho^L_{nn'} = \sum_m c^\dagger_{mn} c_{n'm} \tag{5}$$

and

$$\rho^R_{mm'} = \sum_n c^\dagger_{mn} c_{nm'}. \tag{6}$$

Also the following decomposition [of the basic state (n, m) of the composite object] can be considered:

$$c_{nm} = \int \frac{d\boldsymbol{k}}{(2\pi)^{\frac{3}{2}}} \langle n|\tau_{\boldsymbol{k}}|m\rangle c_{\boldsymbol{k}}, \tag{7}$$

with $\tau_{\boldsymbol{k}} = \exp(i\boldsymbol{k}\cdot\boldsymbol{R})$, where $\boldsymbol{R}$ is a guiding-center coordinate of a single particle in the external magnetic field,

$$[R_x, R_y] = -i. \tag{8}$$

We took l_B (magnetic length) $= 1$, and $\{|n\rangle\}$ are single-particle states (orbitals) in a fixed LL.

The parameter $\boldsymbol{k}$ denotes the momentum of the composite object, i.e., a CF. Physically, the state of the CF with vortex orbital m and electron orbital n can be described by a superposition of the (commutative) momentum $\boldsymbol{k}$ states, the weights of which depend on the effective distance between orbitals (the size of the dipole), $|\boldsymbol{k}_{\rm eff}|$, because $\tau_{\boldsymbol{k}} = \exp(i\boldsymbol{k}\cdot\boldsymbol{R})$ is the translation operator.

The introduced decomposition implies the following expressions for the L and R densities in the inverse space:

$$\rho^L_{nn'} = \sum_m c^\dagger_{mn} c_{n'm} = \int \frac{d\boldsymbol{q}}{2\pi} \langle n'|\tau_{\boldsymbol{q}}|n\rangle \rho^L_{\boldsymbol{q}}, \tag{9}$$

where

$$\rho^L_{\boldsymbol{q}} = \int \frac{d\boldsymbol{k}}{(2\pi)^2} c^\dagger_{\boldsymbol{k}-\boldsymbol{q}} c_{\boldsymbol{k}} \exp\left(i\frac{\boldsymbol{k}\times\boldsymbol{q}}{2}\right). \tag{10}$$

Note the inverse order of indices, n and n', on the left- and right-hand sides of (9). Similarly,

$$\rho^R_{mm'} = \sum_n c^\dagger_{mn} c_{nm'} = \int \frac{d\boldsymbol{q}}{2\pi} \langle m|\tau_{\boldsymbol{q}}|m'\rangle \rho^R_{\boldsymbol{q}}, \tag{11}$$

where

$$\rho^R_{\boldsymbol{q}} = \int \frac{d\boldsymbol{k}}{(2\pi)^2} c^\dagger_{\boldsymbol{k}-\boldsymbol{q}} c_{\boldsymbol{k}} \exp\left(-i\frac{\boldsymbol{k}\times\boldsymbol{q}}{2}\right). \tag{12}$$

The density $\rho^L_{\boldsymbol{q}}$ satisfies the GMP algebra,

$$\left[\rho^L_{\boldsymbol{q}}, \rho^L_{\boldsymbol{q}'}\right] = 2i \sin\left(\frac{\boldsymbol{q}\times\boldsymbol{q}'}{2}\right)\rho^L_{\boldsymbol{q}+\boldsymbol{q}'}, \tag{13}$$

while $\rho^R_{\boldsymbol{q}}$, as a density of "holes,"

$$\left[\rho^R_{\boldsymbol{q}}, \rho^R_{\boldsymbol{q}'}\right] = -2i \sin\left(\frac{\boldsymbol{q}\times\boldsymbol{q}'}{2}\right)\rho^R_{\boldsymbol{q}+\boldsymbol{q}'}, \tag{14}$$

i.e., GMP algebra for particles with opposite electric charge.

Thus we can realize the basic algebra of the electron density projected to a LL as an algebra of the same density expressed via operators that represent overall neutral objects (dipoles), i.e., CFs. We expect that the CF representation will capture the basic physics of the problem, and already at the mean-field level will give meaningful results (stable FL if we apply Hartree-Fock [6]).

2. Hamiltonian and constraints

The basic Hamiltonian in the second-quantized notation,

$$\mathcal{H} = \frac{1}{2}\sum_{m_1,\ldots,m_4} V_{m_1,m_2;m_3,m_4} a^\dagger_{m_1} a^\dagger_{m_2} a_{m_4} a_{m_3}, \tag{15}$$

can be represented by the following Hamiltonian in CF representation [6]:

$$\mathcal{H} = \frac{1}{2}\sum_{\substack{m_1,\ldots,m_4\\ n_1,n_2}} V_{m_1,m_2;m_3,m_4} c^\dagger_{n_1 m_1} c^\dagger_{n_2 m_2} c_{m_4 n_2} c_{m_3 n_1}. \tag{16}$$

That this is possible can be seen because by mapping bilinear $a^\dagger_p a_q$ into $\sum_k c^\dagger_{kp} c_{qk}$, we are preserving the basic algebra of fermionic (electron) bilinears, but what may happen is that new (in the enlarged space) operators (including the Hamiltonian) can map physical states into superpositions of physical and unphysical states. Thus we have to ensure that physical states are mapped into physical (sub)space: the Hamiltonian has to commute with the constraint(s) (that determine the physical subspace of the enlarged space). From (3) we see that in this case the constraint that defines the physical space is $\rho^R_{nn} = 1$ and $[\mathcal{H}, \rho^R_{nn}] = 0$. In the inverse space [6],

$$\mathcal{H} = \frac{1}{2}\int \frac{d\boldsymbol{q}}{(2\pi)^2} \tilde{V}(|\boldsymbol{q}|) : \rho^L(\boldsymbol{q})\rho^L(-\boldsymbol{q}) :, \tag{17}$$

we have $[\rho^L_{\boldsymbol{q}}, \rho^R_{\boldsymbol{q}'}] = 0$, and thus $[\mathcal{H}, \rho^R_{\boldsymbol{q}}] = 0$, as required for the constraint $\rho^R_{\boldsymbol{q}} = 0$. In the following section, where we study electron systems at half-fillings, the constraints will not be so simple, and we have to ensure the commutation with $\mathcal{H}$ at least in the physical subspace (with the help of constraints).

