Lectures on Quantum Field Theory

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March, 1998

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Preface

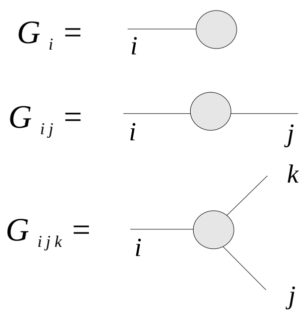
These lecture notes form the material for the graduate course QFT 2, the second of three quantum field theory courses in the graduate program in High Energy Theory at the Institute of Physics in Belgrade. The course presents the functional formalism of Schwinger-Dyson equations, generating functionals and Feynman path integrals. The topics covered include: perturbation theory, loop expansion, Euclidean field theory, solitons, instantons, vacuum polarization, geometry of gauge theories, quantization of gauge theories, anomalies, the background field method, renormalization, the renorm group, Borel summation, large N expansion and quantum field theory at non zero temperature. These general tools are presented on a host of models including: The Schwinger model, 't Hooft model, sine-Gordon model, Liouville field theory, CP_N model, Ising model, Landau-Ginsburg model, Heisenberg model, Wilson model, and the Kogut-Susskind model.

Lecture 1

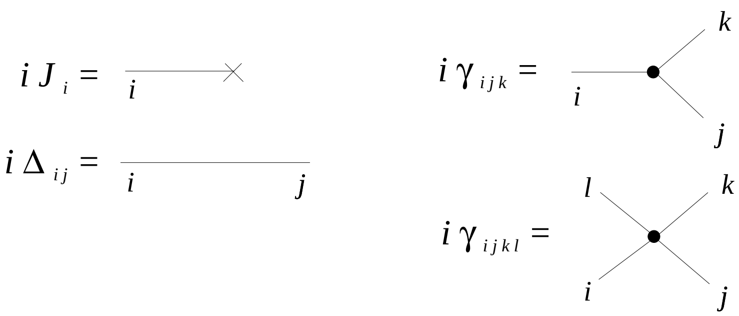
Linearity and Combinatorics

1.1 Schwinger–Dyson Equations

We look at a given quantum field theory (QFT) with one-particle states ϕ_i. The theory is completely known if we are given all the n-particle transition amplitudes (Green's functions) $G_i, G_{ij}, G_{ijk} \cdots$ We shall diagramaticaly depict these as the following set of blobs:

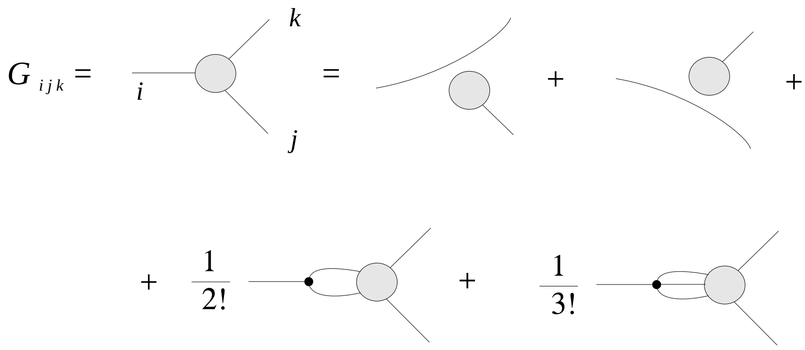


Instead of knowing an infinity of Green's functons we seek a smaller set of basic amplitudes out of which we may build all the Green's functions. For the moment let us suppose that we are given these basic amplitudes — also known as Feynman rules.

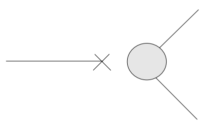


The basic one-point amplitude is often called a source or external field. The corresponding two-point amplitude is the propagator, while all the higher amplitudes designate interaction vertices. The reasons for these names will become evident later. We are usualy going to be interested in dynamics without external fields, however, as we shall soon see, many of our calculations will be simplified if we keep the sources till the very end.

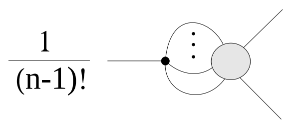
Green's functions are total amplitudes for a given process and they represent a sum of the amplitudes for all distinct ways in which this process can happen. This additive property is the central property of all quantum theories, and is mathematicaly stated in terms of the Schwinger–Dyson (SD) equations. To write down the SD equation for a given Green's function we follow a simple rule: Pick any leg, follow it into the blob and list all possible outcomes consistent with the Feynman rules. For example, let us write the appropriate SD equation for G_{ijk} for a model with only cubic and quartic interactions and no sources.



If we were keeping sources we would have an additional piece

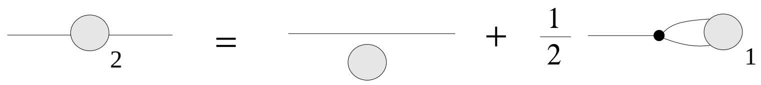


On the other hand, if we also had an n-particle vertex we would in addition have

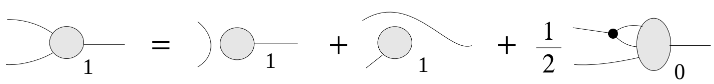


The numerical factors multiplying a given blob are just symmetry factors. The n-vertex has $(n-1)$ internal legs which can be permuted $(n-1)!$ times. To avoid over-counting we multiply by $\frac{1}{(n-1)!}$ as above.

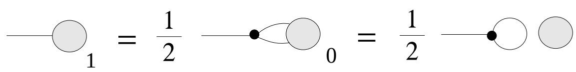
The SD equations are easily seen to represent an infinite set of coupled equations, and are extremely difficult to solve. We shall first present a systematic approximation scheme called perturbation theory. If the interaction aplitudes γ are small, then one can perform an expansion in powers of γ, $i.e.$ in numbers of vertices. We, therefore, iterate the SD equations and disregard diagrams with more than a certain number of vertices. For example, we will calculate the two-point Green's function to two vertices in a model with a purely cubic interaction and no source terms. We have



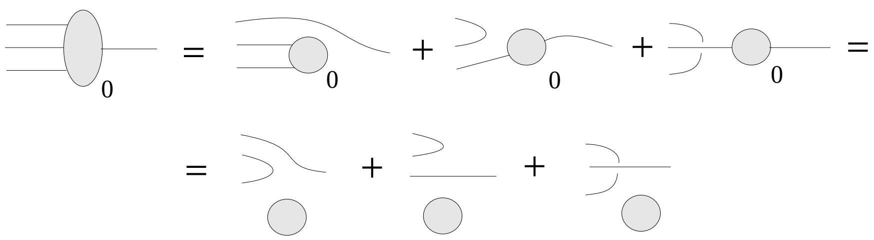
Now the 3-point Green's function obeys



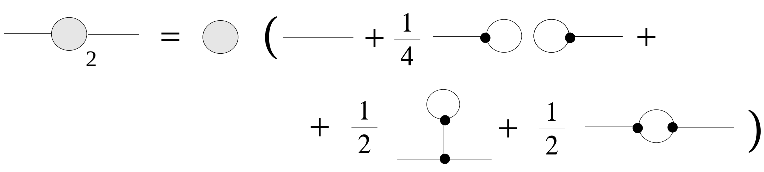
To proceed further we need the tadpole to one vertex



as well as the 4-point function to no vertices



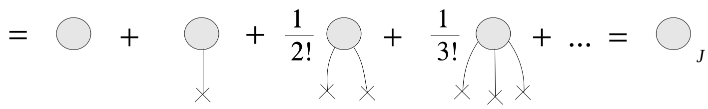
Putting this back into the expression for the two-point amplitude we get



1.2 Generating Functionals

Now we will present a compact way to write all the SD equations at once. To do this we introduce the generating functional

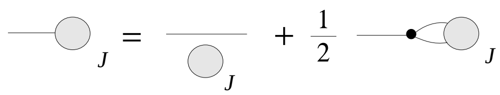
$$Z[J] = \sum_{m=0}^{\infty} \frac{i^m}{m!} G_{i_1 i_2 \cdots i_m} J_{i_1} J_{i_2} \cdots J_{i_m} = \tag{1.1}$$



Note that this is just the vacuum diagram in the presence of sources. Through its derivatives $Z[J]$ generates all the Green's functions, for example

$$G_{ijk} = \frac{1}{i^3} \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} \frac{\partial}{\partial J_k} Z[J] \bigg|_{J=0} \ . \tag{1.2}$$

If we do not set $J = 0$ at the end we get the corresponding Green's function in the presence of an external field. Now we return to the SD equations. For simplicity let us look at a model with a purely cubic interaction. The tadpole SD equation with sources is just



In terms of $Z[J]$ this is simply the differential equation

$$\frac{1}{i} \frac{\partial}{\partial J_i} Z[J] = i \triangle_{ij} \left(i J_i + \frac{1}{2} i \, \gamma_{jkl} \, \frac{1}{i} \frac{\partial}{\partial J_k} \, \frac{1}{i} \frac{\partial}{\partial J_l} \right) Z[J] \ . \tag{1.3}$$

We multiply both sides with $\triangle_{ij}^{-1}$ and the SD equation may then be compactly written as [1]

$$\left(\frac{\partial S}{\partial \phi_i} \bigg|_{\phi = \frac{1}{i}\frac{\partial}{\partial J}} + J_i \right) Z[J] = 0 \ , \tag{1.4}$$

where we have introduced the quantum action functional

$$S[\phi] = \frac{1}{2} \, \phi_i \, \triangle_{ij}^{-1} \, \phi_j + \frac{1}{3!} \, \gamma_{ijk} \, \phi_i \phi_j \phi_k \ . \tag{1.5}$$

For a general theory the SD equation retains the same form, while the appropriate quantum action becomes

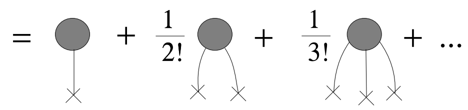
$$S[\phi] = \frac{1}{2} \, \phi_i \, \triangle_{ij}^{-1} \, \phi_j + \frac{1}{3!} \, \gamma_{ijk} \, \phi_i \phi_j \phi_k + \frac{1}{4!} \, \gamma_{ijkl} \, \phi_i \phi_j \phi_k \phi_l + \dots \tag{1.6}$$

As we see $S[\phi]$ is the generating functional for the Feynman rules. Later we will see that it is strongly related to the classical action $I[\phi]$, hence its name. Note that 1.4 is just the tadpole SD equation. It is a homogenous linear differential equation. In the next lecture we shall find its formal solution. What is important for us now is that equation 1.4 completely determines $Z[J]$, and hence our whole QFT.

[1] Note that we are assuming that $\triangle_{ij}$ is invertible. We shall deal with the more general case when we study gauge theories.

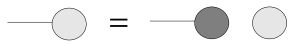
We end this section by introducing yet another generating functional — the generating functional for connected graphs

$$iW[J] = \sum_{m=1}^{\infty} \frac{i^m}{m!} G^{(c)}_{i_1 i_2 \cdots i_m} J_{i_1} J_{i_2} \cdots J_{i_m} = \tag{1.7}$$



Note that from the above definition it follows that $W[0] = 0$. We have denoted the connected Green's functions by dark grey blobs.

Why should we look at connected diagrams? Disconnected diagrams obviously represent independent processes. By knowing just the connected diagrams we in fact know everything. We shall now derive the important relation between $Z[J]$ and $W[J]$. To do this we look at the one-point Green's function (or tadpole) identity



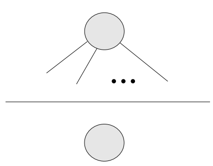
In terms of generating functionals this becomes

$$\frac{1}{i}\frac{\partial}{\partial J_i} Z = \left(\frac{1}{i}\frac{\partial}{\partial J_i} iW\right) Z \ . \tag{1.8}$$

This, along with $W[0] = 0$, gives us the required relation between generating functionals:

$$Z[J] = Z[0] e^{iW[J]} \ . \tag{1.9}$$

As an aside, let us talk about normalized Green's functions (correlators). From 1.9 we see that all Green's functions have as a common factor the vacuum diagrams $Z[0]$ multiplying them. We thus define correlators to be

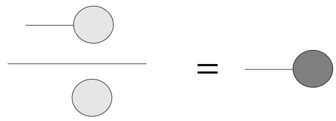


This notation is reminiscent of the operator formalism in which

$$< \phi_i \phi_j \cdots \phi_n > = \frac{< 0 \mid T(\phi_i \phi_j \cdots \phi_n) \mid 0 >}{< 0 \mid 0 >} \ . \tag{1.10}$$

The most important such quantity is called the average field [2] — the quantum average of ϕ_i

[2] In the literature this is often called the classical field. We will reserve the name classical field only for solutions of the classical equations of motion.



where the last equality follows from our tadpole identity. We thus have

$$\varphi_i = \frac{\partial W}{\partial J_i} \ . \tag{1.11}$$

For a general correlator we find

$$< \phi_i \phi_j \cdots \phi_n >= \frac{1}{Z}\, G_{ij\cdots n} = \frac{1}{Z}\left(\frac{1}{i}\frac{\partial}{\partial J_i}\right)\left(\frac{1}{i}\frac{\partial}{\partial J_j}\right)\cdots\left(\frac{1}{i}\frac{\partial}{\partial J_n}\right) Z \tag{1.12}$$

This can be written compactly as

$$< F[\phi] >= \frac{1}{Z}\, F[\frac{1}{i}\frac{\partial}{\partial J}]\, Z \tag{1.13}$$

1.3 Free Field Theory

We finish this lecture by looking at free field theories — theories without interactions. The quantum action in this case is simply $S_0 = \frac{1}{2}\,\phi_i \, \triangle^{-1}_{ij}\, \phi_j$. It follows that $Z_0[J]$ obeys

$$\left(\triangle^{-1}_{ij}\,\frac{1}{i}\frac{\partial}{\partial J_j} + J_i \right) Z_0[J] = 0 \ , \tag{1.14}$$

or equivalently

$$\frac{1}{i}\frac{\partial Z_0}{\partial J_i} + \triangle_{ij} J_j Z_0 = 0 \ . \tag{1.15}$$

This equation is easily solved, and we get

$$Z_0[J] = Z_0[0]\, \exp\left(-\frac{1}{2}\, J_i \, \triangle_{ij}\, J_j \right) \tag{1.16}$$

If only interacting theories would be as simple. . .

EXERCISES

1.1 Using the SD equations calculate the tadpole to two vertices for a theory with cubic and quartic interactions.

1.2 Show that the SD equation for the connected generating functional $W[J]$ may be written as

$$\left(\left.\frac{\partial S}{\partial \phi_i}\right|_{\phi=\varphi+\frac{1}{i}\frac{\partial}{\partial J}} + J_i \right) 1 = 0 \ .$$

1.3 Solve free field theory in two different ways: First by using the SD equation for $W_0(J)$, and second by calculating all the connected diagrams of the theory.

Lecture 2

Further Combinatoric Structure

2.1 Classical Field Theory

Before we continue developing the basic formalism of QFT's we will look at the associated classical theory and cast it in a diagrammatic form. The action in an external field is $I[\phi] + J_i\phi_i$, and leads to the equations of motion

$$\left.\frac{\partial I}{\partial \phi_i}\right|_{\phi=\Phi} + J_i = 0 \ , \tag{2.1}$$

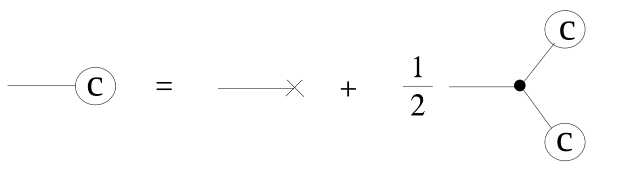
where Φ is the sought-after solution to these equations. Again, for simplicity, we will look at a model with a purely cubic interaction. The equation of motion is then

$$J_i + \triangle^{-1}_{ij}\Phi_j + \frac{1}{2}\gamma_{ijk}\,\Phi_j\Phi_k = 0 \ . \tag{2.2}$$

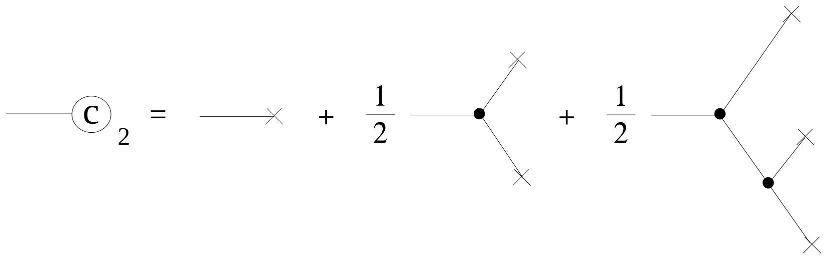
Thus

$$\Phi_i = i\,\triangle_{ij}\left(iJ_j + \frac{1}{2}\,i\gamma_{ijk}\,\Phi_j\Phi_k\right) \ . \tag{2.3}$$

In terms of our diagrammatic rules this is simply



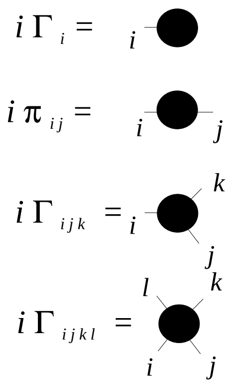
Again, the simplest way to untangle this kind of recursive relation is to solve things perturbatively. For example, to two vertices we easily find



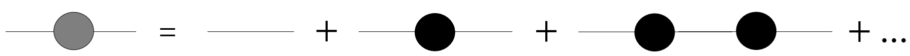
As we see these graphs are just as in the quantum theory, except that here we have no loops. For obvious reasons these are called the tree diagrams. We'll soon see that loop diagrams represent quantum corrections to the classical theory which is given in terms of tree diagrams. Free field theories have no vertices, and hence no loops. They are in fact purely classical.

