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Radioactive decay seen as temporal canonical ensemble

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Abstract: The operator of time formalism is applied to radioactive decay. It appears that the proposed approach offers a better insight and understanding of the phenomena in a way that the decay exponential-law becomes the Boltzmann distribution in Gibbs treatment of canonical ensemble. The radioactive decay is seen as temporal canonical ensemble where the radioactive constant appears as the analog of the absolute temperature multiplied by Boltzmann constant. The stochastic character of decay process becomes plausible in the proposed approach and the explanation why decay is characterized by fixed quantity, and not by some parameter, is offered.

Keywords: Time operator; Radioactive decay; Canonical ensemble

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1. Introduction

Perhaps the most intriguing fact about the radioactive decay is that such a process is stochastic in its nature. This means that we are unable to predict when some radioactive system (particle) will decay. We can only say what is the probability that it will decay at the moment t or later, and this is given by the well-known exponential law $e^{-\lambda t}$, where λ is radioactive constant characteristic for the decaying system under consideration [1]. The unpredictability of when the decay will occur is not the only thing that puzzles us regarding the decay. There are questions about what influences decay in general, how, if possible, can we alter λ and can if we can anyhow prepare the systems to decay faster/slower. These topics are going to be addressed in the present article, and this will be done in a manner that is not, up to my knowledge, previously considered.

Let us jump to the conclusion and say that it is not the Hamiltonian that governs the decay, but some other observable. More concretely, the observable that is, so to say, dynamical analog of the time, here denoted by $\hat{G} = G(\hat{q}, \hat{p})$, dictates, by its spectrum, at which moments the particle can decay. This observable appears in the equation that is the time analog of the Schrödinger equation [2], so it appears to be, for the decay, what the Hamiltonian is for the canonical ensemble.

In order to show how the decay process can be treated as a temporal canonical ensemble, the formalism of the operator of time, that we have proposed in [2–6], will be used. There is a whole variety of topics and approaches related to the operator of time, e.g., [7–9] and references therein. Our approach is similar to the one in [10], and references therein, and [11], and its crucial point is to treat the time and energy like the coordinate and momentum are usually treated. This means that another Hilbert space, where the operators of time and energy act, is introduced, just as it is done for each degree of freedom in the standard formulation of quantum mechanics. In this way the Pauli's objection, saying that if $\hat{H}$ is bounded from below, which is the case for all physical systems, then there is no self-adjoint operator of time conjugate to $\hat{H}$ in the sense of the Weyl commutation relations, is avoided. Then, the same commutation relation that holds for the coordinate and momentum is imposed for the energy and time, which leads to unbounded spectrum of these operators. Finally, the original and the so-called second Schrödinger equation, that we have introduced in [2], as constraints in the overall Hilbert space select physically meaningful states that have non-negative energy and time.

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2. Operators of time and energy

As it is done for every spatial degree of freedom, a separate Hilbert space $\mathcal{H}_t$, where operators of time $\hat{t}$ and energy $\hat{s}$ act non-trivially, can be introduced. So, for the case of one degree of freedom, there are $\hat{q}\otimes\hat{I}$, $\hat{p}\otimes\hat{I}$, $\hat{I}\otimes\hat{t}$ and $\hat{I}\otimes\hat{s}$, that act in $\mathcal{H}_q\otimes\mathcal{H}_t$, and for these self-adjoint operators the following commutation relations hold:

$$\frac{1}{i\hbar}[\hat{q}\otimes\hat{I},\hat{p}\otimes\hat{I}] = \hat{I}\otimes\hat{I},$$
$$\frac{1}{i\hbar}[\hat{I}\otimes\hat{t},\hat{I}\otimes\hat{s}] = -\hat{I}\otimes\hat{I}.$$