B. Electron system of (an isolated) half-filled Landau level

1. Physical states

In the case of the electron system at half-filling, the correlation hole, i.e., the superposition of two Laughlin quasihole constructions, if considered as an independent and well-defined degree of freedom, should carry bosonic statistics. This is a departure from the simple introduction of the unphysical degrees of freedom in the case of the bosonic system at filling factor 1. These degrees of freedom, in that case, carry fermionic statistics and are entering the description via the simple constraint $\rho^R_{nn} = 1$ that ensures easy implementation of the SU(N) invariance; we can transform the basis only in the R sector, and the physical state will be invariant with respect to that particular transformation. Thus, as expected and required, if we are transforming a physical state, we have a usual, unitary implementation of SU(N), and only L degrees of freedom are affected.

We may wonder if it is possible at all to implement the SU(N) invariance if we have bosonic unphysical (additional) degrees of freedom that enter the CF description. If we do not consider CFs and CHs (to account for the PH symmetry in a half-filled Landau level) and a Dirac-type theory [10], we can attempt a description that is similar to that in the bosonic $\nu = 1$ case by considering only CFs, more precisely single-particle operators with two indexes, c_{mn}, but only of one kind, and the following associated physical states (Slater determinants) in the enlarged space:

$$|\Psi_{\rm phy}\rangle = |n_1, \ldots, n_{N_\phi/2}\rangle = \sum_{\sigma\in S_{N_\phi/2}}{}' c^\dagger_{\sigma(m_1)n_1} \cdots c^\dagger_{\sigma(m_{N_\phi/2})n_{N_\phi/2}} |0\rangle, \tag{18}$$

where the prime over the sum means that the sum is over permutations of distinct states, $m_1 \neq m_2 \neq \cdots \neq m_{N_\phi/2}$, i.e., indexes connected with a basis in a LL: $\{|n_1\rangle, |n_2\rangle, \ldots, |n_{N_\phi/2}\rangle, |m_1\rangle, |m_2\rangle, \ldots, |m_{N_\phi/2}\rangle\}$. In (18) we have electrons that occupy states from subspace $V = \{|n_1\rangle, |n_2\rangle, \ldots, |n_{N_\phi/2}\rangle\}$, i.e., half of the available states in a LL. The second half, $V_\perp = \{|m_1\rangle, |m_2\rangle, \ldots, |m_{N_\phi/2}\rangle\}$ (states from the orthogonal subspace), are occupied by "hard-core" bosonic correlation holes.

We may consider both bosonic correlation holes and real (associated with a Chern band, i.e., an LL) fermionic holes. When we talk about PH symmetry, we refer to fermionic holes. But due to the implied constraint, in Eq. (18), on the density and occupation by correlation holes, their densities (occupation numbers) are constrained to be equal, and in that sense the PH symmetry (active exchange of electrons and real holes) can be associated with the exchange of L and R (correlation hole) in the Hamiltonian and constraint(s) that are expressed via densities.

In Appendix A we discuss how the SU(N) invariance can be implemented if one consider simultaneous transformation(s) on L and R; a transformation on only one type of index is nontrivial (nonunitary). In an isolated half-filled LL

we have PH symmetry, and in that case we may expect that the change of basis affects both L (particle) and R (hole), because of the intertwined physics of particles and (real) holes. This is unavoidable, and it is the only option we have both in the Dirac (manifestly invariant PH symmetric with CFs and CHs) or CF-only representation and the description of the problem discussed here.

2. *Hamiltonian and constraints*

The realization of the SU(N) symmetry and the quasiparticle description relies on the requirement that

$$\rho^L_{nn} + \rho^R_{nn} = 1, \tag{19}$$

i.e., the hard-core constraint that we introduced at the beginning of the subsection in the description of the basic physical states (18). The constraint is part of the formulation of the problem; it specifies the half-filling condition. What is special with respect to the previous introduction of the additional degrees of freedom that should represent correlation holes is that here correlation holes describe hole degrees of freedom of the half-filled problem. In that sense, our approach can be considered as only effective, not microscopic, considering how the quasiparticle description is introduced (with respect to the bosonic case at $\nu = 1$). But once the constraint is assumed, we can build our description, i.e., an effective Hamiltonian, by requiring that the Hamiltonian commutes with the constraint, as we will describe shortly. On the other hand, there are reasons why such a constraint is appropriate. First, it includes particle and hole degrees of freedom (CFs and CHs) in such a way that the PH symmetry is accounted for. Second, it leads to a PH symmetric form of the Hamiltonian, which is known as a dipole representation of the underlying quasiparticle physics introduced by Shankar and Murthy, but with an additional factor of 4 that reduces the strength of the Coulomb interaction. That factor is likely the one that was missing in the interpretation of surface acoustic wave experiments by the HLR theory and the dipole-based theory [8]. An additional and important reason for the form of the constraint, as we will show in the following, is that the constraint can be used to obtain a final form of the Hamiltonian that features the boost invariance. The boost invariance should characterize an effective description at least at the Fermi level as an invariance in the system that does not have (bare) mass (i.e., a kinetic term) in its microscopic description. As we will show, the calculated Fermi liquid parameter, F_2, on the basis of that Hamiltonian and constraint, is in very good agreement with the numerical experiment of Ref. [11].

To get the form of the Hamiltonian that satisfies these requirements, we start with the microscopic form of the Hamiltonian [the same as for bosons in (17)] where we abandon the requirement for normal ordering and make the following substitution:

$$\rho^L(\boldsymbol{q}) \to \frac{\rho^L(\boldsymbol{q}) - \rho^R(\boldsymbol{q})}{2}. \tag{20}$$

Therefore,

$$H = \frac{1}{8}\int \frac{d\boldsymbol{q}}{(2\pi)^2}\tilde{V}(|\boldsymbol{q}|)[\rho^L(-\boldsymbol{q}) - \rho^R(-\boldsymbol{q})][\rho^L(\boldsymbol{q}) - \rho^R(\boldsymbol{q})]. \tag{21}$$

Here $\tilde{V}(|\boldsymbol{q}|) = \frac{1}{|\boldsymbol{q}|}\exp(-\frac{|\boldsymbol{k}|^2}{2})[L_n(\frac{|\boldsymbol{k}|^2}{2})]^2$, where L_n represents the Laguerre polynomial for a fixed LL index n. Thus we modified the Hamiltonian in the particle representation to the one that features PH symmetry in such a way that the exchange $\rho^L(\boldsymbol{q}) \leftrightarrow \rho^R(\boldsymbol{q})$ does not change the form of the Hamiltonian. Also,

$$[H, [\rho^L(\boldsymbol{q}) + \rho^R(\boldsymbol{q})]]|_{[\rho^L(\boldsymbol{k})+\rho^R(\boldsymbol{k})]=0} = 0, \tag{22}$$

i.e., the constraint commutes with the Hamiltonian in the physical space, as required.