2.2 The Effective Action

Let us now continue developing the formalism of QFT's. We introduce the concept of one-particle irreducible (1PI) diagrams. A 1PI diagram can't be cut into two disconnected pieces by severing a single internal line. Unlike the full and connected Green's functions we define the 1PI diagrams without external propagators, and denote them as



As may be guessed from this notation, the 2-point diagram plays a specific role. To see this we consider the connected two-point Green's function (called the full propagator)

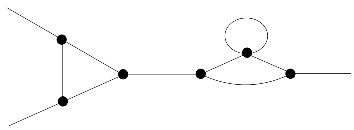


Equivalently we have

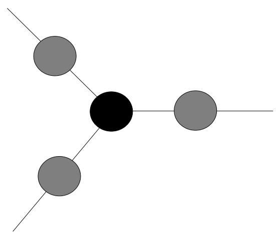
$$G_2^{(c)} = i \triangle + (i\triangle)(i\pi)(i\triangle) + (i\triangle)(i\pi)(i\triangle)(i\pi)(i\triangle) + \ldots = i(\triangle^{-1} + \pi)^{-1} \; . \quad (2.4)$$

Therefore, the effect of quantum fluctuations is to exchange the bare propagator $\triangle$ for the full propagator $(\triangle^{-1} + \pi)^{-1}$. π_{ij} is called the self-energy. All the other

1PI diagrams are often called vertex functions. The reason for this nomenclature may best be seen if we consider a general diagram such as



As we see it is contained in



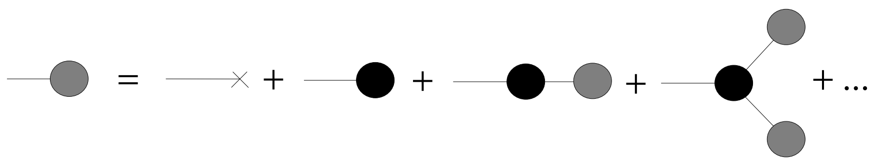
Hence, it is easy to see that just as the connected two-point Green's function plays the role of the full propagator, so $\Gamma_i, \Gamma_{ijk}, \Gamma_{ijkl}, \ldots$ play the roles of the full vertices. From the example above we find that the general graphs of our theory based on the quantum action

$$S[\phi] = \frac{1}{2}\,\phi_i\,\triangle^{-1}_{ij}\,\phi_j + \frac{1}{3!}\,\gamma_{ijk}\,\phi_i\phi_j\phi_k + \ldots \tag{2.5}$$

are the same as the tree graphs of the theory based on the effective action

$$\Gamma[\phi] = \Gamma_i\phi_i + \frac{1}{2}\,\phi_i(\triangle^{-1} + \pi)_{ij}\phi_j + \frac{1}{3!}\,\Gamma_{ijk}\,\phi_i\phi_j\phi_k + \ldots \tag{2.6}$$

Hence, the quantum theory based on $S[\phi]$ is the same as the classical theory based on $\Gamma[\phi]$. Note that even though S may have a finite number of vertices (polynomial action) the corresponding effective action is in general non-polynomial. To show the above connection explicitly let us look at the identity



Equivalently, this is simply

$$\varphi_i = i\,\triangle_{ij}\left(iJ_j + i\Gamma_j + i\pi_{jk}\varphi_k + \frac{1}{2}\,i\Gamma_{jkl}\,\varphi_k\varphi_l + \ldots\right)\,. \tag{2.7}$$

Defining the effective action as above, *i.e.*

$$\Gamma[\varphi] = \sum_{m=1}^{\infty}\frac{1}{m!}\,\Gamma_{i_1,i_2\ldots i_m}\,\varphi_{i_1}\varphi_{i_2}\cdots\varphi_{i_m}\ , \tag{2.8}$$

where $\Gamma_{ij} = (\triangle^{-1} + \pi)_{ij}$, we find that the average field φ_i satisfies

$$\frac{\partial \Gamma}{\partial \varphi_i} + J_i = 0 \ . \tag{2.9}$$

As promised this is indeed the classical equation of motion coming from the effective action. If we remember the definition of the average field

$$\varphi_i = \frac{\partial W}{\partial J_i} \ , \tag{2.10}$$

then equations 2.9 and 2.10 directly give

$$W[J] = \Gamma[\varphi] + J_i \varphi_i \ , \tag{2.11}$$

i.e. the effective action is the Legendre transformation of the generating functional of the connected graphs.

2.3 Path Integrals

As we have seen, the generating functional $Z[J]$ satisfies the SD equation

$$\left(\left.\frac{\partial S}{\partial \phi_i}\right|_{\phi=\frac{1}{i}\frac{\partial}{\partial J}} + J_i \right) Z[J] = 0 \ . \tag{2.12}$$

This is a linear differential equation. We will therefore Fourier transform it to obtain an algebraic equation. We write

$$Z[J] = \int [d\phi] \, \widetilde{Z}[\phi] \, e^{iJ_i\phi_i} \ , \tag{2.13}$$

where the integration measure is simply $[d\phi] = \prod_i d\phi_i$. Substituting this into the SD equation above, we get

$$\begin{aligned} 0 &= \int [d\phi] \, \widetilde{Z}[\phi] \left(\frac{\partial S}{\partial \phi_i} + \frac{1}{i}\frac{\partial}{\partial \phi_i} \right) e^{iJ_i\phi_i} = \\ &= \int [d\phi] \, e^{iJ_i\phi_i} \left(\frac{\partial S}{\partial \phi_i} - \frac{1}{i}\frac{\partial}{\partial \phi_i} \right) \widetilde{Z}[\phi] \ . \end{aligned} \tag{2.14}$$

Therefore

$$\frac{\partial S}{\partial \phi_i} \widetilde{Z}[\phi] - \frac{1}{i}\frac{\partial \widetilde{Z}}{\partial \phi_i} = 0 \ , \tag{2.15}$$

which gives $\widetilde{Z}[\phi] = e^{iS[\phi]}$. Finally, we obtain the solution for the SD equation 2.12 in the form of a path integral

$$Z[J] = \int [d\phi] \, e^{i(S[\phi] + J_i\phi_i)} \ . \tag{2.16}$$

As an asside, let us note that in our nomenclature we are anticipating the fact that in $d \neq 0$ dimensions of space-time the index i takes on continuous values. Thus we talk of functionals (rather than functions) and path integrals (rather than ordinary multiple integrals). In later lectures we shall indicate where path integrals differ from ordinary multiple integrals.

Differentiating equation 2.16 we find

$$G_{ij\ldots n} = \int [d\phi]\, \phi_i \phi_j \cdots \phi_n \, e^{iS} \ . \tag{2.17}$$

For correlators (normalized Green's functions) we simply divide by $Z = Z[0]$ and obtain

$$< \phi_i \phi_j \cdots \phi_n >= \frac{\int [d\phi]\, \phi_i \phi_j \cdots \phi_n \, e^{iS}}{\int [d\phi]\, e^{iS}} \ . \tag{2.18}$$

These expressions correspond to Green's functions for the case of theories with no external fields. If an external field is present it is added in the usual way: $S \to S + J_i \phi_i$.

The path integral expression for $Z[J]$ can be taken as the starting point in the development of QFT. For example, there are several way of deriving Feynman rules directly from the path integral. If we split the quantum action into a free part and an interaction $S[\phi] = \frac{1}{2}\, \phi_i \, \triangle_{ij}^{-1} \, \phi_j + S_{int}[\phi]$, then we have

$$Z[J] = \int [d\phi] \, \exp i \left(\frac{1}{2} \phi_i \, \triangle_{ij}^{-1} \, \phi_j + J_i \phi_i + S_{int}[\phi] \right) \ . \tag{2.19}$$

The direct way to calculate this perturbatively is to Taylor expand $e^{iS_{int}[\phi]}$. We are then left with evaluating integrals of the type

$$\int [d\phi]\, \phi_i \phi_j \cdots \phi_n \, e^{i(\frac{1}{2}\phi_i \triangle_{ij}^{-1} \phi_j + J_i \phi_i)} \ , \tag{2.20}$$

which are just equal to

$$(\frac{1}{i}\frac{\partial}{\partial J_i})(\frac{1}{i}\frac{\partial}{\partial J_j}) \cdots (\frac{1}{i}\frac{\partial}{\partial J_n}) Z_0[J] \ , \tag{2.21}$$

with $Z_0[J] = Z_0[0] e^{-\frac{i}{2} J_i \triangle_{ij} J_j}$. To derive this we have just used the simple identity

$$\frac{1}{i}\frac{\partial}{\partial J_i} e^{iJ_i \phi_i} = \phi_i \, e^{J_i \phi_i} \ . \tag{2.22}$$

Another way to proceed is to first use the above identity and find

$$Z[J] = \exp \left(i S_{int}[\frac{1}{i}\frac{\partial}{\partial J}] \right) \, Z_0[J] \ , \tag{2.23}$$

and now to Taylor expand. A third way is to use the Hori formula

$$Z[J] = \exp \left(\frac{i}{2} \frac{\partial}{\partial \phi_i} \triangle_{ij} \frac{\partial}{\partial \phi_j} \right) \, \exp i \left(S_{int}[\phi] + J_i \phi_i \right) \Big|_{\phi = 0} \ . \tag{2.24}$$

Hori's result is conceptualy nice since it shows that pairs of vertices must be joined by propagators $i\triangle_{ij}$ which is just what Feynman diagrams do. However, to calculate $Z[J]$ we still have to Taylor expand.

As we see all three methods are essentialy similar. Calculation with them is straightforward though tedious. In practice symmetry factors are most easily obtained from the SD equations. The real use for the path integral formalism is in deriving semi-classical results (asymptotic or loop expansion), analyzing symmetries (Ward identities, anomalies), and in quantizing gauge field theories (Faddeev–Popov method, BRST quantization, Batalin–Vilkovisky quantization). We will begin with these aplications of path integrals in the next lecture.

So far we have chosen S to be dimensionless. In order to give it the standard dimension for an action we introduce the Planck constant $\hbar$, and scale

$$S[\phi] \to \frac{1}{\hbar} S[\phi, \hbar] \ . \tag{2.25}$$

For convenience we also take $J_i \to \frac{1}{\hbar} J_i$. The path integral is now

$$Z[J] = \int [d\phi] \, \exp \frac{i}{\hbar} \left(S[\phi, \hbar] + J_i \phi_i \right) \ . \tag{2.26}$$

Let us anticipate a result of the next lecture: In the $\hbar \to 0$ limit the above path integral is dominated by fields whose values are in the vicinity of fields that satisfy

$$\frac{\partial S[\phi, \hbar = 0]}{\partial \phi_i} + J_i = 0 \ . \tag{2.27}$$

These are in fact the solutions of the classical equations of motion

$$\frac{\partial I[\phi]}{\partial \phi_i} + J_i = 0 \ , \tag{2.28}$$

hence we immediately get $S[\phi, \hbar = 0] = I[\phi]$. The full quantum action is thus

$$S[\phi, \hbar] = I[\phi] + M[\phi, \hbar] \ , \tag{2.29}$$

where M, which we shall call the measure term, is such that it vanishes in the $\hbar \to 0$, *i.e.* semi-classical, limit. For now this is all that we know about M[1]. In a later lecture we shall turn to the operator formalism of QFT for help in determining the measure.

Life is much simpler when we choose units in which $\hbar = 1$, and through most of these lectures this is what we are going to do. As we have seen, it is very simple to restore $\hbar$ dependence at the very end of our calculations through the above scaling.

[1]One often writes the integrand of the path integral as an exponent of the classical action. In this case the M term is simply absorbed into the path integral measure $d\mu = [d\phi] \exp(i/\hbar M[\phi, \hbar])$. In this notation lack of knowledge about M translates into lack of knowledge about the precise form of the path integral measure.

EXERCISES

2.1 Calculate Φ to two vertices for a theory with cubic and quartic interactions.

2.2 Calculate the effective action for a free theory.

2.3 Show that the SD equation for Γ is

$$\left(\frac{\partial \Gamma}{\partial \varphi_i} - \left. \frac{\partial S}{\partial \phi_i} \right|_{\phi = \varphi + \frac{1}{i} W'' \frac{\partial}{\partial \varphi}} \right) 1 = 0 \ .$$

2.4 Solve the path integral for $Z[J]$ in the case of free field theory. Discuss convergence of the integral. This is the root of the $i\epsilon$ prescription in the functional formalism.

2.5 Prove the Hori formula.

2.6 Re-do the calculation of Problem 1.1.1 directly in the path integral formalism by using the each of the three methods introduced above.

2.7 Derive the equation of motion satisfied by the average field $\varphi_i =< \phi_i >$. For the case of a purely cubic interaction show that this equation differs from the classical equation of motion by a $o(\hbar)$ term.

Lecture 3

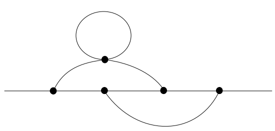
Using the Path Integral

3.1 Semi-Classical Expansion

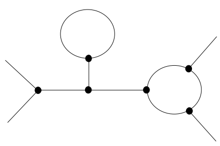
In order to look at semi-classical expansions we re-introduce $\hbar$. As we have seen we have

$$Z[J] = \int [d\phi] \, \exp \frac{i}{\hbar} \left(S[\phi] + J_i \phi_i \right) \ . \tag{3.1}$$

For the rest of this lecture we will work with a broad class of interesting models for which $S = I$, *i.e.* the action has no explicit $\hbar$ dependence. Under the scaling $I \to \frac{I}{\hbar}$ we have $\triangle \to \hbar \triangle$, while the interactions scale as $\gamma \to \frac{1}{\hbar}\gamma$. A general Feynman diagram has E external legs, I internal legs,V_3 3-point vertices, V_4 4-point vertices, etc. It also has L loops.



E = 2
I = 7
L = 3
V_3 = 4
V_4 = 1



E = 4
I = 7
L = 2
V_3 = 5
V_4 = 0

These numbers aren't independent but satisfy certain toplogical relations, namely

$$L = I + 1 - V_3 - V_4 - \ldots \tag{3.2}$$
$$E + 2I = 3V_3 + 4V_4 + \ldots \tag{3.3}$$

as can easily be seen to hold on the above examples. A given diagram is thus proportional to

$$\hbar^{E+I-V_3-V_4-\ldots} = \hbar^{E-1}\hbar^{L} \ . \tag{3.4}$$

For a given Green's function E is fixed. We thus see that an expansion in loops is just an expansion in powers of $\hbar$, *i.e.* a semi-classical expansion. As can be seen from 3.1 $\hbar \to 0$ is in fact an asymptotic expansion of the path integral for the generating functional. Loop expansion is the second systematic aproximation scheme that we have encountered — the first was perturbation theory. Loop expansion is often a more physical aproximation due to the fact that it represents a semi-classical expansion.