The other commutators vanish. The operators of time $\hat{t}$ and energy $\hat{s}$ have continuous spectrum $\{-\infty,+\infty\}$, just as the operators of coordinate and momentum $\hat{q}$ and $\hat{p}$ have. The eigenvectors of $\hat{t}$ are $|t\rangle$ for every $t\in\mathbf{R}$. (The question related to the norm and the measurement of $|t\rangle$ is analyzed in [10].) In $|t\rangle$ representation, the operator of energy is given by $i\hbar\frac{\partial}{\partial t}$ and its eigenvectors $|E\rangle$ in the same representation are $e^{\frac{1}{i\hbar}Et}$ for every $E\in\mathbf{R}$. In [3], we have shown how the unbounded spectrum of the operator of energy is regulated by the Schrödinger equation. Let us stress here that the Schrödinger equation, that appears as a constraint in $\mathcal{H}_q\otimes\mathcal{H}_t$, selects physically meaningful states with non-negative energies, due to the bounded from below spectrum of Hamiltonian. In [2], we have introduced the so-called second Schrödinger equation:

$$\hat{t}|\psi\rangle = G(\hat{q},\hat{p})|\psi\rangle. \tag{1}$$

In (1), one demands that the operator of time and its dynamical counterpart $G(\hat{q},\hat{p})$ act equally on the states of quantum mechanical system, just like in the original Schrödinger equation $\hat{s}|\psi\rangle = H(\hat{q},\hat{p})|\psi\rangle$ one demands that the operator of energy and the Hamiltonian, as its dynamical counterpart, act equally on the states of quantum mechanical system. (That this is the Schrödinger equation could be verified by taking its $|q\rangle\otimes|t\rangle$ representation.) As the original Schrödinger equation, the Eq. (1) represents constraint in $\mathcal{H}_q\otimes\mathcal{H}_t$, as well. The typical solution of (1) is $|\psi_t\rangle\otimes|t\rangle$, where $G(\hat{q},\hat{p})|\psi_t\rangle = t|\psi_t\rangle$ and $\hat{t}|t\rangle = t|t\rangle$. It is the time analog of $|\psi_E\rangle\otimes|E\rangle$ that solves the original Schrödinger equation if $H(\hat{q},\hat{p})|\psi_E\rangle = E|\psi_E\rangle$ and $\hat{s}|E\rangle = E|E\rangle$. (In $|q\rangle\otimes|t\rangle$ representation, the last state becomes $\psi_E(q)e^{\frac{-1}{i\hbar}Et}$.)

Due to the Big Bang as the beginning of time, it seems reasonable to assume that $G(\hat{q},\hat{p})$ has bounded from below spectrum, just like the Hamiltonian. In what follows, for the sake of simplicity, we shall assume that $G(\hat{q},\hat{p})$ and $H(\hat{q},\hat{p})$ have discrete eigenvalues t_i and E_i, and the solutions of original and the so-called second Schrödinger equations will be denoted by $|E_i\rangle\otimes|E_i\rangle$ and $|t_i\rangle\otimes|t_i\rangle$, respectively.

3. Results and discussions

In the proposed framework of $\mathcal{H}_q\otimes\mathcal{H}_t$, the well-known statements regarding canonical ensemble are as follows. Suppose the system is characterized by the Hamiltonian of the form $\hat{H} = \sum_i E_i|E_i\rangle\langle E_i|$. The canonical ensemble is the statistical operator:

$$\hat{\rho}_E = \frac{1}{Z}\sum_i e^{-\beta E_i}|E_i\rangle\langle E_i|\otimes|E_i\rangle\langle E_i|. \tag{2}$$

In the last expression, the first $|E_i\rangle\langle E_i|$ is the projector on the eigenstate of Hamiltonian for eigenvalue E_i, that is $|E_i\rangle$ which belongs to $\mathcal{H}_q$, while the second one is the projector on the eigenvector of $\hat{s}$ for the same eigenvalue E_i, that is $|E_i\rangle$ which belongs to $\mathcal{H}_t$. The canonical partition function Z_E is determined by the normalization condition:

$$Z_E = \mathrm{Tr}\sum_i e^{-\beta E_i}|E_i\rangle\langle E_i|\otimes|E_i\rangle\langle E_i|. \tag{3}$$

Usually, β is seen as $(k_BT)^{-1}$, where k_B is the Boltzmann constant and T is the absolute temperature, but β^{-1} can be taken as temperature *per se*.