The Hamiltonian in (21) possesses a single-particle term, H_1 [12],

$$H = H_1 + :H:, \tag{23}$$

next to the purely interacting term, $:H:$. If we interpret the mass in H_1 at k_F as the effective mass, m^*, of a FL description, we need an additional interaction, a term next to the bare one, i.e., $:H:$ in order to achieve (a) the FL description of the system, and (b) the description that is also invariant under boosts, i.e., whose Hamiltonian is purely interacting at the Fermi level. We will come back to these requirements with more explanations below the final form of the Hamiltonian in Eq. (27).

To implement this, we may add a term that needs to represent an interaction, but at the same time be equal to zero in the physical space (not to add or spoil energetics encoded in H based on the bare, Coulomb interaction). In the inverse space, that term may be of the following form:

$$\int \frac{d\boldsymbol{q}}{(2\pi)^2}\, C(|\boldsymbol{q}|)\exp\left(-\frac{\boldsymbol{q}^2}{2}\right)\left[L_n\left(\frac{\boldsymbol{q}^2}{2}\right)\right]^2 \times [\rho^L(-\boldsymbol{q}) + \rho^R(-\boldsymbol{q})][\rho^L(\boldsymbol{q}) + \rho^R(\boldsymbol{q})]. \tag{24}$$

On the other hand, in the space of the LL orbitals, we may consider the following term:

$$\sum_{n,n'} \delta_{n,n'}\left(\rho^L_{nn} + \rho^R_{nn}\right)\left(\rho^L_{n'n'} + \rho^R_{n'n'}\right), \tag{25}$$

based on the constraint expressed on the space of orbitals $\{|n\rangle\}$, which is not zero but, due to the constraint, simply a constant in the physical space, i.e., a constant equal to the number of orbitals. By comparing two expressions that constrain the form of the required interaction term, we can conclude that $C(|\boldsymbol{q}|)$ in (24) should be a constant, independent of $\boldsymbol{q}$. Namely, if we regularize the expression in (25) in the thermodynamic limit by omitting the terms that do not conserve momentum in the inverse space, and may represent local single-particle potentials, we find that the remaining term that represents an interaction invariant under translation is

$$H_C = C\int \frac{d\boldsymbol{q}}{(2\pi)^2}\, \exp\left(-\frac{\boldsymbol{q}^2}{2}\right)\left[L_n\left(\frac{\boldsymbol{q}^2}{2}\right)\right]^2 \times [\rho^L(-\boldsymbol{q}) + \rho^R(-\boldsymbol{q})][\rho^L(\boldsymbol{q}) + \rho^R(\boldsymbol{q})], \tag{26}$$

i.e., a δ-function interaction projected into an isolated LL. This term is equal to zero in the physical space.

The complete Hamiltonian that describes the low-energy physics at the Fermi level and incorporates the boost

invariance is

$$\mathcal{H}_C = H + H_C, \tag{27}$$

where the constant C is chosen such that the second derivative with respect to momentum of the total single-particle dispersion in the single-particle term of $\mathcal{H}_C$ at the Fermi level is equal to zero.

In short, the FL assumption and assertion is that the effective physics can be expressed via quasiparticle excitations near the Fermi surface with energy,

$$\tilde{\epsilon}(\boldsymbol{p}) = \frac{\boldsymbol{p}^2}{2m^*} + \int \frac{d\boldsymbol{p}'}{(2\pi)^2} f(\boldsymbol{p}, \boldsymbol{p}') n(\boldsymbol{p}'), \tag{28}$$

where m^* is the effective quasiparticle mass, f is the effective interaction among quasiparticles, and $n(\boldsymbol{p})$ is the occupation number at momentum $\boldsymbol{p}$ of quasiparticles. To identify f in the scope of our approximation, we will consider for f the interaction based on the Fock approximation and approach to $\mathcal{H}_C$ (which will coincide with the bare interaction). Furthermore, as usual in the FL description, we are concerned with excitations near the Fermi surface, and we assume that f is only a function of the directions of $\boldsymbol{p}$ and $\boldsymbol{p}'$. Thus we consider FL parameters,

$$F_l = \frac{m^* f_l}{2\pi}, \tag{29}$$

where

$$f_l = \frac{1}{2\pi} \int_0^{2\pi} d\theta f(\theta) \exp(il\theta). \tag{30}$$

The very important aspect of the FL description is the connection between the bare mass of underlying particles $m_b \equiv m_e$ and the quasiparticle description. We may use the Galilean invariance under boost by momentum $\boldsymbol{q}$, which induces the change in the energy density equal to $n_e \frac{|\boldsymbol{q}|^2}{2m_b}$, and we equate this energy to that implied by the FL description, to obtain (see, for example, [8])

$$\frac{1}{m_b} = \frac{1}{m^*} + \frac{f_1}{2\pi}. \tag{31}$$

Thus, in a FL description, to have a consistent theory, we need to have a special (additional) interaction defined by f_1 that will compensate for the renormalization of the mass by interaction (from m_b to m^*) in order to reach a quasiparticle picture. Furthermore, in our case, in the microscopic underlying (electron) description, there is no single-particle term (and bare mass m_b) in the solely interacting (microscopic) Hamiltonian defined inside a LL. To ensure the boost invariance in the purely interacting theory, we need

$$0 = \frac{1}{m^*} + \frac{f_1}{2\pi} \quad \text{(i.e., } F_1 = -1\text{)}. \tag{32}$$

Thus the mass m^* is expressed and defined solely by the Landau interaction (parameter). Concretely, in our case, the introduced H_C is necessary to complete the effective FL description near the Fermi surface. It plays the role of f_1, an extra, necessary interaction. As already emphasized, it also (by a definite value of constant C) eliminates the (total) mass term at the Fermi level in the Fock approximation. Thus, in our case, the Landau interaction f is given in the Fock approximation by the interaction defined by the normal ordering of $\mathcal{H}_C$ for the interacting particles at the Fermi level.