We look at the phase $I[\phi] + J_i\phi_i$ for two near by paths ϕ and $\phi + \delta\phi$. In general these paths have different phases that more or less cancel. However if $\delta\left(I[\phi] + J_i\phi_i\right) = 0$ (stationary phase) then the contributions of near by paths add. The dominant contribution to $Z[J]$ thus comes from fields in the vicinity of the classical solution Φ given by

$$\left.\frac{\partial I}{\partial \phi_i}\right|_{\phi=\Phi} + J_i = 0 \; . \tag{3.5}$$

Because of this it makes sense to expand the action around the classical solution. Thus $\phi_i = \Phi_i + \eta_i$ gives

$$I[\phi] + J_i\phi_i = I[\Phi] + J_i\Phi_i + \frac{1}{2}\,\eta_i \left.\frac{\partial^2 I}{\partial\phi_i\partial\phi_j}\right|_{\phi=\Phi} \eta_j + \ldots \tag{3.6}$$

We shall keep only terms that are at most quadratic in η. Note that linear terms are absent due to the equation of motion 3.5. Using $[d\phi] = [d\eta]$ we get

$$Z[J] \simeq e^{\frac{i}{\hbar}(I[\Phi]+J_i\Phi_i)} \det\left(\left.\frac{\partial^2 I}{\partial\phi_i\partial\phi_j}\right|_{\phi=\Phi}\right)^{-\frac{1}{2}} \; . \tag{3.7}$$

The second term just comes from the Gaussian integration. Factors of π have been absorbed into the definition of the measure $[d\phi]$. If we write $I = \frac{1}{2}\,\phi_i \, \triangle^{-1}_{ij}$ $\phi_j + I_{int}[\phi]$, then we have

$$\left.\frac{\partial^2 I}{\partial\phi_i\partial\phi_j}\right|_{\phi=\Phi} = \triangle^{-1}_{ik}\left(\delta_{kj} - X_{kj}[\Phi]\right) \; , \tag{3.8}$$

where we have introduced the useful notation

$$-X_{kj}[\Phi] = \triangle_{kl}\left.\frac{\partial^2 I_{int}}{\partial\phi_l\partial\phi_j}\right|_{\phi=\Phi} \; . \tag{3.9}$$

We now have

$$\det\left(\left.\frac{\partial^2 I}{\partial\phi_i\partial\phi_j}\right|_{\phi=\Phi}\right) = \det(\triangle^{-1})\det(1 - X) \; . \tag{3.10}$$

Using the identity $\det M = \exp\left(\mathrm{tr}\;\ln M\right)$ we get

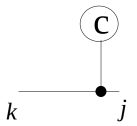
$$W[J] \simeq I[\Phi] + J_i\Phi_i + \frac{i\hbar}{2}\,\mathrm{tr}\;\ln\left(1 - X[\Phi]\right) \; . \tag{3.11}$$

Note that we have droped a constant term. It is not important since it doesn't contribute to the Green's functions, since these follow from W by differentiation. Expanding the logarithm we find

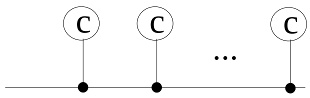
$$W[J] \simeq I[\Phi] + J_i\Phi_i + \frac{i\hbar}{2}\sum_{m=1}^{\infty}\frac{1}{m}\;\mathrm{tr}\;\left(X[\Phi]^m\right) \; . \tag{3.12}$$

To see the meaning of this let us look at purely cubic theory $I_{int} = \frac{1}{3!}\gamma_{ijk}\,\phi_i\phi_j\phi_k$. In this case we have

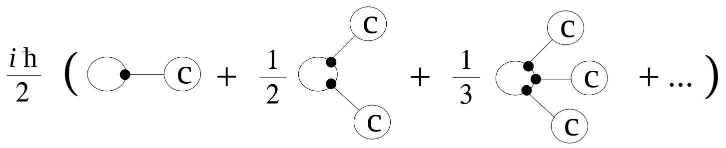
$$X_{kj} = -\triangle_{kl}\,\gamma_{ljm}\Phi_m = \tag{3.13}$$



and therefore X^m is simply depicted by



Finally, the trace glues the two free legs. The $o(\hbar)$ contribution to $W[J]$ is just



As we have seen, the classical field is given as a sum of tree diagrams. Here the Φ's are glued to a single loop, hence this is a one loop correction to $W[J]$. This we already knew since it is of order $\hbar$. However, these are all the one loop diagrams with the appropriate symmetry factors. It follows that asymptotic expansion automatically gives the one loop result.

We will now calculate the higher loop corrections using the background field method. We again write the field as $\phi = \Phi + \eta$, but now we present an exact treatment of the η field. The generating functional is

$$Z[J] = \int [d\eta]\, e^{i(I[\Phi+\eta]+J_i(\Phi_i+\eta_i))} = e^{i(I[\Phi]+J_i\Phi_i)} \int [d\eta]\, e^{i(I[\Phi+\eta]-I[\Phi]+J_i\eta_i)} \ . \tag{3.14}$$

We now introduce the action for η in the presence of the background field Φ according to

$$I[\Phi;\eta] = I[\Phi+\eta] - I[\Phi] + J_i\eta_i \ . \tag{3.15}$$

The background field is a solution of the classical equations of motion. Because of this the above action doesn't depend on the J's. In fact we have

$$I[\Phi;\eta] = \frac{1}{2}\eta_i \left.\frac{\partial^2 I[\phi]}{\partial\phi_i\partial\phi_j}\right|_{\phi=\Phi}\eta_j + \frac{1}{3!}\left.\frac{\partial^3 I[\phi]}{\partial\phi_i\partial\phi_j\partial\phi_k}\right|_{\phi=\Phi}\eta_i\eta_j\eta_k + \dots \tag{3.16}$$

$$= \frac{1}{2}\eta_i\hat{\triangle}^{-1}_{ij}\eta_j + \frac{1}{3!}\hat{\gamma}_{ijk}\,\eta_i\eta_j\eta_k + \dots \ . \tag{3.17}$$

Hats indicate the background dependent propagator and vertices. To calculate the original generating functional $Z[J]$ we simply need to evaluate the vacuum diagram for the background field action. We do this in the usual way

$$\int [d\eta]\, e^{iI[\Phi;\eta]} = \int [d\eta]\, e^{i(I[\Phi;\eta]+K_i\eta_i)}\Big|_{K=0} = \tag{3.18}$$

$$= \left(\det \hat{\triangle}^{-1}\right)^{-\frac{1}{2}} \; e^{iI_{int}[\Phi;\frac{1}{i}\frac{\partial}{\partial K}]}\, e^{\frac{i}{2}K_i\hat{\triangle}_{ij}K_j}\Big|_{K=0} . \tag{3.19}$$

Therefore we get

$$Z[J] = e^{i(I[\Phi]+J_i\Phi_i)} \left(\det \hat{\triangle}^{-1}\right)^{-\frac{1}{2}} \; e^{iI_{int}[\Phi;\frac{1}{i}\frac{\partial}{\partial K}]}\, e^{\frac{i}{2}K_i\hat{\triangle}_{ij}K_j}\Big|_{K=0} . \tag{3.20}$$

As we have seen the first two terms give us the tree and one loop contributions to $W[J]$. Therefore, the effect of two loops and higher are encoded in

$$e^{iI_{int}[\Phi;\frac{1}{i}\frac{\partial}{\partial K}]}\, e^{\frac{i}{2}K_i\hat{\triangle}_{ij}K_j}\Big|_{K=0} . \tag{3.21}$$

3.2 Ward Identities

Now we come to the second topic of this lecture — symmetries. The path integral formalism is specialy usefull for deriving the consequences of symmetries of a QFT. Under an infinitesimal field redefinition

$$\phi_i \to \phi'_i = \phi_i + \epsilon F_i[\phi] \; , \tag{3.22}$$

we have

$$I[\phi] \quad\to\quad I[\phi'] = I[\phi] + \epsilon \frac{\partial I}{\partial \phi_i} F_i[\phi] + o(\epsilon^2) \tag{3.23}$$

$$J_i\phi_i \quad\to\quad J_i\phi'_i = J_i\phi_i + \epsilon J_i F_i[\phi] \; . \tag{3.24}$$

At the same time the measure changes according to

$$\begin{aligned}
[d\phi] \to [d\phi'] &= \det\left(\frac{\partial \phi'_i}{\partial \phi_j}\right)[d\phi] = \\
&= \exp\left(\operatorname{tr}\;\ln\left(\delta_{ij} + \epsilon\frac{\partial F_i}{\partial \phi_j}\right)\right)[d\phi] = \\
&= \exp\left(\operatorname{tr}\;\left(\epsilon\frac{\partial F_i}{\partial \phi_j}\right) + o(\epsilon^2)\right)[d\phi] = \\
&= [d\phi]\left(1 + \epsilon\frac{\partial F_i}{\partial \phi_i} + o(\epsilon^2)\right) \; .
\end{aligned} \tag{3.25}$$

We now put this into the path integral

$$\begin{aligned} Z[J] &= \int [d\phi']\, e^{iI[\phi']+iJ_i\phi'_i} = \\ &= Z[J] + \epsilon \int [d\phi] \left(\frac{\partial F_i}{\partial \phi_i} + i \left(\frac{\partial I}{\partial \phi_i} + J_i \right) F_i[\phi] \right) e^{iI[\phi]+iJ_i\phi_i} \ . \end{aligned} \tag{3.26}$$

Therefore we have derived the identity

$$0 = \int [d\phi] \left(\frac{\partial F_i}{\partial \phi_i} + i \left(\frac{\partial I}{\partial \phi_i} + J_i \right) F_i[\phi] \right) e^{iI[\phi]+iJ_i\phi_i} \ . \tag{3.27}$$

Now let us look at various consequences of this identity. If $\phi \to \phi'$ is a classical symmetry of the theory, then the action is invariant, *i.e.* $\frac{\partial I}{\partial \phi_i} F_i[\phi] = 0$. Therefore, we must have

$$0 = \int [d\phi] \left(\frac{\partial F_i}{\partial \phi_i} + iJ_iF_i[\phi] \right) e^{iI[\phi]+iJ_i\phi_i} \ . \tag{3.28}$$

If $\phi \to \phi'$ is a symmetry of the quantum theory then along with the action the measure must be invariant. Thus $\frac{\partial F_i}{\partial \phi_i} = 0$, which gives us the Ward identity

$$0 = \int [d\phi] J_i F_i[\phi] e^{iI[\phi]+iJ_i\phi_i} \ , \tag{3.29}$$

which may also be written as

$$J_i F_i[\frac{1}{i}\frac{\partial}{\partial J}] Z[J] = 0 \ . \tag{3.30}$$

Ward identities are very important. They are expressions of symmetry in the formalism of QFT's. Let us end by giving one further consequence of 3.27. Let $F_i[\phi]$ be a constant. The measure is invariant, so we get

$$0 = \int [d\phi] \left(\frac{\partial I}{\partial \phi_i} + J_i \right) e^{iI[\phi]+iJ_i\phi_i} \ , \tag{3.31}$$

which is just the SD equation for $Z[J]$. We may write it directly in the more familiar form

$$0 = \left(\left. \frac{\partial I}{\partial \phi_i} \right|_{\phi = \frac{1}{i}\frac{\partial}{\partial J}} + J_i \right) Z[J] = \ . \tag{3.32}$$

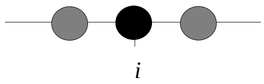
Note that SD equations follow even more trivialy from the path integral formalism since 3.31 may be written as

$$0 = \int [d\phi] \frac{\partial}{\partial \phi_i} e^{iI[\phi]+iJ_i\phi_i} \ , \tag{3.33}$$

which obviously holds for theories where fields fall off fast enough at infinity.

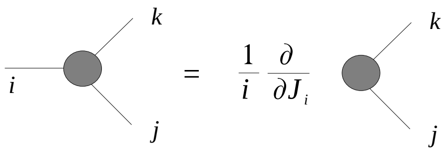
EXERCISES

3.1 Prove that $\frac{\partial}{\partial\varphi_i}$ acting on the full propagator gives



3.2 Write out the digramatic form of the SD equations for the effective action for a theory with cubic and quartic interactions. Use the result of the previous problem.

3.3 Note that



Using the identity from Problem 3.1 calculate the above connected Green's function. Do the same for the four-point function.

3.4 Differentiate the 1PI tadpole SD equation of Problem 3.2 (using the identity of Problem 3.1) to get the SD equations for the 1PI 2-point and 3-point functions.

3.5 Look at a theory with cubic and quartic interactions, where the cubic interaction is proportional to g, and the quartic interaction to g^2. Using the results of the previous problem calculate the $o(g^5)$ piece of Γ_{ijk}. Assume there is no symmetry breaking, *i.e.* $\varphi_i = 0$.

3.6 Prove the two topological relations for Feynman diagrams given in this lecture.

3.7 Calculate $W[J]$ to two loops.

3.8 Derive the identity for $\phi \to \phi' = \phi_i + c_i$ for the path integral

$$\int [d\phi]\, F[\phi]\, e^{iI[\phi]+iJ_i\phi_i} \ .$$

Show that $F[\phi] = \phi_{i_1}\phi_{i_2}\cdots\phi_{i_n}$ gives the SD equation for the (n+1)-point Green's function.

3.9 Prove that the expansion in Problem 3.5 is in fact the two loop result.

Lecture 4

Fermions

4.1 Grassmann Numbers

So far we have considered bosonic field theories. All the Green's functions were symmetric under the interchange of any two indices. For fermions we will have antisymmetry, for example $G_{ij} = -G_{ji}$. If our whole formalism of generating functionals and sources is to work for fermions we must have

$$\frac{\partial}{\partial\eta_i}\frac{\partial}{\partial\eta_j}Z[\eta] = -\frac{\partial}{\partial\eta_j}\frac{\partial}{\partial\eta_i}Z[\eta] \ . \tag{4.1}$$

Our convention will be to denote a generic Fermi field by ψ and its source by η. We are thus forced to consider a theory of anticommuting sources

$$\eta_i\eta_j = -\eta_j\eta_i \ . \tag{4.2}$$

In fact the fields ψ themselves must anticommute. Such objects are obviously not numbers — they are called Grassmann numbers.

Let us now consider free field theory for fermions. The action is given by the quadratic form

$$S_0 = \frac{1}{2}\,\psi_i K^{-1}_{ij}\psi_j \ , \tag{4.3}$$

where K^{-1}_{ij} must be antisymmetric. It follows that the indices must take on an even number of values, *i.e.* $i = 1, 2, \ldots, 2k$. A $2k \times 2k$ antisymmetric matrix K can be brought to a block diagonal form by a symplectic rotation $G \in \mathrm{Sp}(2k)$, so that

$$G^T K^{-1} G = \begin{pmatrix} 0 & -\lambda_1 & & & & & & \\ \lambda_1 & 0 & & & & & & \\ & & 0 & -\lambda_2 & & & & \\ & & \lambda_2 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & -\lambda_k \\ & & & & & & \lambda_k & 0 \end{pmatrix} . \tag{4.4}$$

This is the analogue of diagonalization for symmetric matrices. We define new fields according to

$$\psi_i = \frac{1}{\sqrt{2}}\left(\psi_{2i-1} + \psi_{2i}\right) \tag{4.5}$$

$$\psi^i = \frac{1}{\sqrt{2}}\left(\psi_{2i-1} - \psi_{2i}\right) . \tag{4.6}$$

In terms of these new fields the action is

$$S_0[\psi,\bar\psi] = \psi^i {S^{-1}}_i^{\ j}\psi_j = \bar\psi S^{-1}\psi \ , \tag{4.7}$$

where

$${S^{-1}}_i^{\ j} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_k \end{pmatrix} . \tag{4.8}$$

We have written the free action in terms of two Fermi fields ψ and $\bar\psi$ (note that each has indices that take on k values). The advantage of using such a pair of fields is that the kinetic term is again given in terms of a symmetric matrix S^{-1}. Hence free fermions have the action

$$S_0 = \bar\psi S^{-1}\psi \ , \tag{4.9}$$

where S^{-1} is a symmetric matrix. In general it is not diagonal, but when diagonalized we would get 4.8. We now need to add external fields or sources. Putting in sources is trivial, we take

$$S \to S + \bar\eta\psi + \bar\psi\eta \ , \tag{4.10}$$

where η and $\bar\eta$ are the sources corresponding to $\bar\psi$ and ψ. Note that the action is a number. As we can easily see, what we have writen is OK since products of an even number of Grassmann numbers are commuting objects *i.e.* numbers. Commuting objects are Grassmann-even, while anticommuting objects are Grassmann-odd.