The Boltzmann distribution $e^{-\beta E_i}$ is a probability distribution over various states $|E_i\rangle\otimes|E_i\rangle$. According to the second law of thermodynamics, the state of equilibrium maximizes the entropy, and maximization of entropy leads to Gibbs prescription for the statistical operator (2), *i. e.*, to Boltzmann distribution. Among all ensembles with the same mean value of $\hat{H}$ or $\hat{s}$, the canonical ensemble has the maximal entropy. The expression relating temperature, internal energy $\langle\hat{H}\rangle$ and entropy $-\langle\hat{\rho}ln\hat{\rho}\rangle$ is:

$$\beta = (k_BT)^{-1} = -\frac{\partial\langle\hat{H}\rangle}{\partial\langle\hat{\rho}ln\hat{\rho}\rangle} = -\frac{\partial\mathrm{Tr}\hat{\rho}\hat{H}}{\partial\mathrm{Tr}\hat{\rho}ln\hat{\rho}}. \tag{4}$$

For the canonical ensemble (2), it holds:

$$[\hat{\rho}_E,\hat{s}] = 0, \tag{5}$$

the meaning of which is that $\hat{\rho}$ does not depend on time:

$$\frac{\partial\hat{\rho}_E}{\partial\hat{t}} = 0, \tag{6}$$

since $\hat{s}$ is the generator of time translation.

On the other side, radioactivity is described by the well-known formula:

$$N(t_i) = N_0e^{-\lambda t_i}. \tag{7}$$

Here, $N(t_i)$ is the number of decaying systems (particles) present at the moment t_i if there were N_0 at the moment t_0. (For the sake of simplicity, we shall consider time as having discrete values t_i.) Equivalent description of the decay process is to ask what is the probability that some system will last until t_i, when the decay occurs, and the corresponding expression for this is:

$$\frac{e^{-\lambda t_i}}{\sum_i e^{-\lambda t_i}}. \tag{8}$$

If $\hat{G}$ has a discrete spectrum, then the solutions of (1) are $|t_i\rangle \otimes |t_i\rangle$. By using the probability distribution (8), one can form a statistical operator:

$$\hat{\rho}_t = \frac{1}{\sum_i e^{-\lambda t_i}} \sum_i e^{-\lambda t_i} |t_i\rangle\langle t_i| \otimes |t_i\rangle\langle t_i|. \tag{9}$$

This statistical operator represents temporal canonical ensemble that describes the decay process. There is a complete analogy between $\hat{\rho}_E$ and $\hat{\rho}_t$, the meaning of which is that as the probability distribution $e^{-\beta E_i}$ determines how likely it is to find the system under consideration having the energy E_i, the probability distribution $e^{-\lambda t_i}$ determines how likely it is to find that the system under consideration will last until the moment t_i. The analogy is complete in the sense that the canonical ensemble and the radioactive decay appear to be essentially the same phenomena for the two conjugate observables $\hat{s}$ and $\hat{t}$. This does not come as a surprise after noticing the similarity between the original and the so-called second Schrödinger equation and the fact that both, $\hat{H}$ and $\hat{G}$, which determine solutions of these equations, have bounded from below spectra. Having this in mind, instead of approaching heuristically as was done here, one can start with $\hat{\rho}_E$, substitute E_i with t_i and β with λ, and by proceeding in the reverse order arrive to the decay formula (7). What was said for the canonical ensemble and its derivation, exactly the same can be said for the temporal canonical ensemble. Namely, among all ensembles with the same mean value of $\hat{G}$ or $\hat{t}$, the temporal canonical ensemble has the maximal entropy, *i. e.*, maximization of entropy leads to the Gibbs prescription for the statistical operator (9).