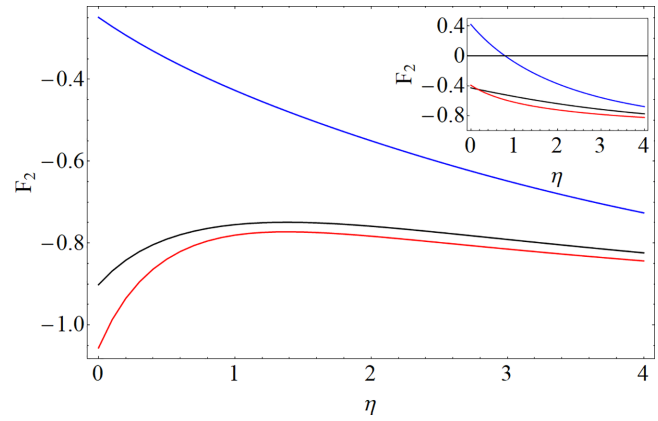

FIG. 1. Fermi liquid parameter F_2 as defined in Eq. (34) with (35), in the case of the LLL (blue), the second LL (black), and the third LL (red). The inset shows the calculated values for F_2 if the interaction with C_η is absent in (35).

To probe the FL description implied by $\mathcal{H}_C$, we consider a generalized Coulomb interaction [11],

$$V_\eta(q) = \frac{1}{|\boldsymbol{q}|} \exp(-|\boldsymbol{q}|\eta), \tag{33}$$

which models the effect of the finite thickness of samples in experiments, but also stabilizes FL behavior [11]. The calculated F_2—the FL parameter at angular momentum $l = 2$—on the basis of the effective Hamiltonian $\mathcal{H}_C$ (in the Fock approximation) is given in Fig. 1. We calculated

$$F_2 = -\frac{\int d\theta \cos(2\theta) f(\theta)}{\int d\theta \cos(\theta) f(\theta)}, \tag{34}$$

where $f(\theta)$ is defined by the total interaction at $\boldsymbol{q} = \boldsymbol{p} - \boldsymbol{p}'$, for which $|\boldsymbol{p}| = |\boldsymbol{p}'| = k_F = 1$, $\boldsymbol{p}' = \hat{e}_x$, $\boldsymbol{p} \cdot \boldsymbol{p}' = \cos(\theta)$, and thus $|\boldsymbol{q}| = 2|\sin(\frac{\theta}{2})|$ with

$$f(\theta) = \left[V_\eta(|\boldsymbol{q}|) \sin^2\left(\frac{\sin(\theta)}{2}\right) + 8C_\eta \cos^2\left(\frac{\cos(\theta)}{2}\right) \right] \times \exp\left(-\frac{\boldsymbol{q}^2}{2}\right) \left[L_n\left(\frac{\boldsymbol{q}^2}{2}\right) \right]^2, \tag{35}$$

where C_η is the value of C at a particular η necessary to eliminate the mass term at k_F. The calculated F_2 (Fig. 1) is in an agreement with the trends of the numerical experiment of Ref. [11]; it predicts the absence and presence of the Pomeranchuk instability in the LLL and third LL, respectively, for the pure Coulomb interaction, as in Ref. [11], but it does not predict one in the second LL, which exists according to the analysis in [11]. In the inset of Fig. 1, the values of F_2 are given for the usual Hamiltonian H in the dipole representation, in the absence of the single-particle term, i.e., for $:H:$. We can see that they differ considerably from the expectations based on $\mathcal{H}_C$ and the results of the numerical experiment in [11]; the inclusion of the C-interaction together with the requirement for boost invariance is essential to get an agreement with the

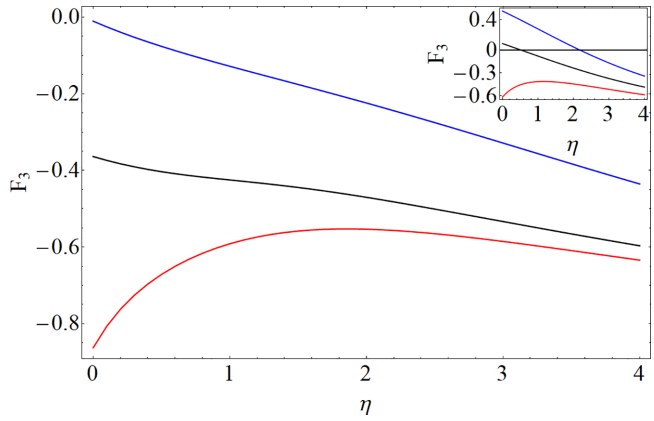

FIG. 2. Fermi liquid parameter F_3 as defined in Eq. (34) with (35), in the case of the LLL (blue), the second LL (black), and the third LL (red). The inset shows the calculated values for F_3 if the interaction with C_η is absent in (35).

results of Ref. [11]. In Figs. 2 and 3 we present the values of Fermi liquid parameters, F_3 and F_4.

III. THE BILAYER CASE: PAIRING BETWEEN TWO LLLs

To get a better understanding of the underlying quasiparticle—a dipole in the new representation—we consider the quantum bilayer problem, i.e., the problem with two half-filled LLLs. The bilayer consists of two layers. Each layer represents a half-filled LLL, but with no PH symmetry, because of the interlayer interaction. The two-component physics can be represented inside a single LLL by the following Hamiltonian with electron density operators, $\rho_\sigma(\boldsymbol{q})$, $\sigma = \uparrow, \downarrow$:

$$\mathcal{H}_e = \int \frac{d\boldsymbol{q}}{(2\pi)^2} \left\{ \sum_\sigma \frac{1}{2} V(|\boldsymbol{q}|) \; :\rho_\sigma(\boldsymbol{q})\, \rho_\sigma(-\boldsymbol{q}): + V_{\uparrow\downarrow}(|\boldsymbol{q}|)\, \rho_\uparrow(\boldsymbol{q})\, \rho_\downarrow(-\boldsymbol{q}) \right\}. \tag{36}$$

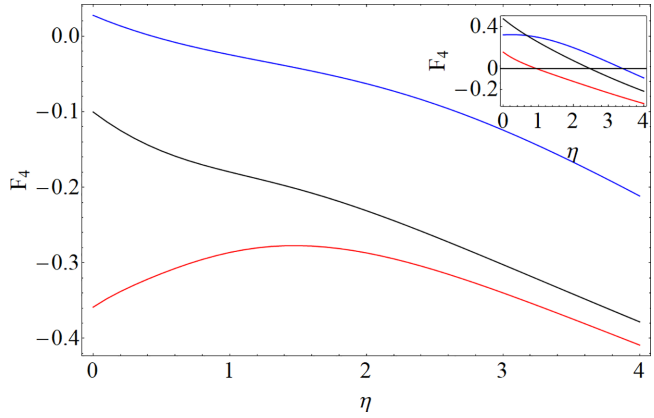

FIG. 3. Fermi liquid parameter F_4 as defined in Eq. (34) with (35), in the case of the LLL (blue), the second LL (black), and the third LL (red). The inset shows the calculated values for F_4 if the interaction with C_η is absent in (35).