For a general fermionic theory the generating functional is given by (note the order)

$$Z[\eta,\bar\eta] = \sum_{m,n} \frac{i^{m+n}}{m!n!}\,\eta_{j_n}\cdots\eta_{j_1}\eta^{i_m}\cdots\eta^{i_1}\,{G_{i_1\cdots i_m}}^{j_1\cdots j_n} \ , \tag{4.11}$$

hence, for example,

$$G_i^{\ j} = \left(\frac{1}{i}\frac{\partial}{\partial\eta^i}\right)\left(\frac{1}{i}\frac{\partial}{\partial\eta_j}\right)Z[\eta,\bar\eta] \ . \tag{4.12}$$

The SD equation (now a pair of equations, one for ψ and one for $\bar\psi$) is

$$\left(\left.\frac{\partial S}{\partial\psi}\right|_{\psi=\frac{1}{i}\frac{\partial}{\partial\bar\eta},\,\bar\psi=\frac{1}{i}\frac{\partial}{\partial\eta}} + \bar\eta\right) Z[\eta,\bar\eta] = 0 \tag{4.13}$$

$$\left(\left.\frac{\partial S}{\partial\psi}\right|_{\psi=\frac{1}{i}\frac{\partial}{\partial\bar\eta},\,\bar\psi=\frac{1}{i}\frac{\partial}{\partial\eta}} + \eta\right) Z[\eta,\bar\eta] = 0 \ . \tag{4.14}$$

We would like to Fourier transform this and obtain the path integral for $Z[\eta, \bar{\eta}]$. To do that we would first need to know how to define integrals of Grassmann numbers. However since $(\frac{\partial}{\partial\eta})^2 = 0$ we can't define the inverse operation to differentiation! Hence, there is no analogue of an indefinite integral over Grassmann variables. What is surprising is that one may define the notion of definite integrals. If we look back to our bosonic theory we see that all we needed (for path integrals) were definite integrals of the type $\int_{-\infty}^{+\infty} dx\, f(x)$. The central property of these integrals is

$$\int dx\, f(x) = \int dx\, f(x + C) \ , \tag{4.15}$$

for any constant C — for simplicity we stop writing the boundaries of integration explicitly. This is easily derived from properties of ordinary integrals over the whole real line. Can we turn things around? Can we take the above property and from it derive all the other properties of integrals? Obviously for ordinary numbers the answer is no. Surprisingly, for Grassmann numbers the answer is yes! Thus, for a single Grassmann variable η we impose

$$\int d\eta\, f(\eta) = \int d\eta\, f(\eta + \theta) \ , \tag{4.16}$$

for any Grassman number θ. Owing to the fact that $\eta^2 = 0$ we may write the general function $f(\eta)$ as

$$f(\eta) = x + \eta\xi \ , \tag{4.17}$$

where x is Grassmann-even, and ξ is Grassmann-odd. It follows that

$$\int d\eta \ = \ 0 \tag{4.18}$$

$$\int d\eta\, \eta \ = \ 1 \ . \tag{4.19}$$

Note now that we have $\int d\eta\, f(\eta) = \frac{\partial}{\partial\eta} f(\eta)$ — fermionic differentiation and integration are one and the same operation. This trivialy generalizes to the case of more than one Grassman variable. To conclude this mathematical asside let us recapitulate: We have seen what (algebraic) properties of integrals were needed in order to get our bosonic path integral. Insisting on these properties in the fermionic case has led us to a unique definition of an associate integration over fermionic variables. This integration turns out to be the same operation as differentiation.

The only bosonic path integral we could do explicity was the Gaussian integral. Over real fields we had

$$\int [d\phi]\, e^{i\, \frac{1}{2}\, \phi\triangle^{-1}\phi} = \left(\det \triangle^{-1}\right)^{-\frac{1}{2}} \ , \tag{4.20}$$

while for complex fields this is just

$$\int [d\phi d\bar{\phi}]\, e^{i\, \bar{\phi}\triangle^{-1}\phi} = \left(\det \triangle^{-1}\right)^{-1} \ , \tag{4.21}$$

The fermionic analogue of this is

$$\int [d\psi d\bar{\psi}]\, e^{i\,\bar{\psi}S^{-1}\psi} = \det S^{-1} \ , \tag{4.22}$$

i.e. the reciprocal of the bosonic result. This is easily proven. We diagonalize S^{-1} to get 4.8. Orthogonal transformations leave the volume element unchanged, so

$$\int [d\psi d\bar{\psi}]\, e^{i\,\bar{\psi}S^{-1}\psi} = \prod_i \int d\psi_i d\psi^i\, e^{i\,\lambda_i \psi^i \psi_i} \ . \tag{4.23}$$

This is trivial to integrate since

$$\int d\psi d\bar{\psi}\, e^{i\,\lambda\bar{\psi}\psi} = \int d\psi d\bar{\psi}(1 + i\lambda\bar{\psi}\psi) = i\lambda \ . \tag{4.24}$$

We pull in factors of i into our definition of the path integral measure, so that $[d\psi d\bar{\psi}] = \prod_j \frac{d\psi^j d\psi_j}{i}$, and we recover the result of equation 4.22.

In QFT we indeed only know how to solve Gaussian integrals. The reason path integrals turn out to be so usefull is that we may change variables easily, and in doing this simplify the expresions we are calculating. Let us see how one changes variables in a Grassmann integral. We first demonstrate things for the case of a single Grassman variable ψ. The general change of variables is

$$\psi \to \psi' = a\psi + \theta \ , \tag{4.25}$$

under which $d\psi \to d\psi' = J\, d\psi$. This gives us

$$1 = \int d\psi'\, \psi' = \int J\, d\psi\, (a\psi + \theta) \ , \tag{4.26}$$

hence $J = \frac{1}{a} = \frac{d\psi}{d\psi'}$. This is the inverse of the bosonic Jacobian. For $2k$ variables $\psi_i \to \psi'_i$ gives

$$\int d\psi'_1 \cdots d\psi'_{2k} = \frac{\partial}{\partial\psi'_1} \cdots \frac{\partial}{\partial\psi'_{2k}} = \left(\frac{\partial\psi_{i_1}}{\partial\psi'_1} \cdots \frac{\partial\psi_{i_{2k}}}{\partial\psi'_{2k}}\right) \frac{\partial}{\partial\psi_{i_1}} \cdots \frac{\partial}{\partial\psi_{i_{2k}}} \ . \tag{4.27}$$

Because of antisymmetry we indeed get

$$d\psi'_1 \cdots d\psi'_{2k} = \left|\frac{\partial\psi}{\partial\psi'}\right| d\psi_1 \cdots d\psi_{2k} \ . \tag{4.28}$$

The fermionic Jacobian thus equals

$$J_F = \left|\frac{\partial\psi}{\partial\psi'}\right| \ . \tag{4.29}$$

EXERCISES

4.1 Calculate the generating functional $Z_0[\eta, \bar{\eta}]$ for free fermions. From this find $W_0[\eta, \bar{\eta}]$ as well as the effective action.

4.2 Show that

$$\int [d\phi][d\psi d\bar{\psi}] \, \exp i \left(\frac{1}{2} \, \phi_i \, \triangle_{ij}^{-1} \, \phi_j + \psi^i {S^{-1}}_i^{\ j} \psi_j \right) = 1$$

if $S^2 = \triangle$. The theory coming from the above action is obviously trivial, but let us play around with it anyway. Show that this action is invariant under

$$\begin{array}{rcl} \phi & \rightarrow & \phi + \theta S \psi \\ \psi & \rightarrow & \psi \\ \bar{\psi} & \rightarrow & \bar{\psi} - \theta \phi \ , \end{array}$$

where θ is a Grassmann constant that parametrizes the transformation. This is an example of a supersymmetry — a symmetry that mixes Bose and Fermi fields. Derive the Ward identity that follows from the above symmetry being valid at the quantum levesl. Look at the Ward identity diagrammatically.

Lecture 5

Euclidean Field Theory

So far in our development of the formalism of QFT we have mostly been doing combinatorics. All our manipulations with amplitudes have been through various forms of SD equations. Up to that point the formalism offers no restrictions on what the basic amplitudes should be. We just assumed that they were given complex numbers. This situation changed somewhat when we started to solve the SD equations via the path integral. We identified the quantum action S — the generating functional of the basic amplitudes — with the classical action (up to $o(\hbar)$ terms coming from the measure). Thus, disregarding the measure for the moment, we see that $\triangle_{ij}, \gamma_{ijk}, \gamma_{ijkl}, \cdots$, as well as J_i are all given *real* numbers.

We shall call the formalism that has been studied so far Minkowskian field theory. Now let us consider a different kind of beast that we designate Euclidean field theory. Once again, the basis of the formalism is linearity. In the Euclidean case we take all the basic amplitudes to be imaginary. To be concrete — the way we will do this is by making the following susbstitution in our Minkowski theory:

$$\begin{aligned} iJ_i &\rightarrow \bar{J}_i \\ i\triangle_{ij} &\rightarrow \bar{\triangle}_{ij} \\ -i\gamma_{ijk} &\rightarrow \bar{\gamma}_{ijk} \\ -i\gamma_{ijkl} &\rightarrow \bar{\gamma}_{ijkl} \ . \end{aligned} \tag{5.1}$$

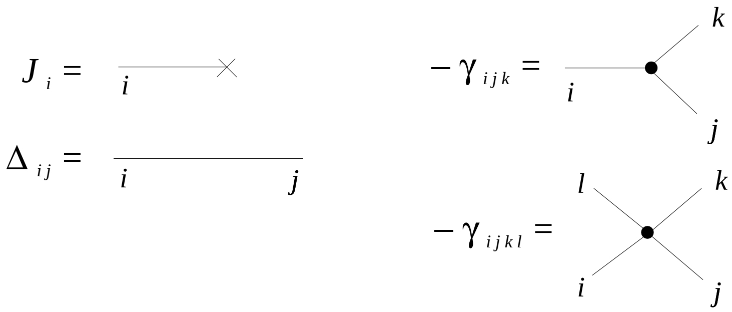
The barred expressions are now real. The Euclidean action is the generating functional of these new amplitudes, *i.e*

$$\bar{S} = \frac{1}{2}\,\phi_i \bar{\triangle}^{-1}_{ij}\,\phi_j + \frac{1}{3!}\,\bar{\gamma}_{ijk}\,\phi_i\phi_j\phi_k + \dots \ . \tag{5.2}$$

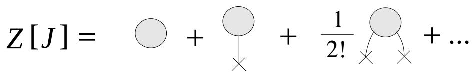
The fields ϕ are the same in both formalisms — after all they are just dummy variables that we integrate over in the path integral. In terms of the action the correspondence between the two formalisms is just

$$iS + iJ_i\phi_i \rightarrow -\bar{S} + \bar{J}_i\phi_i \ . \tag{5.3}$$

Euclidean and Minkowski theories represent two very different formalisms. At this point the only thing that they have in common is the underlying linearity. Using the above correspondences we will give a fast tour of the basic structure of Euclidean field theory. To simplify notation we will no longer write bars over Euclidean expressions. Bars will be used only when we compare Euclidean formulas with their Minkowsi cousins. The correspondence between algebraic expressions and diagrams is:



The generating functional is defined to be



hence

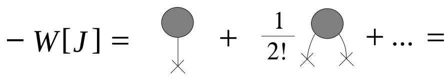
$$Z[J] = \sum_{m=0}^{\infty} \frac{1}{m!} G_{i_1 \ldots i_m} J_{i_1} \cdots J_{i_m} \tag{5.4}$$

$$G_{i_1 \ldots i_m} = \frac{\partial}{\partial J_{i_1}} \cdots \frac{\partial}{\partial J_{i_m}} Z[J] \bigg|_{J=0} . \tag{5.5}$$

The generating functional is determined from the action by its SD equation

$$\left(\frac{\partial S}{\partial \phi_i} \bigg|_{\phi = \frac{\partial}{\partial J}} - J_i \right) Z[J] = 0 . \tag{5.6}$$

Connected diagrams are generated by



$$= \sum_{m=1}^{\infty} \frac{1}{m!} G^{(c)}_{i_1 \ldots i_m} J_{i_1} \cdots J_{i_m} , \tag{5.7}$$

where

$$G^{(c)}_{i_1 \ldots i_m} = - \frac{\partial}{\partial J_{i_1}} \cdots \frac{\partial}{\partial J_{i_m}} W[J] |_{J=0} . \tag{5.8}$$

In particular, the average field satisfies

$$\varphi_i = -\frac{\partial W}{\partial J_i} \ . \tag{5.9}$$

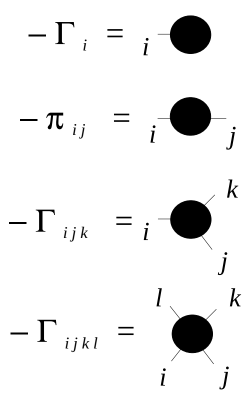
The relation between generating functionals $Z[J]$ and $W[J]$ is now

$$Z[J] = Z[0]e^{-W[J]} \ . \tag{5.10}$$

The SD equation for $Z[J]$ can now be solved by a Laplace transform, and we get the Euclidean path integral

$$Z[J] = \int [d\phi] e^{-S+J_i\phi_i} \ . \tag{5.11}$$

We next proceed to the 1PI diagrams and the effective action. The 1PI diagrams are given by



The effective action is

$$\Gamma[\varphi] = \sum_{m=1}^{\infty} \frac{1}{m!} \Gamma_{i_1\cdots i_m} \varphi_{i_1} \cdots \varphi_{i_m} \ , \tag{5.12}$$

where $\Gamma_{ij} = (\triangle^{-1} + \Pi)_{ij}$. The relation between W and Γ is now given by the Legendre transform

$$W[J] = \Gamma[\varphi] - J_i\varphi_i \ , \tag{5.13}$$

with

$$\frac{\partial W}{\partial J_i} = -\varphi_i \tag{5.14}$$

$$\frac{\partial \Gamma}{\partial \varphi_i} = J_i \tag{5.15}$$

Similarly one can find the Euclidean field theory expressions for fermions. As we shall see, Euclidean field theory is just statistical mechanics. For this reason we continue this lecture with a brief review of thermodynamics.

5.1 Thermodynamics

The central relation of thermodynamics (that embodies the first two laws) is

$$\delta Q \equiv TdS = dU - PdV \tag{5.16}$$

We augment this with the (macroscopic) equations of state that characterize the system, namely

$$P = P(T, V) \tag{5.17}$$
$$U = U(T, V) \tag{5.18}$$

which are determines experimentaly. In principle that's all there is to thermodynamics. All the relations between macroscopic variables follow from this. In practise there is a way to simplify calculations. We do this by introducing the so-called thermodynamic potentials. The first is the internal energy U. The natural variables in terms of which the energy is best given are not T, V as in the equation of state but S, V. This follows directly from our central thermodynamic relation. Therefore, our first potential is $U(S, V)$. From 5.16 we also see that the temperature T is a conjugate variable to the entropy S. Since the entropy is not something we directly measure it is useful to construct a quantity related to the energy whose natural variables are T, V. We know the way to do this — just perform a Legendre transform. We define the free energy to be

$$F = U - TS \ . \tag{5.19}$$

It follows that indeed $F = F(T, V)$. Similarly, one can also define the Gibbs potental to be

$$\Gamma = U - TS + PV = F + PV \ . \tag{5.20}$$

Obviously $\Gamma = \Gamma(T, P)$. The fourth and last thermodynamic potential, the enthalpy $E(S, P)$, is not so useful.

Statistical mechanics makes the connection between microphysics and thermodynamics. One starts from the Hamiltonian $\mathcal{H}(\phi; T, V)$, where ϕ are microscopic dynamical variables. If $\mathcal{H}$ has an explicit T dependence this means that it describes an effective theory, not the fundamental microphysics. We then calculate the partition function

$$Z = \sum_{\{\phi\}} e^{-\frac{1}{T}\mathcal{H}} \ . \tag{5.21}$$

Note that we are being sensible in measuring temperature in units of energy — *i.e.* we have set the Boltzmann constant equal to 1. The conection with thermodynamics follows from the relation

$$Z = e^{-\frac{1}{T}F} \ . \tag{5.22}$$

Calculating Z we determine the free energy, and from this all the other thermodynamic quantities. All of the above has been illustrated on the case of gases. If we look at the thermodynamics of magnetic systems we have the same formulas with the following change $V \to H$, $P \to M$. The external variable is then no longer V (the volume of the gas) but H (the external magnetic field). At the same time the system's response is no longer P (pressure of the gas) but M (the magnetization).

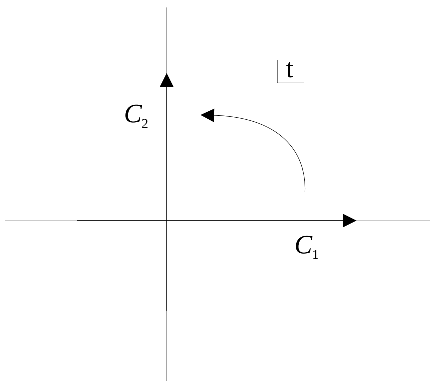
The Euclidean field theory formalism is nothing but statistical mechanics. The translation is just

$$S = \mathcal{H} \tag{5.23}$$
$$\hbar = T \tag{5.24}$$
$$J_i = H_i \tag{5.25}$$
$$W(\hbar, J) = F(T, H) \tag{5.26}$$
$$\Gamma(\hbar, \varphi) = \Gamma(T, M) \tag{5.27}$$

Going from our basic amplitudes we have derived the path integral result for the partition function. In this way we have given a non-standard derivation of statistical mechanics.