Obviously, in analogy with the canonical partition function Z_E, one can introduce its temporal counterpart:

$$Z_t = \mathrm{Tr} \sum_i e^{-\lambda t_i} |t_i\rangle\langle t_i| \otimes |t_i\rangle\langle t_i|. \tag{10}$$

Moreover:

$$\lambda = -\frac{\partial\langle\hat{G}\rangle}{\partial\langle\hat{\rho} ln \hat{\rho}\rangle} = -\frac{\partial \mathrm{Tr}\hat{\rho}\hat{G}}{\partial \mathrm{Tr}\hat{\rho} ln\hat{\rho}}, \tag{11}$$

so one can relate the radioactive constant to the mean value of $\hat{G}$. For instance, by using the example given in [2], if $G(\hat{q},\hat{p})$ is:

$$G(\hat{q},\hat{p}) = \frac{\hbar}{m^2c^4}\left(\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega\hat{q}^2\right),$$

then the solutions of the so-called second Schrödinger equation (1) are $|\psi_n\rangle \otimes |t_n\rangle$, $n \in \mathrm{N}$, where $|\psi_n\rangle$ are the well-known solutions of the eigenvalue problem for the Hamiltonian of harmonic oscillator and:

$$t_n = \frac{\hbar^2\omega}{m^2c^4}\left(n + \frac{1}{2}\right).$$

By introducing $d = \frac{\hbar^2\omega}{m^2c^4}$, the relation between λ and $\langle\hat{G}\rangle$ is then:

$$\langle\hat{G}\rangle = \frac{1}{2}\cot h\left(\frac{\lambda d}{2}\right). \tag{12}$$

So, this example shows that by measuring the half-life, one can find the mean value of $\hat{G}$. Let us also mention that if, instead of the non coherent mixture (9), one takes the adequate coherent mixture of involved states, with the coefficients that are square roots of the probabilities, then one gets the same results for measurement of time and all the other compatible observables. Similar argument would hold for the standard canonical ensemble in relation to the measurements of the energy and the compatible observables. However, the coherent mixtures (pure states) and the non coherent mixtures (mixed states) can be distinguished if one measures the observables that, so to say, are not diagonal when represented in the basis of vectors that are coherently mixed.

From the obvious fact that commutator $[\hat{\rho}_t, \hat{t}]$ vanishes, it follows:

$$\frac{\partial\hat{\rho}_t}{\partial\hat{s}} = 0, \tag{13}$$

since $\hat{t}$ is a generator of energy translation. The meaning of this equation is that the radioactive decay, seen as the temporal canonical ensemble, does not depend on the (internal) energy, what is well known from the experience.

As the absolute temperature appears in $\beta = (k_BT)^{-1}$ after the Boltzmann constant is introduced, with the appropriate constant l, one can define the persistence P by:

$$\lambda = (lP)^{-1}.$$

However, there is a difference between the absolute temperature and the persistence (or β and λ). Namely, it is possible to change the temperature of the canonical ensemble by putting it in a contact with the other one. During thermalisation, systems exchange energy. If one system changes its state from the one with energy E_1 to the state with E_2, then the other system can change its state, too. If the difference in the energy between the energy levels of the second system does not match $E_2 - E_1$, then the part of the energy can come from or go to the kinetic energy of the systems under consideration. In this process, system can instantaneously change its momentum. On the other side, similar changes of the states with the sharp values of time are limited by the impossibility to exceed the speed of light. If one system changes its state from the one with duration of existence t_1 to the state with t_2, then the other system has to change its state in a way by exactly matching $t_2 - t_1$. Otherwise, since the coordinate is for the time what the momentum is for the energy (appropriate components of a quadri vector), the instantaneous change in the coordinate, by some finite amount, would be necessary. Since this can contradict the fact that the velocity greater than c is impossible, the process analog to the thermalisation and the changes in the persistence of the temporal canonical ensembles are hard to manage. This is the reason why the radioactive elements are characterized by the radioactive constant, while β, or T as the parameter that characterizes the canonical ensemble (2), can vary.