We begin the description of the system in the new representation with the following Hamiltonian:

$$\mathcal{H} = \int \frac{d\boldsymbol{q}}{(2\pi)^2} \left\{ \sum_\sigma \frac{1}{8} V(|\boldsymbol{q}|) \left[\rho_\sigma^L(-\boldsymbol{q}) - \rho_\sigma^R(-\boldsymbol{q})\right] \times \left[\rho_\sigma^L(\boldsymbol{q}) - \rho_\sigma^R(\boldsymbol{q})\right] + \frac{V_{\uparrow\downarrow}(|\boldsymbol{q}|)}{4} \left[\rho_\uparrow^L(-\boldsymbol{q}) - \rho_\uparrow^R(-\boldsymbol{q})\right]\left[\rho_\downarrow^L(\boldsymbol{q}) - \rho_\downarrow^R(\boldsymbol{q})\right] \right\}, \tag{37}$$

with constraints $[\rho_\sigma^L(\boldsymbol{k}) + \rho_\sigma^R(\boldsymbol{k})] = 0; \sigma = \uparrow, \downarrow$.

We chose the form of the Hamiltonian as the one that will conform to the requirements, $[\mathcal{H}, [\rho_\sigma^L(\boldsymbol{q}) + \rho_\sigma^R(\boldsymbol{q})]] = 0; \sigma = \uparrow, \downarrow$, that hold if the constraints $[\rho_\sigma^L(\boldsymbol{k}) + \rho_\sigma^R(\boldsymbol{k})] = 0; \sigma = \uparrow, \downarrow$ are applied. The constraints define physical spaces in two layers, and they also constrain the form of the Hamiltonian in the description with enlarged space(s). Just as in the single-layer case, we can add the terms that are zero on the physical space of the form of H_C in each layer to ensure the boost invariance of the system at the Fermi level. Our main interest is the effect of the interlayer interaction. To get a transparent representation of the underlying physics, within the physical space, we can add effectively zero terms, and transform the operators that define the interlayer interaction in the following way:

$$\begin{aligned} &\left[\rho_\uparrow^L(-\boldsymbol{q}) - \rho_\uparrow^R(-\boldsymbol{q})\right]\left[\rho_\downarrow^L(\boldsymbol{q}) - \rho_\downarrow^R(\boldsymbol{q})\right] \\ &\quad - \left[\rho_\uparrow^L(-\boldsymbol{q}) + \rho_\uparrow^R(-\boldsymbol{q})\right]\left[\rho_\downarrow^L(\boldsymbol{q}) + \rho_\downarrow^R(\boldsymbol{q})\right] \\ &\quad = -2\left[\rho_\uparrow^L(-\boldsymbol{q})\, \rho_\downarrow^R(\boldsymbol{q}) + \rho_\uparrow^R(-\boldsymbol{q})\, \rho_\downarrow^L(\boldsymbol{q})\right]. \end{aligned} \tag{38}$$

Now the effective form of the interlayer interaction represents a view of the underlying physics: excitonic binding of electrons and holes, i.e., CFs and CHs as emphasized in the recent work in Ref. [13], which we know is a completely justified view of the physics at small distances. But here we have only one effectively neutral quasiparticle operator "c" which is CF and CH at the same time—a simple dipole; the excitonic pairing that is implied by (38) is effectively Cooper pairing of the underlying quasiparticles "c" from each layer. Indeed, in the BCS mean-field treatment, we have an obvious instability described by the order parameter $\Delta_{\boldsymbol{k}}$,

$$\Delta_{\boldsymbol{k}} = \int \frac{d\boldsymbol{q}}{(2\pi)^2} V_{\uparrow\downarrow}(|\boldsymbol{q} - \boldsymbol{k}|) \frac{\Delta_{\boldsymbol{q}}}{2E_{\boldsymbol{q}}}, \tag{39}$$

where

$$V_{\uparrow\downarrow}(q) = \frac{\exp(-qd)}{q} \exp\left(-\frac{q^2}{2}\right), \tag{40}$$

and $E_{\boldsymbol{q}}$ is the Bogoliubov quasiparticle energy, defined considering the complete Hamiltonian that contains the boost invariance, and d is the distance between the layers.

In Fig. 4, solutions for $\Delta_{\boldsymbol{k}}$ at $k = k_F$ for s, p, and d waves are plotted as a function of distance. For a large interval of d, the results are in qualitative agreement with the most recent numerical results of Ref. [13], done on the sphere, when we identify CF-CH (excitonic) pairing of the same reference with the s-wave Cooper pairing of dipoles in the representation that we presented here. Nevertheless, a question can be raised

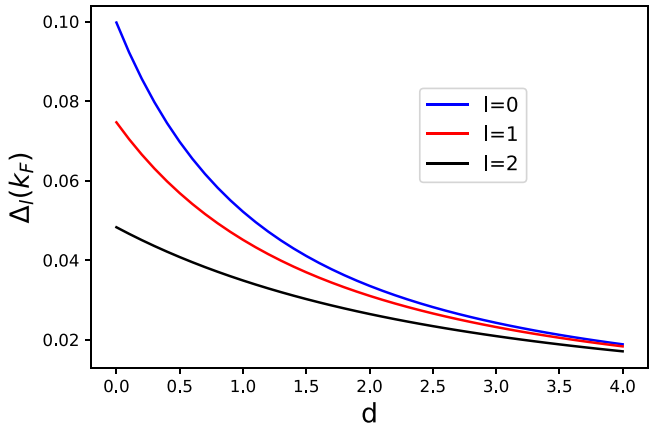

FIG. 4. The solutions for Δ_k calculated self-consistently using (39), in the case of s (blue), p (red), and d (black) wave at $k = k_F$.

whether for larger (or even intermediate, $d \sim l_B$) distances, an effective description given by the Hamiltonian in (37) is more appropriate, because at very large distances we expect two decoupled Fermi liquids, for which a dipole representation, with dipole densities $\rho^L_\sigma(\boldsymbol{q}) - \rho^R_\sigma(\boldsymbol{q}), \sigma = \uparrow, \downarrow$ that are interacting, seems quite appropriate (according to our reasoning and comments in Sec. II B). Then the mean-field solutions $\Delta^s_{\boldsymbol{k}}$ for the dominant s-wave excitonic instability, for this setup, will necessarily behave near $k = 0$ as $\Delta^s_{\boldsymbol{k}} \sim |\boldsymbol{k}|^2$, and thus they will exemplify an anomalous behavior and may lead to the prediction of a new (intermediate) phase detected in numerical experiments on a torus [14,15].