5.2 Wick Rotation

We end this lecture by describing the formal connection between Euclidean and Minkowski theories (in $d \geq 1$) as an analytic continuation in time. This procedure is called Wick rotation. We start with a Minkowski theory. If we think of time as a complex variable then the action is given as an integral over the real axis in the complex t plane



Wick rotation is just the rotation from contour C_1 (real t axis) to C_2 (imaginary t axis). To be concrete let's look at the action of a scalar field

$$I = \int_{-\infty}^{\infty} dt \int d\vec{x} \left(\frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\partial_i\phi)^2 - V(\phi) \right) \,. \tag{5.28}$$

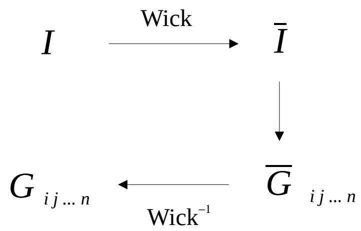
Wick rotation then gives

$$iI \to -\frac{1}{i} \int_{-i\infty}^{+i\infty} dt \int d\vec{x} \left(\frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\partial_i\phi)^2 - V(\phi) \right) = -\bar{I} \,, \tag{5.29}$$

hence writing $\tau = it$ we get

$$\bar{I} = \int_{-\infty}^{+\infty} d\tau \int d\vec{x} \left(\frac{1}{2}(\partial_\tau \phi)^2 + \frac{1}{2}(\partial_i \phi)^2 + V(\phi) \right) \ . \tag{5.30}$$

Note that $\bar{I}$ is bounded from bellow (if the potential $V(\phi)$ is bounded from bellow). In the language of statistical mechanics this just means that the energy is bounded from bellow. The status of the precise definition of the Euclidean path integral is much better than of its Minkowski cousin. In Euclidean theory we are dealing with exponentialy damped integrals, while in the Minkowski case we work with oscillating expressions. For this reason we often calculate Minkowskian Green's functions (after all that is what interests us) in a roundabout way:



Nobody guarantees that this diagram commutes. The fact that the action is analytic in t doesn't automatically imply that all the Green's functions are analytic. Still, usually, this is the only thing we know how to do.

At the end we mention the most important property of Wick rotation. The transformation

$$t \to \tau = it \ , \tag{5.31}$$

takes Minkowski geometry into Euclidean geometry

$$ds^2 = dt^2 - d\vec{x}^2 \to -d\bar{s}^2 = d\tau^2 + d\vec{x}^2 \ . \tag{5.32}$$

This finaly explains why we have called the two formalisms Minkowskian and Euclidean. It is easy to see that this is not a 1-1 map. Far from it. For example $ds = 0$ gives us all the possible trajectories of massless particles (the lightcone), while $d\bar{s} = 0$ just gives the origin. Why then do we Wick rotate? Often this is the *only* thing we know how to do. Perturbatively we do not get into trouble, and most of the time perturbative results are all that we have at our disposal.

Lecture 6

Ferromagnets and Phase Transitions

6.1 Models of Ferromagnets

A ferromagnet is a collection of spins s_n distributed on a d dimensional lattice (for example a hyper cubic lattice) with points indexed by n. We start by introducing a set of spin models whose Hamiltonian is

$$\mathcal{H} = -\frac{1}{2} s_n J_{nm} s_m - H_n s_n \ , \tag{6.1}$$

where

$$J_{nm} = \begin{cases} J & \text{if } n \text{ and } m \text{ are nearest neighbours} \\ 0 & \text{otherwise} \end{cases} \tag{6.2}$$

These models differ in the distribution functions $\rho(s)$ that tell us what values of spin are allowed. We will look at

$$\rho(s) = \begin{cases} \frac{1}{2}\,\delta(s^2-1) & \text{Ising model} \\ \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}s^2} & \text{Gaussian model} \\ C(g)\, e^{-g(s^2-1)^2-\frac{1}{2}s^2} & \text{Landau–Ginsburg model} \end{cases} \tag{6.3}$$

All of these distributions are normalilzed, *i.e.* $\int_{-\infty}^{+\infty} ds\,\rho(s) = 1$. Note that for $g \to 0$ the Landau–Ginsburg model becomes the Gaussian model, while for $g \to \infty$ it goes over into the Ising model.

The partition function is

$$Z[H] = \int d\mu\, e^{-\beta\mathcal{H}} \ , \tag{6.4}$$

where $\beta = \frac{1}{T}$, and the integration measure is given by

$$d\mu = \prod_n ds_n\, \rho(s_n) \ . \tag{6.5}$$

6.2 The Mean Field Approximation

To be specific we look at the Ising model. We may write it as

$$Z[H] = \sum_{\{s\}} \exp\left(\beta s_n (H_n + \frac{1}{2} J_{nm} s_m)\right) \ . \tag{6.6}$$

Let us solve this in the mean field approximation. We approximate the partition function by

$$Z[H] \approx \sum_{\{s\}} \exp\left(\beta s_n (H_n + \frac{1}{2} J_{nm} M_m)\right) \ , \tag{6.7}$$

where we have introduced the magnetization

$$M_n = \langle s_n \rangle = \frac{1}{Z} \frac{1}{\beta} \frac{\partial}{\partial H_n} Z \ . \tag{6.8}$$

The approximate partition function now factorizes

$$\begin{aligned} Z[H] &\approx \prod_n \sum_{s_n} \exp\left(\beta s_n (H_n + \frac{1}{2} J_{nm} M_m)\right) = \\ &= \prod_n 2 \cosh \beta (H_n + \frac{1}{2} J_{nm} M_m) \ . \end{aligned} \tag{6.9}$$

Using this and equation 6.8 we find that (in this approximation) the magnetization satisfies

$$M_n = \tanh \beta (H_n + \frac{1}{2} J_{nm} M_m) \ . \tag{6.10}$$

This completely determines M_n and with it the mean field approximation to the partition function. Using these last two formulas we will now calculate the free energy F and Gibbs potential Γ. For the free energy we find

$$F = -\frac{1}{\beta} \ln Z = -\frac{N}{\beta} \ln 2 - \frac{1}{\beta} \sum_n \ln \cosh \beta (H_n + \frac{1}{2} J_{nm} M_m) \ , \tag{6.11}$$

where N is the number of lattice points, *i.e.* the volume. The natural variables for the Free energy are T and H_n. To get this we would need to solve equation 6.10 for M_n, however, we can't do this in closed form. What we *can* do is solve this equation for H_n. Using this we can then explicitly calculate the Gibbs potential, whose natural variables are T and M_n.

$$\Gamma = F + H_n M_n \tag{6.12}$$

$$M_n = -\frac{\partial F}{\partial H_n} \tag{6.13}$$

$$H_n = \frac{\partial \Gamma}{\partial M_n} \tag{6.14}$$

We first solve 6.10 for H_n. To do this we invert $y = \tanh x$ to get $x = \frac{1}{2}\ln\frac{1+y}{1-y}$. Thus

$$H_n = -\frac{1}{2}J_{nm}M_n + \frac{1}{2\beta}\ln\frac{1+M_n}{1-M_n} \ . \tag{6.15}$$

Now, $y = \tanh x$ implies $\cosh x = \frac{1}{\sqrt{1-y^2}}$, so that we get

$$\begin{aligned}\Gamma = &-\frac{N}{\beta}\ln 2 - \frac{1}{2}M_n J_{nm} M_m + \\ &+ \frac{1}{2\beta}\sum_n \ln(1-M_n^2) + \frac{1}{2\beta}\sum_n M_n \ln\frac{1+M_n}{1-M_n} \ .\end{aligned} \tag{6.16}$$

Simplifying this we find

$$\begin{aligned}\Gamma = &-\frac{N}{\beta}\ln 2 - \frac{1}{2}M_n J_{nm} M_m + \\ &+ \frac{1}{2\beta}\sum_n (1+M_n)\ln(1+M_n) + \frac{1}{2\beta}\sum_n (1-M_n)\ln(1-M_n) \ .\end{aligned} \tag{6.17}$$

For the case of a homogenous external magnetic field $H_n = H$ we have $M_n = M$, and so

$$\begin{aligned}\frac{\Gamma}{N} = &-\frac{1}{\beta}\ln 2 - \frac{1}{2}qJM^2 + \\ &+ \frac{1}{2\beta}(1+M)\ln(1+M) + \frac{1}{2\beta}(1-M)\ln(1-M) \ .\end{aligned} \tag{6.18}$$

q is called the coordination number, and it is just the number of nearest neighbours that a point has on a given lattice. For a hyper cubic lattice in d dimensions $q = 2d$.

6.3 Transfer Matrices

The Ising model in one dimension is exactly soluble. We will solve it using the transfer matrix technique. Let us look at N spins on a circle in a homogenous magnetic field H.

$$Z_N(H) = \sum_{\{s\}} e^{\beta J(s_1 s_2 + s_2 s_3 + \cdots + s_N s_1) + \beta H(s_1 + s_2 + \cdots + s_N)} \ . \tag{6.19}$$

We define the transfer matrix to be

$$T_{ss'} = \exp\left(\beta J s s' + \beta H s\right) \ , \tag{6.20}$$

thus

$$T = \begin{pmatrix} e^{\beta J + \beta H} & e^{-\beta J + \beta H} \\ e^{-\beta J - \beta H} & e^{\beta J - \beta H} \end{pmatrix} \ . \tag{6.21}$$

It follows that we have

$$Z_N(H) = \sum_{\{s\}} T_{s_1 s_2} T_{s_2 s_3} \cdots T_{s_N s_1} = \text{ tr } T^N = {t_1}^N + {t_2}^N \ , \tag{6.22}$$

where t_1 and t_2 are the eigenvalues of the transfer matrix. We are interested in the thermodynamic limit $N \to \infty$. If t_1 is the larger eigenvalue then obviously $Z(H) =\longrightarrow {t_1}^N$. The free energy per spin becomes

$$\frac{F}{N} \to -\frac{1}{\beta} \ln t_1 \ . \tag{6.23}$$

All that is left is to calculate the largest eigenvalue of T. The characteristic equation

$$\begin{vmatrix} e^{\beta J + \beta H} - t & e^{-\beta J + \beta H} \\ e^{-\beta J - \beta H} & e^{\beta J - \beta H} - t \end{vmatrix} = 0 \ , \tag{6.24}$$

gives

$$t^2 - 2\, e^{\beta J} \cosh \beta H \ t + 2 \sinh 2\beta J = 0 \ . \tag{6.25}$$

Solving this we find

$$t_{1/2} = e^{\beta J} \cosh \beta H \pm e^{-\beta J} \sqrt{1 + e^{4\beta J} \sinh^2 \beta H} \ , \tag{6.26}$$

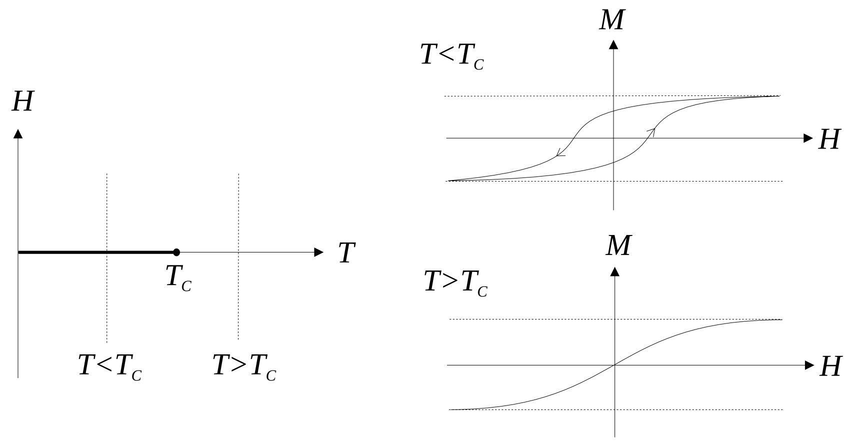
so that finaly we get

$$\frac{F}{N} \to -\frac{1}{\beta} \ln \left(e^{\beta J} \cosh \beta H + e^{-\beta J} \sqrt{1 + e^{4\beta J} \sinh^2 \beta H} \right) \ . \tag{6.27}$$

We end this section by giving a brief comment on why the ferromagnetic models are not trivial. There are two answers to this question. The first is that, in general, the measures are non-trivial, hence these models are not free even though their Hamiltonians are quadratic. The exception is the Gaussian model whose measure is itself the exponent of a quadratic. The second point we may see, for example, on the Gaussian model. This *is* a free field theory, however, the partition function is given in terms of $\triangle$, the inverse of the kinetic term. This is not something that we can easily calculate in closed form.

6.4 Landau–Ginsburg Theory

Ferromagnets often display an interesting phenomenon called spontaneous magnetization — a non-zero magnetization even when there is no external magnetic field present. In field theory language we are dealing with symmetry breaking. Ferromagnetic models will give us the simplest illustration of this important phenomenon. As we shall see, this interesting behaviour is associated with a phase transition. There exists a critical temperature T_c. Bellow this temperature we have spontaneous magnetization, above it we don't.



The line along the T-axis from 0 to T_c is a line of first order phase transitions (there is a discontinuity in the order parameter — in this case the magnetization). The point T_c is the critical point — here we have a second order phase transition. In the vicinity of the critical point the magnetization is small, so we can Taylor expand the Gibbs potential (effective action in field theory language). For an isotropic ferromagnet in a homogenous field we have

$$\Gamma[\vec{M}, T] \approx A(T) + B(T)\vec{M}^2 + C(T)(\vec{M}^2)^2 \ . \tag{6.28}$$

The Gibbs potential satisfies $\frac{\partial\Gamma}{\partial\vec{M}} = \vec{H}$, so that at $\vec{H} = 0$ we have

$$(2B + 4C\vec{M}^2)\vec{M} = 0 \ . \tag{6.29}$$

We need to choose B and C as functions of $T - T_c$ such that for $T > T_c$ the only solution is $\vec{M} = 0$, while for $T < T_c$ we have solutions with $\vec{M} \neq 0$. The simplest choice is to take

$$B(T) = b(T - T_c) \tag{6.30}$$

$$C(T) = c \ , \tag{6.31}$$

where b and c are positive constants. Thus, the simplest phenomenological model that correctly describes the above phase transitions is given by

$$\Gamma[\vec{M}, T] = b(T - T_c)\vec{M}^2 + c(\vec{M}^2)^2 \ . \tag{6.32}$$

If the external field is no longer homogenous then $\vec{M} = \vec{M}(\vec{x})$ and we have

$$\Gamma[\vec{M}, T] = \int d^3x \left(\frac{1}{2}(\nabla M_a)^2 + \frac{1}{2}b(T - T_c)(M_a)^2 + c\left((M_a)^2\right)^2 \right) \ . \tag{6.33}$$

The first term is the simplest way to encode the tendency for nearby spins to align with one another. This is the Landau–Ginsburg model. Knowing Γ we can now directly calculate all thermodynamic quantities. Landau and Ginsburg have given us a qualitatively correct description of phase transitions. Yet, they simply postulated the answer. We next need to derive the correct phenomenology starting from the appropriate microphysics.

6.5 Towards Loop Expansion

In this section we want to apply our loop expansion scheme to ferromagnetic models. To make things a bit more interesting we introduce another, wider, class of spin models — the vector models. For these models the Hamiltonian is given by

$$\mathcal{H} = -\frac{1}{2}\,\vec{s}_n J_{nm} \vec{s}_m - \vec{H}_n \cdot \vec{s}_n \ , \tag{6.34}$$

where the spins $\vec{s}_n$ are N component vectors constrained to be of unit length. For $N = 1$ this is just our old Ising model. For $N = 3$ we have the Heisenberg model. In this case the spin distribution function is $\rho(\vec{s}) = \frac{1}{2\pi}\,\delta(\vec{s}^2 - 1)$.