The radioactive decay offers the concrete example where the formalism related to the so-called second Schrödinger equation, that has been introduced in [2], finds its applicability. There, we have introduced the 'dynamical' counterpart of the time $\hat{G}$ and, from the above given, its importance becomes obvious for it is for the decay what the Hamiltonian is for the standard canonical ensemble. The eigenstates of G determine at which instance of time the decay can happen, which is the analog of the eigenstates of Hamiltonian, that determine the energy levels of the considered system. On the other side, one may prefer to simplify the formalism by neglecting the second Hilbert space ($\mathcal{H}_t$) and the operators that act there. By doing this, there would be no significant consequences toward the main argument of this article, which is that the radioactive decay can be seen as the (temporal) canonical ensemble. However, by excluding the second Hilbert space and the related operators, one would lose the possibility to consider the Schrödinger and the so-called second Schrödinger equation since, so to say, left sides of these equations are attached to $\mathcal{H}_t$, while the right sides are related to $\mathcal{H}_q$. We have proceeded our argumentation here in $\mathcal{H}_q \otimes \mathcal{H}_t$ since we wanted to be as formal and complete as possible.

Regarding the comparison of our approach to the radioactive decay with the other ones, let us say that we are not treating the decay as a dynamical process that is governed by some Hamiltonian. In our proposal, Hamiltonian is related to the energy and the standard canonical ensemble, while $\hat{G}$ is connected to the temporal canonical ensemble, *i. e.*, radioactive decay. Among others, this means that we do not start with the pure state of two coupled systems and then reach the formal description of the statistical character of the radioactive decay by neglecting one of the involved systems. In other words, what one gets by tracing out the degrees of freedom of one of the coupled systems is the mixture of the second kind, while we see the radioactive decay as the mixture of the first kind. We believe that the proper formal description of the decay process should use mixtures of the first kind since they are more appropriate for the objective phenomena.

In proposed formalism, the exponential decay law appears to be the probability distribution characteristic for canonical ensembles. In this way, the Boltzmann distribution, used in the Gibbs treatment of the canonical ensembles, gets on its universality. The normalized Boltzmann distribution gives the probability to find the system in some state, say $|t_a\rangle\langle t_a| \otimes |t_a\rangle\langle t_a|$. As is the case for every mixed state, *a priori* it is not possible to know the state of some particular system from the ensemble described by (9). This is why the radioactive decay looks like the stochastic process. Namely, we do not know when some system will decay because we do not know in which state $|t_a\rangle\langle t_a| \otimes |t_a\rangle\langle t_a|$ the system is. We only know the probability to find the system in $|t_a\rangle\langle t_a| \otimes |t_a\rangle\langle t_a|$, *i.e.*, the probability that the system will not decay before t_a. In other words, the radioactive decay is not essentially different from any other mixed state regarding the randomness and the stochasticity. It is not possible to predict when some system will decay as it is not possible to be certain about the energy of some system that belongs to the canonical ensemble.

4. Conclusions

We have applied the operator of time formalism in order to discuss the radioactive decay. We have shown that this formalism offers the new insight into the decay in a sense that the decay exponential-law is seen as the Boltzmann distribution in the Gibbs treatment of the canonical ensemble. In other words, the radioactive decay is seen as the temporal canonical ensemble. Within our approach, the radioactive constant is the analog of the absolute temperature multiplied by the Boltzmann constant, and this led us to the introduction of the new quantity, which we have called persistence, which is the analog of the absolute temperature in the treatment of the canonical ensemble. Finally, we have offered a new explanation of the stochastic character of the decay process.

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