IV. DISCUSSION AND CONCLUSIONS

To further gauge the FL nature of our system (in the proposed formalism), we can consider the quantum Boltzmann equation for the Wigner function,

$$\nu(\boldsymbol{k},\boldsymbol{r}) = \int d\boldsymbol{s}\, \exp(i\boldsymbol{k}\boldsymbol{s}) \mathrm{Tr}\left\{\rho \Psi^\dagger\left(\boldsymbol{r}+\frac{\boldsymbol{s}}{2}\right)\Psi\left(\boldsymbol{r}-\frac{\boldsymbol{s}}{2}\right)\right\}, \tag{41}$$

where ρ is the density matrix of the system, and $\Psi, \Psi^\dagger$ are second-quantized operators that are defined, in the long-distance approximation (in the usual way) on the space of commuting coordinates, as

$$\Psi(\boldsymbol{x}) = \int \frac{d\boldsymbol{k}}{(2\pi)^2} \exp(i\boldsymbol{k}\boldsymbol{x}) c_{\boldsymbol{k}},$$
$$\Psi^\dagger(\boldsymbol{x}) = \int \frac{d\boldsymbol{k}}{(2\pi)^2} \exp(-i\boldsymbol{k}\boldsymbol{x}) c^\dagger_{\boldsymbol{k}}. \tag{42}$$

Applying $i\frac{\partial\rho}{\partial t} = [\mathcal{H}_C, \rho]$, i.e., the von Neumann equation, we arrive at the following equation for $\nu(\boldsymbol{k},\boldsymbol{r})$:

$$i\frac{\partial \nu(\boldsymbol{k},\boldsymbol{r})}{\partial t} = \int d\boldsymbol{s}\, \exp(i\boldsymbol{k}\boldsymbol{s}) \mathrm{Tr}\left\{\rho\left[\Psi^\dagger\left(\boldsymbol{r}+\frac{\boldsymbol{s}}{2}\right)\Psi \times \left(\boldsymbol{r}-\frac{\boldsymbol{s}}{2}\right), \mathcal{H}_C\right]\right\}. \tag{43}$$

By considering shifts in $\boldsymbol{r}$ in single-particle correlators to linear order, we can derive an effective expression presented in Appendix B. Only in the small-momentum limit, i.e., the limit in which the change of momentum (that couples with the $\boldsymbol{r}$ coordinate) is small, does the expression take the form of the usual quantum Boltzmann equation for a description of a FL with a boost invariance,

$$\frac{\partial \nu}{\partial t} + \partial_{\boldsymbol{r}}\nu\partial_{\boldsymbol{k}}\epsilon - \partial_{\boldsymbol{k}}\nu\partial_{\boldsymbol{r}}\epsilon = \text{additional (negligible insmall-momentum transfer) terms.} \tag{44}$$

Here

$$\epsilon = -\int d\boldsymbol{q} \exp\left(-\frac{\boldsymbol{q}^2}{2}\right)\left(\frac{V(\boldsymbol{q})}{4} - 2C\right)(\boldsymbol{q}\times\boldsymbol{k})^2 \nu(\boldsymbol{k}+\boldsymbol{q}), \tag{45}$$

i.e., the Fock contribution to the quasiparticle dispersion that also includes the C-interaction contribution, i.e., the interaction defined in (26). The C-interaction ensured that no term with a finite mass ($= \frac{1}{M}\boldsymbol{k}\nabla_{\boldsymbol{r}}\nu$) appears in the quantum Boltzmann equation and on the Fermi level to the order that was considered. Overall, the description based on the dipole representation is in accordance with Ref. [16] and the quantum Boltzmann equation for CFs in the absence of the projection to a LL (based on the Chern-Simons field-theoretical approach), if we associate the processes behind smooth variations of the Fermi surface with small-momentum transfer variations in the dipole representation that lead to FL behavior. We leave a detailed comparison and analysis of the quantum Boltzmann equation for future work.

Thus the C-interaction ensures the boost invariance even if we go beyond the ordinary (Hartree)-Fock mean-field approach. Our description incorporates also the PH symmetry, and it does not contain a potential bias to CFs like the use of the Rezayi-Read state [17] in [11]. Thus the discrepancy between our prediction for the absence of the Pomeranchuk instability in the second LL with respect to Ref. [11] may come from the explicit PH symmetry breaking in their analysis. The PH symmetry breaking can be associated with LL mixing, which may drive the Pomeranchuk instability and also stabilize p-wave at larger distances in the bilayer system.

Our proposal maintains the PH symmetry with the application of the constraint in (19). At the same time, it places correlation holes in the place of real holes, which seems to be an unusual circumstance—correlation holes should bind to particles (or vice versa). But this is a reflection of the effective physics at the Fermi level where particles are shifted from their correlation holes by an amount proportional to $k_F l_B^2 = l_B$, i.e., on the order of the average distance between LL orbitals. This is an important consequence of the projection to a fixed LL of the Fermi sea correlations among quasiparticles [17].

Therefore, our proposal takes into account the requirements for the boost invariance and PH symmetry in the theory that captures the effective physics at the Fermi level of dipole quasiparticles.

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APPENDIX A: THE SU(N) INVARIANCE IN THE HALF-FILLED CASE

In this Appendix, we will discuss the action of the SU(N) symmetry in the theory for the half-filled LL. The symmetry must exist, but, as we will describe, not as an independent transformation on L and R indexes. An arbitrary, fixed SU(N) transformation on one type of index only is not unitary. However, we show that, in principle, we can complete a transformation, defined by the same SU(N) matrix on the other index, and after a projection to the physical space we can reach a unitary realization of the SU(N) symmetry. In describing this action, we want to show how the transformations on both sectors, i.e., the indexes, are intertwined, and we provide a description of the unitary realization of the SU(N) symmetry on the physical space described by Eq. (A9). On the other hand, we can regard (A9) as an expected, natural (unitary) realization of the SU(N) symmetry on the space defined by Eq. (18), in which only one set of indexes (L or R) is independent and defines the other.