The partition function is

$$Z[\vec{H}] = e^{-\beta F[\vec{H}]} = \int \prod_n \left(d\vec{s}_n\, \rho(\vec{s}_n)\right) \exp\left(\frac{1}{2}\beta \vec{s}_n J_{nm} \vec{s}_m + \beta \vec{H}_n \cdot \vec{s}_n\right) \ . \tag{6.35}$$

As it stands this expression for the partition function is not amenable to loop expansion. We derived our formulas for loop expansion for the case of theories with a trivial measure. Here, the spin distribution functions lead to a non-trivial measure. However, by using the Gaussian path integral identity

$$\exp\left(\frac{1}{2}\beta \vec{s}_n J_{nm} \vec{s}_m\right) = \int [d\vec{\phi}] \exp\left(-\frac{1}{2\beta}\vec{\phi}_n J^{-1}_{nm} \vec{\phi}_m + \vec{\phi}_n \cdot \vec{s}_n\right) \ , \tag{6.36}$$

we simplify the $\vec{s}_n$ dependence of $Z[\vec{H}]$. It now factorizes into single spin integrals like

$$e^{Q(\vec{\phi})} = \int d\vec{s}\, \rho(\vec{s})\, e^{\vec{s}\cdot\vec{\phi}} \ . \tag{6.37}$$

We now have

$$e^{-\beta F[\vec{H}]} = \int [d\vec{\phi}] \exp\left(-\frac{1}{2\beta}(\vec{\phi}_n - \beta \vec{H}_n) J^{-1}_{nm} (\vec{\phi}_m - \beta \vec{H}_m) + \sum_n Q(\vec{\phi}_n)\right) \ . \tag{6.38}$$

This is a very difficult integral to do, however it is written as an integral over a trivial measure. For this reason we can evaluate it approximately using standard loop expansion. The tree level approximation is given as exercise 6.8. We will calculate the one loop contribution to Γ in a later lecture. Note that, even though loop expansion is well defined, in this case it doesn't correspond to an asymptotic expansion in $1/\beta$. In fact in this case there is no expansion parameter. The reason for this is that we derived 6.38 by doing an integration. The integrand thus corresponds to an effective model. All of this can be seen immediately from the integrand's complicated β dependence.

EXERCISES

6.1 Look at the Ising model in a homogenous field $H_n = H$. In the mean field approximation show that:

(a) For $T < T_c = q\frac{J}{2k}$ there is a spontaneous magnetization.

(b) For $T < T_c$ show that $M(H)$ is a Hysteresis curve.

(c) Show how $M(0)$ depends on T in the vicinity of T_c. Give the result in term of $t = \frac{T-T_c}{T_c}$.

6.2 Calculate the Gibbs potential $\Gamma(\beta, M_n)$ for the Gaussian model in the mean field approximation.

6.3 Show that the Ising ferromagnet in 1 dimension doesn't have a phase transition, *i.e.* there is no spontaneous magnetization.

6.4 Do the Gaussian model exactly. Find Γ and compare with the mean field approximation. Discuss the results.

6.5 Look at a model of Ising-like spins ($s = \pm 1$) whose Hamiltonian is

$$\mathcal{H} = -\frac{1}{2}\frac{J}{N}\sum_{n,m} s_n s_m - \sum_n H_n s_n \ .$$

Here all pairs of spins interact with the same energy. Write the partition function for this model as a path integral in such a way that one may perform an asymptotic expansion for $N \to \infty$. Show that in this limit the mean field result becomes exact. The mean field result is exact for *any* theory in the limit $d \to \infty$. Try to give a hand-waving argument for this.

6.6 Calculate $\vec{M}(\vec{x})$ for the Landau–Ginsburg model if the magnetic field is $\vec{H} = \vec{H}_0\,\delta^3(\vec{x})$. Show that the magnetization is

$$\vec{M} = \frac{\vec{H}_0}{4\pi}\frac{1}{r}\,e^{-\frac{r}{\xi}} \ ,$$

where $\xi = (2b(T - T_c))^{-1/2}$ is the correlation length. This is the length over which spins influence each other. At the critical point ξ diverges.

6.7 Show that for the Ising model

$$Q(\phi) = \ln\cosh\phi \ ,$$

while for the Heisenberg model we get

$$Q(\vec{\phi}) = \ln\left(\frac{\sinh|\vec{\phi}|}{|\vec{\phi}|}\right) \ .$$

6.8 Calculate the tree level approximation of equation 6.38.

(a) Show that to this approximation we have

$$\beta\Gamma[\vec{M}] = -\frac{1}{2}\beta\vec{M}_n J_{nm}\vec{M}_m + \sum_n R(\vec{M}_n) \ ,$$

where $R(\vec{M}) = -Q(\vec{\phi}) + \vec{M}\cdot\vec{\phi}$, and $\vec{M} = \frac{\partial Q}{\partial\vec{\phi}}$.

(b) Calculate this explicitly for the Ising and Heisenberg models.

(c) Expand around $\vec{M} = 0$ and show that one gets the Landau–Ginsburg potential.

Lecture 7

The Propagator

7.1 Scalar Propagator

The first six lectures have been a fast tour of the basic formalism of QFT (and statistical mechanics). Hopefully one now has an overview of the "forest". It is time to get to know some of the "trees" in more detail. In this lecture we look at the space-time propagation of free particles. Scalar free field theory in d dimensions is given by the action

$$S = \int dx \left(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 \right) \ . \tag{7.1}$$

As we have seen in exercise 2.4, in order to do the associated path integral we need to regularize this by taking $m^2 \to m^2 - i\varepsilon$. This gives an exponential damping factor to the path integral. At the end we set ε to zero. From the above action we find

$$\triangle^{-1}(x,y) = -(\partial_x^2 + m^2 - i\varepsilon)\delta(x-y) \ . \tag{7.2}$$

The propagator is just the inverse of this kinetic operator, therefore

$$-(\partial_x^2 + m^2 - i\varepsilon) \triangle (x,y) = \delta(x-y) \ . \tag{7.3}$$

Translation invariance gives us $\triangle(x,y) = \triangle(x-y)$, so that by Fourier transforming the above equation we find

$$\triangle(x-y) = \int \frac{dk}{(2\pi)^d} \frac{1}{k^2 - m^2 + i\varepsilon} e^{-ik\cdot(x-y)} \tag{7.4}$$

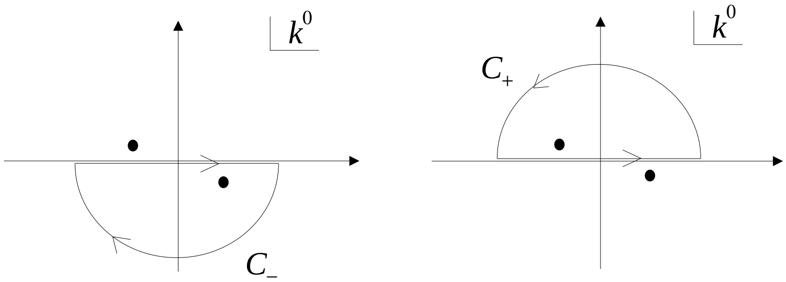
Let us evaluate this expression.

$$\triangle(x) = \int \frac{d\vec{k}}{(2\pi)^{d-1}} e^{i\vec{k}\cdot\vec{x}} \int \frac{dk^0}{2\pi} \frac{e^{-ik^0 t}}{k^{0 2} - E^2 + i\varepsilon} \ , \tag{7.5}$$

where $E = \sqrt{m^2 + \vec{k}^2}$ is the energy of a relativistic particle of mass m. The k^0 integral can be done by contour integration. The poles of the integrand are located at $k^0 = \pm E \mp i\varepsilon$. Note that the $i\varepsilon$ term lifts the poles off the contour of integration and makes the k^0 integral well defined. As we shall see, this precise way of moving the poles (or equivalently of deforming the contour of integration) gives the correct causality properties to the propagator. For $t > 0$ we can close the contour in the negative k^0 plane. Therefore we have

$$I = \oint_{C_-} \frac{dz}{2\pi} \frac{e^{-izt}}{z^2 - E^2 + i\varepsilon} = -\frac{i}{2} \frac{e^{-iEt}}{E} \; . \tag{7.6}$$

The minus sign comes because the contour is traversed in the negative direction.



Similarly, for $t < 0$, we integrate over the contour C_+. In this case the other pole contributes, and we get

$$I = \int_{C_+} \frac{dz}{2\pi} \frac{e^{-izt}}{z^2 - E^2 + i\varepsilon} = -\frac{i}{2} \frac{e^{iEt}}{E} \; . \tag{7.7}$$

Finally, we may write the propagator as

$$\triangle(x) = \theta(t) \triangle^+ (x) + \theta(-t) \triangle^- (x) \; , \tag{7.8}$$

where we have introduced the auxilliary quantities

$$\triangle^{\pm}(x) = \int \frac{d\vec{k}}{(2\pi)^{d-1}} \frac{1}{2iE} e^{\mp iEt + i\vec{k}\cdot\vec{x}} \; . \tag{7.9}$$

We see that $\triangle$ is Lorentz invariant, complex, and that it propagates signals both into the future and into the past. $\triangle$ is called the Feynman propagator. If we were looking at the classical theory following from 7.1 we would also be interested in finding the propagator. We would use it to solve the inhomogenous equation

$$(\partial_x^2 + m^2)\Phi(x) = J(x) \; . \tag{7.10}$$

In the classical theory, however, we have no $i\varepsilon$ prescription, and the propagator satisfies

$$-(\partial_x^2 + m^2)G(x, y) = \delta(x - y) \; . \tag{7.11}$$

This is again solved by using the Fourier transform. This time, however, the propagator is given in terms of a singular integral — the poles of the integrand lie on the contour of integration. We must choose a specific way to deform the contour of integration. Different choices give us different propagators — solutions of the above equation that have different behaviour at infinity. Most of these solutions aren't even Lorentz invariant. As we know, for example from electrodynamics, the propagator that we finaly use is called the retarded propagator. The retarded propagator is Lorentz invariant, real, and it propagates signals only into the future. There is no mystery why classical field theory uses the retarded propagator, while quantum field theory uses the Feynman propagator. They are used to do different jobs. For example, in classical theory we use the propagator to solve equation 7.10 in terms of real fields Φ. The associated propagator thus has to be real. In quantum field theory the propagator is an amplitude, hence there is no reason why it should be real. The fact that in QFT particles propagate both into the future as well as into the past is a bit more subtle. We will look at this in the next section.

7.2 Random Walk

It is not enough to be able to calculate the propagator. In this section we want to see in more detail just what the propagator does. To get a clearer picture we now look at our scalar particle propagating on a discrete d dimensional space-time, *i.e.* on a lattice. It is usefull to work with the Wick rotated theory as that allows us to treat space and time on an equal footing. The propagator is given as the solution of the Euclidean equation

$$(\partial^2 - m^2) \triangle (x-y) = -\delta(x-y) \ . \tag{7.12}$$

On a hyper cubic lattice of spacing a this discretizes to

$$\frac{1}{a^2}\sum_i \Big(\triangle (a\vec{n} + a\hat{e}_i - a\vec{m}) - 2 \triangle (a\vec{n} - a\vec{m}) + \\ + \triangle(a\vec{n} - a\hat{e}_i - a\vec{m})\Big) - m^2 \triangle (a\vec{n} - a\vec{m}) = -\frac{1}{a^d}\,\delta_{\vec{n},\vec{m}} \ . \tag{7.13}$$

It is now convenient to rescale $\triangle$ according to

$$\triangle(a\vec{n} - a\vec{m}) = \frac{1}{m^2 a^d}\,\triangle_{\vec{n},\vec{m}} \ . \tag{7.14}$$

Note that $\triangle_{\vec{n},\vec{m}}$ is dimensionless. We now have

$$\frac{1}{m^2a^2}\sum_i (\triangle_{\vec{n}+\hat{e}_i,\vec{m}} + \triangle_{\vec{n}-\hat{e}_i,\vec{m}}) - \left(1 + \frac{2d}{m^2a^2}\right)\triangle_{\vec{n},\vec{m}} = -\delta_{\vec{n},\vec{m}} \ . \tag{7.15}$$

For later convenience we will set

$$m^2 = \frac{s}{ha^2} \tag{7.16}$$

$$s + 2dh = 1 \ . \tag{7.17}$$

This allows us to write the above propagator equation in matrix form as

$$(1 - h\Sigma) \triangle = s \ , \tag{7.18}$$

where we have introduced the step matrix

$$\Sigma_{\vec{n},\vec{m}} = \sum_i (\delta_{\vec{n}+\hat{e}_i,\vec{m}} + \delta_{\vec{n}-\hat{e}_i,\vec{m}}) \ . \tag{7.19}$$

Equation 7.18 is easily solved. We find

$$\triangle_{\vec{n},\vec{m}} = s \left(\frac{1}{1 - h\Sigma} \right)_{\vec{n},\vec{m}} = s \sum_{L=0}^{\infty} h^L \left(\Sigma^L \right)_{\vec{n},\vec{m}} \ . \tag{7.20}$$

It is not difficult to see that $\left(\Sigma^L\right)_{\vec{n},\vec{m}}$ just gives the number of paths of L links that connect the points $\vec{n}$ and $\vec{m}$. We now have a further way of writing the propagator, namely

$$\triangle_{\vec{n},\vec{m}} = s \sum_p h^{L(p)} \ , \tag{7.21}$$

the sum being over all paths from $\vec{n}$ to $\vec{m}$. This is the formula for a random walk. A random walker goes from one point to another by taking unit steps. At each point he has equal probability to hop to any of the nearest neigbour points. Call this probability h. The walker also has some probability s to stop his walk. If we calculate the total probability to start at $\vec{n}$ and finally stop at $\vec{m}$ we just get our previous result. Obviously s and h are related. At any given point the walker will either hop to one of the 2d nearest neighbours or stop. Therefore $s + 2dh = 1$, which is indeed what we had. Of course, in our propagator problem $\triangle$, h and s are amplitudes, not probabilities. This is just the relation between Euclidean field theory and Statistical Mechanics. One has the same formalism but a different interpretation. In terms of our original quantities we have

$$\triangle(a\vec{n} - a\vec{m}) = \frac{s}{m^2 a^d} \sum_p e^{-\frac{1}{a}\ln(2d + a^2 m^2)\ell(p)} \ , \tag{7.22}$$

where $\ell(p) = aL(p)$ is the length of the path p. Now we may take the continuum limit and find

$$\triangle = \int [dq] e^{-S[q]} \ , \tag{7.23}$$

where $S[q] = m_0 \int ds$ is the *first quantized* action for a relativistic particle of mass m_0 (ds is just the space-time interval). We also have

$$m_0 = \lim_{a \to 0} \frac{1}{a} \ln(2d) = \infty \ . \tag{7.24}$$

This is the simplest example of renormalization — the bare mass m_0 has to be infinite in order for the physical mass m (given as the pole of the propagator) to be finite. If we write our space-time point as $x = (\tau, \vec{x})$, then equation 7.23 is actually

$$\triangle(\tau, \vec{x} - \vec{y}) = \int_{\vec{x}(0)=\vec{x}}^{\vec{x}(\tau)=\vec{y}} \prod_{\tau \in \mathbb{R}} d\vec{x}(\tau)\, e^{-m_0 \int_0^\tau \sqrt{\dot{\vec{x}} \cdot \dot{\vec{x}}}\, d\tau} \ . \tag{7.25}$$

Contrast this first quantization path integral result with our field theory *i.e. second quantized* path integral for the same propagator

$$\triangle(x - y) = \frac{1}{Z} \int \prod_{z \in \mathbb{R}^d} d\phi(z)\, \phi(x)\phi(y)\, e^{-\int dz \left(\frac{1}{2}(\partial\phi)^2 + \frac{1}{2} m^2 \phi^2\right)} \ . \tag{7.26}$$

To recapitulate: what we have done is to show is that in quantum field theory propagation corresponds to a random walk in space-time. Free particles go where they please — left or right, up or down, into the future or into the past. Each path gives an associated amplitude, and the total amplitude is found by summing over all paths. The free particle drunkenly careens every which way — it is a random walker. If the walker has $s \gg h$ then this implies that the particle is very heavy. On the other hand if $h \gg s$ we are dealing with a very light particle. Massless particles can't stop, *i.e.* they have $s = 0$.

EXERCISES

7.1 How does one have to deform the contour of integration to get the retarded propagator? Compare this to the contour of integration for the Feynman propagator.

7.2 In $d = 2$ one can evaluate the scalar propagator for massless particles in closed form in terms of elementary functions. Do this.

7.3 Show that $(\Sigma^L)_{\vec{n},\vec{m}}$ equals the number of paths of L links that connects $\vec{n}$ and $\vec{m}$.

Lecture 8

The Propagator Continued

8.1 The Yukawa Potential

Let us look at two scalar fields ϕ and A interacting through a cubic coupling $\phi^2 A$. This kind of interaction is called a Yukawa coupling. The action for our model is

$$S = \int dx \left(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - g\phi^2 A + \frac{1}{2}(\partial A)^2 - \frac{1}{2}M^2A^2 \right) \ . \tag{8.1}$$

We are going to assume that $M \gg m$. If our experiments are at energies that are small compared with M then all we see are ϕ particles. Therefore, all that interests us is to look at the generating functional

$$Z[J] = \int [d\phi]\, [dA] e^{iS + i\int dx\, J\phi} \ . \tag{8.2}$$

We don't need to couple A to an external field since our experiments can only probe the Green's function built out of the ϕ field. The integral over A is just a Gaussian and can be done exactly. We find

$$Z[J] = \int [d\phi]\, e^{iS_{eff}[\phi] + i\int dx\, J\phi} \ . \tag{8.3}$$

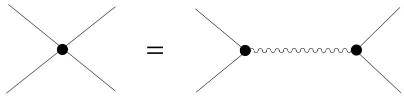
The exact ϕ field dynamics is encoded in the effective action

$$S_{eff}[\phi] = \int dx \left(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 \right) + W_0[-g\phi^2] \ , \tag{8.4}$$

where

$$W_0[J] = -\frac{1}{2}\int dx\, dy\, J(x) \triangle (x - y; M) J(y) \ . \tag{8.5}$$

Because of the W term, the effective theory for ϕ particles is non-local. W represents a non-local ϕ^4 interaction. From our derivation we see that the interaction is mediated by the exchange of an A particle.



Let us now look at the A field propagator. For very large M we have

$$\int \frac{dk}{(2\pi)^d} \frac{e^{-ik\cdot(x-y)}}{k^2 - M^2} \to -\frac{1}{M} \int \frac{dk}{(2\pi)^d} e^{-ik\cdot(x-y)} = -\frac{1}{M^2}\delta(x-y) \ . \qquad (8.6)$$

Well, this is a hand waving argument. To be a bit more precise we first need to Wick rotate to the Euclidean theory. We then introduce an ultra violet cut off Λ and restrict k integration to $|k| < \Lambda$. Now we take the large M limit. Only then do we lift the UV regulator (keeping in mind that $M \gg \Lambda$) and recover the above delta function. Finaly, we Wick rotate back to Minkowski theory. In the large M limit the effective ϕ dynamics is given by

$$S_{eff}[\phi] = S_0[\phi] - \frac{1}{2}g^2 \int dx\, \phi^4 \ . \qquad (8.7)$$

This is a *local* ϕ^4 theory. This action is valid for length scales bigger than $\frac{1}{M}$. To get the moral of the story we turn this argument around. If we have a local field theory (say 8.7) that works down to length scales $\frac{1}{M}$ it may only be an effective theory. The true (fundamental) theory might have further (massive) particles in it that act as a mediators of the ϕ interaction.

To get another perspective on the Yukawa interaction and the associated effective theory, let us look at the induced interaction term 8.5. For two static point sources we have

$$J(x) = g_1\,\delta(\vec{x} - \vec{x}_1) + g_2\,\delta(\vec{x} - \vec{x}_2) \ , \qquad (8.8)$$

and the interaction term equals

$$-\int dx\, dy\, g_1\,\delta(\vec{x} - \vec{x}_1) \triangle (x-y; M) g_2\,\delta(\vec{y} - \vec{x}_2) \ . \qquad (8.9)$$

Integrations over $\vec{x}$ and $\vec{y}$ are trivial. The integration over y^0 yealds another delta function, and this gets rid of the k^0 integration. We find

$$g_1 g_2 \int dx^0 \int \frac{d\vec{k}}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)}}{\vec{k}^2 + M^2} \ . \qquad (8.10)$$

The above expression can be written as $-\int dx^0\, V(\vec{x}_1 - \vec{x}_2)$, where V is called the Yukawa potential — it represents an effective potential giving the attraction of the two charges g_1 and g_2. To calculate the V we go to spherical coordinates. The angular integrals are easily done and we get

$$-\frac{g_1 g_2}{(2\pi)^2 ir} \int_0^{\infty} dk\, \frac{k}{k^2 + M^2} \left(e^{ikr} - e^{-ikr}\right) =$$
$$= -\frac{g_1 g_2}{(2\pi)^2 ir} \int_{-\infty}^{+\infty} dk\, \frac{k}{k^2 + M^2}\, e^{ikr} \ . \qquad (8.11)$$

This integral is easily solved by contour integration. Finaly, the Yukawa potential equals

$$V = -\frac{1}{4\pi} \frac{g_1 g_2}{|\vec{x}_1 - \vec{x}_2|} e^{-M\,|\vec{x}_1 - \vec{x}_2|} \,. \tag{8.12}$$

The Yukawa potential gives an atraction if the two charges g_1 and g_2 have the same sign. In fact, if we did this calculation for fields of various spin we would find that like charges attract for fields with even spin and repel for odd spin. We are familiar with the spin 1 case from electrodynamics where we indeed find that like charges repel. In gravitation the particle masses are the charges. Therefore, charges are necessarily of the same sign. Gravitation is mediated by a spin 2 particle (the graviton) and so the masses attract each other. In both electrodynamics and gravitation the particles that mediate the interactions are massless, so that the above formula just gives the standard $\frac{1}{r}$ potential.

From 8.12 we see that the two charges g_1 and g_2 can form a bound state whose size is roughly $\frac{1}{M}$. Yukawa originaly used this kind of argument to try and explain nuclear forces. He assumed that they were mediated by scalar fields. From the knowledge of the size of nuclear interactions he calculated that the mediating particles should have a mass of approximately 100 MeV. Although such particles were soon found it turns out they are not responsible for the strong interaction. However, the basic reasoning given above does hold and represents an important part of the Standard Model.

8.2 Virtual Particles

In the previous lecture we looked at the scalar field propagator in d dimensions. We will calculate it in closed form for the case of massless particles in $d = 4$. We had done the k^0 integration and found

$$\int \frac{dk^0}{2\pi} \frac{e^{-ik^0 t}}{k^{0\,2} - E^2 + i\varepsilon} = \begin{cases} -\frac{i}{2E}\, e^{-iEt} & \text{if } t > 0 \\ -\frac{i}{2E}\, e^{iEt} & \text{if } t < 0 \end{cases} \tag{8.13}$$

Alternately, this can be written as

$$\int \frac{dk^0}{2\pi} \frac{e^{-ik^0 t}}{k^{0\,2} - E^2 + i\varepsilon} = -\frac{i}{2} \frac{e^{-iE\,|t|}}{E} \,. \tag{8.14}$$

Using this, and writing the remaining integrals in spherical coordinates, we find that the $d = 4$ propagator becomes

$$\triangle(x) = -\frac{\pi i}{(2\pi)^3} \int_0^\infty k^2 dk \int_{-1}^{1} d\cos\theta \, \frac{1}{\sqrt{k^2 + m^2}} \, e^{ikr\cos\theta} \, e^{-i\sqrt{k^2+m^2}\,|t|} \,. \tag{8.15}$$

Doing the θ integration we get

$$-\frac{1}{8\pi^2 r} \int_0^\infty dk \, \frac{k}{\sqrt{k^2 + m^2}} \, e^{-i\sqrt{k^2+m^2}\,|t|} \left(e^{ikr} - e^{-ikr}\right) \,. \tag{8.16}$$

For $m = 0$ this simplifies to

$$-\frac{1}{8\pi^2 r}\left(\int_0^\infty dk\, e^{-ik(\,|t|-r)} - \int_0^\infty dk\, e^{-ik(\,|t|+r)}\right) \ . \tag{8.17}$$

We regularize this by taking $|t| \to |t| - i\varepsilon$. The above integrals are now trivialy done, and we find

$$\triangle_{m=0}(x) = \frac{i}{4\pi^2}\,\frac{1}{x^2 - i\varepsilon} \ . \tag{8.18}$$

Using the identity

$$\frac{1}{A - i\varepsilon} = \frac{1}{A} + \pi i\,\delta(A) \ , \tag{8.19}$$

we find

$$\triangle_{m=0}(x) = \frac{i}{4\pi^2}\,\frac{1}{x^2} - \frac{1}{4\pi}\,\delta(x^2) \ . \tag{8.20}$$

From this result we see that for massless particles propagation along the light cone $x^2 = 0$, *i.e.* at the speed of light, is dominant. However, we also see that there are contributions from all speeds both slower and faster than light. This is not a paradox. The Feynman propagator is not an amplitude for a physical process. It is just a mathematical object out of which we build Green's functions. Even the Green's functions aren't directly physical. As we shall later see, the truly physical, *i.e.* measurable, objects are the scattering matrix amplitudes that follow from the Green's functions through the use of the LSZ reduction formulas. It is all very well to say that Feynman propagators are just mathematical objects that give us the right answer for physicaly observable quantities. However, we have gained a lot of insight from viewing Feynman diagrams as true scattering processes. It is customary to keep this nice visualization and say that what propagates inside the diagrams are *virtual* and not physical particles. In the $m = 0$ case the physical particles move at the speed of light, while virtual particles move according to the Feynman propagator. As we have seen in the previous lecture, virtual particles can go both forward and backword in time, both faster or slower than light.

We have looked at virtual particles for massless fields. The only reason for doing this is that in that case it was possible to calculate the Feynman propagator in closed form in terms of elementary functions. However, it is not difficult to show that propagators for massive fields also get contributions from virtual particles. Virtual particles are also called off shell particles, *i.e.* particles that do not satisfy the mass shell relation $p^2 = m^2$. Off shell contributions to the propagator decrese as we look at propagation at larger distances. Effectively, over macroscopic distances, only the physical particles contribute.

EXERCISES

8.1 Calculate the Yukawa potential for $d = 2$ and $d = 6$.

8.2 Prove that $\frac{1}{A-i\varepsilon} = \frac{1}{A} + \pi i\,\delta(A)$. The best way to do this is to first prove the more symmetric looking identity $\frac{1}{A-i\varepsilon} - \frac{1}{A+i\varepsilon} = 2\pi i\,\delta(A)$.

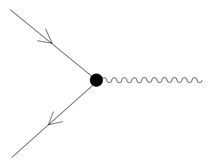
8.3 Consider an effective scalar theory in $d = 4$ whose propagator is

$$\triangle(x-y) = \int \frac{dk}{(2\pi)^4} \frac{e^{-ik\cdot(x-y)}}{k^4}$$

Calculate the potential between two static point sources. When do we have bound states? What is the characteristic size of the bound state? Show that we have confinement, *i.e.* that it takes an infinite ammount of energy to "ionize" the bound state.

8.4 Give a dimensional argument for what the effective scalar propagator has to be in d dimensions in order to have confinement.

8.5 The basic interaction in quantum electrodynamics (QED) is given by



Everything is built out of this. However, at least one of the particles above must be virtual — prove this.

8.6 From the previous exercise we see that virtual particles play a fundamental role in mediating interactions in QFT. Discuss this qualitativly using the Heisenberg relations of quantum mechanics.

Lecture 9

From Operators to Path Integrals

9.1 Hamiltonian Path Integral

We have learned how important quantities like $Z[J]$ may be written as integral expressions. We have called these expressions path integrals, although, strictly, in $d = 0$ (where all our derivations were) these were just ordinary multiple integrals. As we have seen only the physicaly interesting $d \geq 1$ case leads to real path integrals. $d = 1$ is the first non-trivial case. This is just mechanics — fields depend only on the time t. To make things look more like mechanics we shall call our fields $q(t)$. We know how to canonicaly quantize mechanics. In this lecture we will derive the path integral from the canonical formalism.

The classical dynamics of a particle moving along a line is determined by the Hamiltonian $H(p, q) = f_{ab}\, p^a q^b$. Canonical quantization gives $q, p \to \hat{q}, \hat{p}$ so that $[\hat{q}, \hat{p}] = i\hbar$. Note that this doesn't uniquely specify the quantum Hamiltonian. As always one has to specify the ordering prescription. The two simplest ordering prescriptions one could use would be

$$\hat{H}_L = f_{ab}\, \hat{p}^a \hat{q}^b \quad \text{or} \quad \hat{H}_R = f_{ab}\, \hat{q}^b \hat{p}^a \; . \tag{9.1}$$

It is most natural, however, to choose the symmetric or Weyl ordering. For example, $(\hat{q}\hat{p})_W = \frac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q})$. Unlike the other two orderings, Weyl ordering gives us a Hermitean Hamiltonian. Let us look at the transition amplitude for going from $q_i(t_i)$ to $q_f(t_f)$. We have

$$\langle q_f, t_f | q_i, t_i \rangle = \langle q_f | \hat{U}(t_f - t_i) | q_i \rangle \; , \tag{9.2}$$

where

$$\hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}\, t} \tag{9.3}$$

is the time evolution operator. If we subdivide the time interval according to $t_n = t_i + n\varepsilon$ for $n = 0, 1, \ldots, N$ so that $t_f - t_i = N\varepsilon$ — *i.e.* $t_0 = t_i$ and $t_N = t_f$, we find that the above amplitude becomes

$$\int dq_1 \cdots dq_{N-1} \langle q_N|\hat{U}(\varepsilon)|q_{N-1}\rangle\langle q_{N-1}|\hat{U}(\varepsilon)|q_{N-2}\rangle \cdots \langle q_1|\hat{U}(\varepsilon)|q_0\rangle \ . \tag{9.4}$$

All we did is insert $N-1$ resolutions of the identity. We are thus led to calculate the transition amplitudes $\langle q_{n+1}|\hat{U}(\varepsilon)|q_n\rangle$ for short times ε, *i.e.* for large N. For example, for L-ordering we get

$$\begin{aligned}
\langle q_{n+1}|\hat{U}(\varepsilon)|q_n\rangle_L &\approx \langle q_{n+1}|(1-\frac{i}{\hbar}\varepsilon\hat{H}_L)|q_n\rangle = \\
&= \int \frac{dp_n}{2\pi\hbar} e^{\frac{i}{\hbar}p_n q_{n+1}} \langle p_n|(1-\frac{i}{\hbar}\varepsilon\hat{H}_L)|q_n\rangle = \\
&= \int \frac{dp_n}{2\pi\hbar} e^{\frac{i}{\hbar}p_n(q_{n+1}-q_n)} \left(1-\frac{i}{\hbar}\varepsilon H(p_n,q_n)\right) \approx \\
&\approx \int \frac{dp_n}{2\pi\hbar} \exp\frac{i}{\hbar}\varepsilon\left(p_n\frac{q_{n+1}-q_n}{\varepsilon} - H(p_n,q_n)\right) \ .
\end{aligned} \tag{9.5}$$

Similarly, for R-ordering we get

$$\langle q_{n+1}|\hat{U}(\varepsilon)|q_n\rangle_R \approx \int \frac{dp_n}{2\pi\hbar} \exp\frac{i}{\hbar}\varepsilon\left(p_n\frac{q_{n+1}-q_n}{\varepsilon} - H(p_n,q_{n+1})\right) \ . \tag{9.6}$$

From now on we shall use Weyl ordering. This is exactly in between the L and R prescriptions, so it is not surprising that we get

$$\langle q_{n+1}|\hat{U}(\varepsilon)|q_n\rangle_W \approx \int \frac{dp_n}{2\pi\hbar} \exp\frac{i}{\hbar}\varepsilon\left(p_n\frac{q_{n+1}-q_n}{\varepsilon} - H(p_n,\frac{q_n+q_{n+1}}{2})\right) \ . \tag{9.7}$$

For the full transition amplitude we thus get

$$\begin{aligned}
\langle q_f,t_f|q_i,t_i\rangle_W &= \lim_{N\to\infty}\int dq_1\cdots dq_{N-1}\,\frac{dp_0}{2\pi\hbar}\cdots\frac{dp_{N-1}}{2\pi\hbar}\,\cdot \\
&\cdot \exp\left(\frac{i}{\hbar}\sum_{n=0}^{N-1}\varepsilon\left(p_n\frac{q_{n+1}-q_n}{\varepsilon} - H(p_n,\frac{q_n+q_{n+1}}{2})\right)\right) = \\
&= \int [dq\,dp]\exp\left(\frac{i}{\hbar}\int_{t_i}^{t_f} dt\,(p\dot{q} - H(p,q))\right) \ .
\end{aligned} \tag{9.8}$$

The last equality is just the definition of the phase space path integral. From now Weyl ordering will be understood, so we can drop the W subscript. The beauty of the path integral 9.8 is that all of quantum mechanics is given in terms of classical quantities. At the same time, the path integral allows us to make use of many of the properties of ordinary integrals. We will often continue to treat the path integral naively as just another integral. However, it is *not* a Riemann integral. An ordinary Riemann integral is defined as the limit

$$\lim_{N\to\infty}\sum_{n=0}^{N-1}\varepsilon\,f(\xi_n) = \int_a^b dx\,f(x) \ , \tag{9.9}$$

where $\xi_n = \frac{x_n + x_{n+1}}{2}$, $x_0 = a$ and $x_n = b$. The important property of Riemann integrals is that *any* choice of $\xi_n \in [x_n, x_{n+1}]$ gives the same result. As we have seen from equations 9.5, 9.6 and 9.7 there is no analogous property for path integrals and different discretizations give different results. This is important — it is here that path integrals encode information about ordering prescriptions *i.e.* commutation relations. From the mathematical point of view, however, this makes path integrals much more complex than ordinary integrals. For this reason we still lack a general theory of path integrals. In fact, essentaly the only thing we know about them is the defining relation 9.8. Having said this, through most of this book we will continue to be cavalier about subtleties and write formal expressions like

$$[dq] = \prod_{t \in \mathbb{R}} dq(t) \ . \tag{9.10}$$

Generalizations of these results from one to many degrees of freedom are straight forward. The measure is then

$$[dp\, dq] = \prod_{n=1}^{N-1} dq^1{}_n \cdots dq^k{}_n \prod_{n=0}^{N-1} dp_{1\,n} \cdots dp_{k\,n} \ , \tag{9.11}$$

while in the exponent we have the action $\int dt\, (p_a q^a - H(p,q))$.