We may ask, "How can the SU(N) invariance be implemented on the states defined in Eq. (18)?" First, let us consider an artificial but instructive problem of electrons at $\nu = 1$ in a CF representation. The unique physical state can be defined in an enlarged space by a trace on bosonic, artificial degrees of freedom,

$$|\Psi_{\rm phy}\rangle^{\nu=1} = {\sum_{\sigma\in S_{N_\phi}}}' c^\dagger_{\sigma(m_1)n_1}\cdots c^\dagger_{\sigma(m_{N_\phi})n_{N_\phi}}|0\rangle, \tag{A1}$$

and $m_1 \neq m_2 \neq \cdots \neq m_{N_\phi}$. Thus bosonic (artificial) degrees of freedom enter as hard-core bosons into the description. The requirement is a necessary condition for the implementation of the SU(N) invariance, and it reflects a physical expectation that correlation holes for a fermionic system should not overlap. Namely, by introducing

$$c^\dagger_{mn} \to \sum_{m'}^{N_\phi} U_{mm'} c^\dagger_{m'n}, \tag{A2}$$

we may notice that under the hard-core constraint, the SU(N) transformation in the R sector will act locally, i.e., it will induce the permanent number of the SU(N) matrix that will be multiplied by the same state:

$$\hat{g}^R|\Psi_{\rm phy}\rangle^{\nu=1} = \Lambda(g)|\Psi_{\rm phy}\rangle^{\nu=1}. \tag{A3}$$

The SU(N) invariance exists if its action is unitary under simultaneous transformations in the R and L sectors. Thus by allowing a nonunitary (in general) action on L degrees of freedom,

$$\hat{g}^L|\Psi_{\rm phy}\rangle^{\nu=1} = \frac{1}{\Lambda(g)}|\Psi_{\rm phy}\rangle^{\nu=1}, \tag{A4}$$

we can reach the invariance:

$$\hat{g}_{{\rm SU}(N)}|\Psi_{\rm phy}\rangle^{\nu=1} = \hat{g}^L\hat{g}^R|\Psi_{\rm phy}\rangle^{\nu=1} = |\Psi_{\rm phy}\rangle^{\nu=1}. \tag{A5}$$

The previous case is artificial and of no physical importance, but it suggests how the SU(N) invariance can be accommodated in the system of interest, namely a half-filled LL of electrons (if we first apply an independent transformation on the indexes on one of the sectors). We may begin with the usual transformation on R indexes as in (A2). As a result of the hard-core constraint among holes, we have

$$\begin{aligned}|\Psi_{\rm phy}\rangle^R = &\sum_{\{m_1'\neq m_2'\neq\cdots\neq m_{N_\phi/2}'\}}\ \sum_{\sigma\in S_{N_\phi/2}}\\ &\times\left[{\sum_{p\in S_{N_\phi/2}}}' U^{p(m_1)\sigma(m_1')}\cdots U^{p(m_{N_\phi/2})\sigma(m_{N_\phi/2}')}\right]\\ &\times c^\dagger_{\sigma(m_1')n_1}\cdots c^\dagger_{\sigma(m_{N_\phi/2}')n_{N_\phi/2}}|0\rangle,\end{aligned} \tag{A6}$$

where the first sum is over all possible distinct collections of $N_\phi/2$ numbers, i.e., basis vectors. We can denote the number in square brackets by $[\cdots] = \Lambda(\{m_i'\}) = K(\{n_i'\})$, where $\{n_i'\}$ denote basis states from the subspace orthogonal to the one spanned by $\{m_i'\}$. ($|\Psi_{\rm phy}\rangle$ is defined by the set of $\{n_i\}$'s [or $\{m_i\}$'s; see Eq. (18)] and they fix Λ's.) The number $\Lambda(\{m_i'\})$ is symmetric under permutations of $\{m_i'\}$ and can be pulled out of the sum over σ permutations. Thus

$$\begin{aligned}\hat{g}^R|\Psi_{\rm phy}\rangle \equiv |\Psi_{\rm phy}\rangle^R &= \sum_{\{m_1'\neq m_2'\neq\cdots\neq m_{N_\phi/2}'\}}\Lambda(\{m_i'\})\\ &\times\sum_{\sigma\in S_{N_\phi/2}} c^\dagger_{\sigma(m_1')n_1}\cdots c^\dagger_{\sigma(m_{N_\phi/2}')n_{N_\phi/2}}|0\rangle,\end{aligned} \tag{A7}$$

where the resulting state may not be in the physical space, which is spanned by vectors in Eq. (18). Now if we define an SU(N) transformation on the L indexes, in such a way that for each term $\{n_i\} \to \{n_i'\}$ in the generated expansion we divide by $K(\{n_i'\})$, i.e.,

$$\begin{aligned}\hat{g}^L|\Psi_{\rm phy}\rangle \equiv |\Psi_{\rm phy}\rangle^L &= \sum_{\{n_1'\neq n_2'\neq\cdots\neq n_{N_\phi/2}'\}}\frac{1}{K(\{n_i'\})}U^{n_1'n_1}\cdots U^{n_{N_\phi/2}'n_{N_\phi/2}}\\ &\times\sum_{\sigma\in S_{N_\phi/2}} c^\dagger_{\sigma(m_1)n_1'}\cdots c^\dagger_{\sigma(m_{N_\phi/2})n_{N_\phi/2}'}|0\rangle,\end{aligned} \tag{A8}$$

it follows, under application of all hard-core constraints that define the physical space in the enlarged space, which action we will denote by $P_{\rm hc}$, that

$$\begin{aligned}\hat{g}_{{\rm SU}(N)}|\Psi_{\rm phy}\rangle &= P_{\rm hc}(\hat{g}^L\hat{g}^R|\Psi_{\rm phy}\rangle)\\ &= \sum_{\{n_1'\neq n_2'\neq\cdots\neq n_{N_\phi/2}'\}}U^{n_1'n_1}\cdots U^{n_{N_\phi/2}'n_{N_\phi/2}}\\ &\times\sum_{\sigma\in S_{N_\phi/2}} c^\dagger_{\sigma(m_1')n_1'}\cdots c^\dagger_{\sigma(m_{N_\phi/2}')n_{N_\phi/2}'}|0\rangle.\end{aligned} \tag{A9}$$

Any change of basis in enlarged space (that acts on R and L indexes) is represented in physical states by the unitary implementation (A9), as required and expected. The implementation is unitary and represents an expected expansion on physical states, but it cannot be described as a simple action on L indexes.

APPENDIX B: QUANTUM BOLTZMANN EQUATION

The explicit expression for the right-hand side of Eq. (43) to linear order in shifts in single-particle correlators is

$$
\begin{aligned}
i\frac{\partial \nu(\boldsymbol{k},\boldsymbol{r})}{\partial t} = \int \frac{d\boldsymbol{q}}{(2\pi)^2}\frac{\tilde{V}(\boldsymbol{q})}{4}\{&(\boldsymbol{q}\times\boldsymbol{k})(i\boldsymbol{q}\times\nabla_{\boldsymbol{r}})[\nu(\boldsymbol{k},\boldsymbol{r},t)\nu(\boldsymbol{k}+\boldsymbol{q},\boldsymbol{r},t)] - \frac{1}{4}[i\nabla_{\boldsymbol{k}}(i\boldsymbol{q}\times\nabla_{\boldsymbol{r}})\nu(\boldsymbol{k},\boldsymbol{r},t)][\nabla_{\boldsymbol{r}}(i\boldsymbol{q}\times\nabla_{\boldsymbol{r}})\\
&\times\nu(\boldsymbol{k}+\boldsymbol{q},\boldsymbol{r},t)] + \frac{1}{4}[\nabla_{\boldsymbol{r}}(i\boldsymbol{q}\times\nabla_{\boldsymbol{r}})\nu(\boldsymbol{k},\boldsymbol{r},t)][i\nabla_{\boldsymbol{k}}(i\boldsymbol{q}\times\nabla_{\boldsymbol{r}})\nu(\boldsymbol{k}+\boldsymbol{q},\boldsymbol{r},t)]\\
&+ i(\boldsymbol{q}\times\boldsymbol{k})^2[\nabla_{\boldsymbol{r}}\nu(\boldsymbol{k},\boldsymbol{r},t)\nabla_{\boldsymbol{k}}\nu(\boldsymbol{k}+\boldsymbol{q},\boldsymbol{r},t) - \nabla_{\boldsymbol{k}}\nu(\boldsymbol{k},\boldsymbol{r},t)\nabla_{\boldsymbol{r}}\nu(\boldsymbol{k}+\boldsymbol{q},\boldsymbol{r},t)]\\
&+ (\boldsymbol{q}\times\boldsymbol{k})(i\boldsymbol{q}\times z)[\nu(\boldsymbol{k},\boldsymbol{r},t)\,\nabla_{\boldsymbol{r}}\nu(\boldsymbol{k}+\boldsymbol{q},\boldsymbol{r},t) - \nabla_{\boldsymbol{r}}\nu(\boldsymbol{k},\boldsymbol{r},t)\,\nu(\boldsymbol{k}+\boldsymbol{q},\boldsymbol{r},t)]\}\\
\times\int\frac{d\boldsymbol{q}}{(2\pi)^2}(-2C)\{&(\boldsymbol{q}\times\boldsymbol{k})(i\boldsymbol{q}\times\nabla_{\boldsymbol{r}})[\nu(\boldsymbol{k},\boldsymbol{r},t)\nu(\boldsymbol{k}+\boldsymbol{q},\boldsymbol{r},t)] + i(\boldsymbol{q}\times\boldsymbol{k})^2[\nabla_{\boldsymbol{r}}\nu(\boldsymbol{k},\boldsymbol{r},t)\nabla_{\boldsymbol{k}}\nu(\boldsymbol{k}+\boldsymbol{q},\boldsymbol{r},t)\\
&- \nabla_{\boldsymbol{k}}\nu(\boldsymbol{k},\boldsymbol{r},t)\nabla_{\boldsymbol{r}}\nu(\boldsymbol{k}+\boldsymbol{q},\boldsymbol{r},t)] - \frac{i}{8}[\nabla_{\boldsymbol{k}}\nu(\boldsymbol{k},\boldsymbol{r},t)\nabla_{\boldsymbol{r}}(\boldsymbol{q}\times\nabla_{\boldsymbol{r}})^2\nu(\boldsymbol{k}+\boldsymbol{q},\boldsymbol{r},t) + \nabla_{\boldsymbol{k}}(\boldsymbol{q}\times\nabla_{\boldsymbol{r}})^2\nu(\boldsymbol{k},\boldsymbol{r},t)\nabla_{\boldsymbol{r}}\nu(\boldsymbol{k}+\boldsymbol{q},\boldsymbol{r},t)]\\
&+ \frac{i}{8}[\nabla_{\boldsymbol{r}}\nu(\boldsymbol{k},\boldsymbol{r},t)\nabla_{\boldsymbol{k}}(\boldsymbol{q}\times\nabla_{\boldsymbol{r}})^2\nu(\boldsymbol{k}+\boldsymbol{q},\boldsymbol{r},t) + \nabla_{\boldsymbol{r}}(\boldsymbol{q}\times\nabla_{\boldsymbol{r}})^2\nu(\boldsymbol{k},\boldsymbol{r},t)\nabla_{\boldsymbol{k}}\nu(\boldsymbol{k}+\boldsymbol{q},\boldsymbol{r},t)]\\
&+ \frac{i}{4}[\nu(\boldsymbol{k}+\boldsymbol{q},\boldsymbol{r},t)(\boldsymbol{q}\times\boldsymbol{k})(\boldsymbol{q}\times\nabla_{\boldsymbol{r}})\nu(\boldsymbol{k},\boldsymbol{r},t) - \nu(\boldsymbol{k},\boldsymbol{r},t)(\boldsymbol{q}\times\boldsymbol{k})(\boldsymbol{q}\times\nabla_{\boldsymbol{r}})\nu(\boldsymbol{k}+\boldsymbol{q},\boldsymbol{r},t)]\}. \qquad \text{(B1)}
\end{aligned}
$$

If we neglect $(\boldsymbol{q}\times\nabla_{\boldsymbol{r}})$ with respect to $(\boldsymbol{q}\times\boldsymbol{k})$ and contributions higher in $\nabla_{\boldsymbol{r}}$, and also calculate the overall contribution to the mass term ($\sim \boldsymbol{k}\nabla_{\boldsymbol{r}}$) in the small-$\boldsymbol{q}$ limit, we arrive at the simple form in (44).

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