9.2 Lagrangian Path Integral

The phase space path integral 9.8 is still not the final expression that we seek. What we want is to have the transition amplitude in the form of an integral over configuration space only. To get this we need to integrate out the momenta in 9.8. Let us do this for the simplest (and most important) case of theories with a Lagrangian of the form

$$L = \frac{1}{2}\, q^2 - V(q) \ . \tag{9.12}$$

We now have $H = \frac{1}{2}\, p^2 + V(q)$, and the transition amplitude is simply

$$\lim_{N\to\infty} \int dq_1 \cdots dq_{N-1} \frac{dp_0}{2\pi\hbar} \cdots \frac{dp_{N-1}}{2\pi\hbar} \cdot \\ \cdot \exp \frac{i}{\hbar} \varepsilon \sum_{n=0}^{N-1} \left(p_n \frac{q_{n+1} - q_n}{\varepsilon} - \frac{1}{2} p_n^2 - V(\frac{q_{n+1} + q_n}{2}) \right) \ . \tag{9.13}$$

The momentum integrals are just Gaussian, and can be easily done. We find

$$\int \frac{dp_n}{2\pi\hbar} \exp\left(-\frac{i\varepsilon}{2\hbar} p_n^2 + \frac{i}{\hbar}(q_{n+1} - q_n) p_n \right) = \\ = \frac{1}{\sqrt{2\pi\hbar i\varepsilon}} \exp\left(\frac{i\varepsilon}{2\hbar} \left(\frac{q_{n+1} - q_n}{\varepsilon} \right)^2 \right) \ . \tag{9.14}$$

Therefore, we get

$$\begin{aligned}\langle q_f,t_f|q_i,t_i\rangle &= \lim_{N\to\infty}(2\pi\hbar i\varepsilon)^{-N/2}\int dq_1\cdots dq_{N-1}\,\cdot\\ &\cdot\exp\frac{i}{\hbar}\varepsilon\sum_{n=0}^{N-1}\left(\frac{1}{2}\left(\frac{q_{n+1}-q_n}{\varepsilon}\right)^2-V\left(\frac{q_{n+1}+q_n}{2}\right)\right)=\\ &= C\int[dq]\,e^{\frac{i}{\hbar}\int_{t_i}^{t_f}dt\,L(q,\dot q)}\ .\end{aligned}\tag{9.15}$$

This is the sought-after coordinate space path integral. Note that this is the same simple q measure as in 9.8. The difference is only in the (infinite) normalization factor

$$C=\lim_{N\to\infty}\left(2\pi\hbar i\,\frac{t_f-t_i}{N}\right)^{-N/2}\ .\tag{9.16}$$

We can usualy forget this normalization. For example, when we calculate normalized Green's functions we are only dealing with ratios of two path integral expression, so that overall normalizations cancel. Unlike the phase space path integral 9.8, its coordinate space cousin is not always as simple as 9.15 — this expression is valid only for theories whose Lagrangian is of the form 9.12. In general we have quite a difficult momentum integrations to perform. What we get is

$$\langle q_f,t_f|q_i,t_i\rangle=\int d\mu\,e^{\frac{i}{\hbar}\int_{t_i}^{t_f}dt\,L(q,\dot q)}\ ,\tag{9.17}$$

where $d\mu$ is a complicated measure factor that in general depends on $\hbar$. We may write it as

$$d\mu=e^{\frac{i}{\hbar}M[q,\hbar]}\,[dq]\ .\tag{9.18}$$

Thus, the transition amplitude becomes

$$\langle q_f,t_f|q_i,t_i\rangle=\int[dq]\,e^{\frac{i}{\hbar}S}\ .\tag{9.19}$$

The quantum action is

$$S=I+M[q,\hbar]\ .\tag{9.20}$$

All the difficulty is in finding the measure term M, although we now have a definite proceedure of how to do this. The actions I and S must have the same classical limit. It follows that in the $\hbar\to 0$ limit we must have

$$M[q,\hbar]\to 0\ .\tag{9.21}$$

The phase space path integral completely determines the theory. It also represents the starting point for determining the measure term for the coordinate space path integral. On the other hand, as we shall see, the phase space path integral is even more difficult to define rigorously than its coordinate space cousin.

9.3 Quantum Field Theory

In QFT we need the vacuum-to-vacuum transition amplitudes. We shall now show that these amplitudes can also be written down in terms of path integrals. We turn to what we already know — transition amplitudes from $Q(T)$ to $Q'(T')$. Inserting two resolutions of the identity at intermediate times t and t', where $T < t < t' < T'$,

this amplitude may be written as

$$\langle Q', T' | Q, T \rangle = \int dq\, dq'\, \langle Q', T' | q', t' \rangle \langle q', t' | q, t \rangle \langle q, t | Q, T \rangle \ . \tag{9.22}$$

Note that

$$\langle q, t | Q, T \rangle = \langle q | e^{-\frac{i}{\hbar} \hat{H}(t-T)} | Q \rangle = \sum_n \langle q | n \rangle e^{-\frac{i}{\hbar} E_n (t-T)} \langle n | Q \rangle \ . \tag{9.23}$$

We we now regularize the Hamiltonian with the usual $i\epsilon$ prescription, that is we add to it an infinitesimal imaginary term $-\frac{1}{2} i\epsilon \hat{q}^2$. The above expression now has a well defined $T \to -\infty$ limit. In this limit the vacuum energy term dominates, and we get

$$\langle q, t | Q, T \rangle \to \langle q | 0 \rangle \langle 0 | Q \rangle \ . \tag{9.24}$$

For simplicity we have supposed that the vacuum energy vanishes. Similarly we can evaluate $\langle Q', T' | q', t' \rangle$ in the $T' \to \infty$ limit. Finally we find

$$\begin{aligned} &\langle Q', +\infty | Q, -\infty \rangle \\ &\quad = \langle Q' | 0 \rangle \langle 0 | Q \rangle \int \langle 0 | q' \rangle \, dq' \, \langle q' | e^{-\frac{i}{\hbar} \hat{H}(t'-t)} | q \rangle \, dq \, \langle q | 0 \rangle = \\ &\quad = \text{const.} \, \langle 0 | e^{-\frac{i}{\hbar} \hat{H}(t'-t)} | 0 \rangle = \text{const.} \, Z[J] \ . \end{aligned} \tag{9.25}$$

In the last step we have taken $t \to -\infty$ and $t' \to \infty$, in order to make contact with $Z[J] = \langle 0, +\infty | 0, -\infty \rangle$. We have thus found the path integral expression for the generating functional

$$Z[J] = \int d\mu \, e^{\frac{i}{\hbar} \int_{-\infty}^{+\infty} dt \, (L + Jq)} = \int [dq] \, e^{\frac{i}{\hbar} (S + \int_{-\infty}^{+\infty} dt \, Jq)} \ , \tag{9.26}$$

where we have absorbed the above constant into the definition of the path integral measure. This is precisely what we derived previously when we treated the path integral naively. The added bonus is that we now know how to determine the measure $d\mu$, or equivalently the quantum action S.

EXERCISES

9.1 Show that the symmeteric (Weyl) ordering prescription indeed gives 9.7.

9.2 Calculate the quantum action S for a theory with Lagrangian $L = \frac{1}{2}\, G(q)\dot{q}^2$.

9.3 Show that the Green's functions of QFT can be written in the operator formalism as the following expectation values

$$G(t_1, \ldots t_n) = \langle 0 |\, \mathrm{T}\, (\hat{q}(t_1) \cdots \hat{q}(t_n))\, | 0 \rangle \ ,$$

where the time ordering T orders operators from right to left according to increasing time.

9.4 Calculate $\langle q' | (\hat{q}\hat{p} - \hat{p}\hat{q}) | q \rangle$ in two ways. First by inserting appropriate resolutions of the identity, and second by doing the comutator. Compare the two results.

Lecture 10

Path Integral Surprises

10.1 Paths that don't Contribute

Let us try to make sense of the measure $[d\phi]$ for the case of Euclidean free field theory in $d = 1$. Formaly we have

$$[d\phi] = \prod_{\tau \in \mathbb{R}} d\phi(\tau) \ . \tag{10.1}$$

To regularize this formal expression let τ take its values not on the real line, but on a circle, *i.e.* $0 \leq \tau < \beta$. At the end we may take the $\beta \to \infty$ limit. We thus work on the space of periodic fields $\phi(\tau) = \phi(\tau + \beta)$. The action for a free scalar field is in this case equal to

$$S = \int_0^\beta d\tau \left(\frac{1}{2}(\partial_\tau \phi)^2 + \frac{1}{2} m^2 \phi^2 \right) \ . \tag{10.2}$$

Let us first diagonalize the kinetic operator above. To do this we solve the simple eigen equation

$$(-\partial_\tau^2 + m^2) f_n(\tau) = \lambda_n \, f_n(\tau) \ . \tag{10.3}$$

Obviously the summation convention does not apply for mode indices n. We find

$$f_n(\tau) = \frac{1}{\sqrt{\beta}} \exp \frac{2\pi i n \tau}{\beta} \tag{10.4}$$

$$\lambda_n = m^2 + \frac{4\pi^2 n^2}{\beta^2} \ , \tag{10.5}$$

where $n \in \mathbb{Z}$. The eigenfunctions are orthonormalized according to

$$\int_0^\beta d\tau \, f_n^*(\tau) f_m(\tau) = \delta_{nm} \ . \tag{10.6}$$

Now, we may take an arbitrary field $\phi(\tau)$ and write it as a linear combination of these eigenstates. Thus

$$\phi(\tau) = \sum_n \phi_n \, f_n(\tau) \ . \tag{10.7}$$

The reality of $\phi(\tau)$ implies that $\phi_n^* = \phi_{-n}$. Inverting 10.7 gives

$$\phi_n = \int_0^\beta d\tau\, f_n^*(\tau)\,\phi(\tau)\ . \tag{10.8}$$

In terms of modes, our action in an external field J is just

$$S - \int_0^\beta d\tau\, J(\tau)\,\phi(\tau) = \sum_n \left(\frac{1}{2}\lambda_n\, \phi_n^*\phi_n - J_n^*\phi_n\right)\ . \tag{10.9}$$

We can now *define* the path integral measure precisely to be

$$[d\phi] = \prod_n \left(\sqrt{\frac{\lambda_n}{2\pi}}\, d\phi_n\right)\ . \tag{10.10}$$

The normalization is chosen so that

$$Z = \int \prod_n \left(\sqrt{\frac{\lambda_n}{2\pi}}\, d\phi_n\right)\, e^{-\frac{1}{2}\sum_n \lambda_n\, \phi_n^*\phi_n} = 1\ . \tag{10.11}$$

Let us now look at some expectation values. Because of the above normalization we have

$$\langle F[\phi_n]\rangle = \int \prod_n \left(\sqrt{\frac{\lambda_n}{2\pi}}\, d\phi_n\right)\, e^{-\frac{1}{2}\sum_n \lambda_n\, \phi_n^*\phi_n}\, F[\phi_n]\ . \tag{10.12}$$

If we choose to calculate the average of $F[\phi_n] = e^{-\frac{1}{2}\varepsilon\sum_n b_n\lambda_n\, \phi_n^*\phi_n}$ then the resulting integral is still Gaussian, so we get

$$\langle e^{-\frac{1}{2}\varepsilon\sum_n b_n\lambda_n\, \phi_n^*\phi_n}\rangle = \prod_n \left(\frac{1}{\sqrt{1+\varepsilon b_n}}\right)\ . \tag{10.13}$$

The above integral exists if $\varepsilon b_m \geq -1$ is true for all m. If $b_m \geq 0$ and $\sum_m b_m = \infty$ then the right hand side of equation 10.13 is a product of an infinite number of terms that are smaller than one, hence it vanishes. One such example is if we choose $b_m = 1$, for all m. This gives us

$$\langle e^{-\varepsilon S}\rangle = 0\ , \tag{10.14}$$

for all $\varepsilon > 0$. In general mean values just add the contributions of all the fields. We may, therefore, write

$$\langle F\rangle = \langle F\rangle_{\text{infinite action}} + \langle F\rangle_{\text{finite action}}\ . \tag{10.15}$$

The first term is the contribution of fields ϕ whose action is infinite, the second term gives the contribution of all the fields whose action is finite. For $F = e^{-\varepsilon S}$ the first term is obviously zero since the integrand vanishes. We thus find

$$\langle e^{-\varepsilon S}\rangle_{\text{finite action}} = 0\ . \tag{10.16}$$

This is just a sum of terms each of which is non-vanishing. The only way we can get 10.16 to hold is if the fields of finite action have *zero measure*, *i.e.* if they do not contribute to the path integral.

We can further narrow down the support of our path integral — the set of fields ϕ that have non-zero measure. To do this let us look at the truncated action

$$S_N = \sum_{n=-N}^{N} \lambda_n \phi_n^2 \ . \tag{10.17}$$

We can again apply equation 10.13. We will choose to look at the average of

$$F = e^{-\frac{1}{2}\varepsilon \, \frac{1}{2N+1} \, S_N} \ . \tag{10.18}$$

This corresponds to the choice

$$b_n = \begin{cases} \frac{1}{2N+1} & \text{if } |n| \le N \\ 0 & \text{otherwise} \end{cases} \tag{10.19}$$

Equation 10.13 now gives us

$$\begin{aligned} \langle e^{-\frac{1}{2}\varepsilon \, \frac{1}{2N+1} \, S_N} \rangle &= \prod_{n=-N}^{N} \frac{1}{\sqrt{1 + \frac{\varepsilon}{2N+1}}} = \\ &= \left((1 + \frac{\varepsilon}{2N+1})^{2N+1} \right)^{-\frac{1}{2}} \to e^{-\frac{1}{2}\varepsilon} \ , \end{aligned} \tag{10.20}$$

where in the last step we take the $N \to \infty$ limit. Differentiating this with respect to ε and setting $\varepsilon = 0$ we get

$$\frac{1}{2N+1} \langle S_N \rangle \quad \to \quad 1 \tag{10.21}$$

$$\left(\frac{1}{2N+1} \right)^2 \langle S_N^2 \rangle \quad \to \quad 1 \ , \tag{10.22}$$

hence

$$\langle S_N \rangle \quad \to \quad 2N+1 \tag{10.23}$$

$$\delta S_N \quad = \quad \frac{\sqrt{\langle S_N^2 \rangle - \langle S_N \rangle^2}}{\langle S_N \rangle} \sim o(1/N) \ . \tag{10.24}$$

From this we see that the fields ϕ that contribute to the path integral have $S[\phi]$ which grows as the number of degrees of freedom. The fact that finite action configurations do not contribute to the path integral is quite surprising. It is worth deriving this yet again in a different way. We look at

$$Z = \int \prod_n d\phi_n \, e^{-\frac{1}{2} \sum_n \lambda_n \, \phi_n^* \phi_n} \ . \tag{10.25}$$

Writing $\sqrt{\lambda_n}\,|\phi_n| = b_n$, and normalising the measure appropriately we get

$$Z = \prod_n \int_{-\infty}^{+\infty} \frac{db_n}{\sqrt{2\pi}}\, e^{-\frac{1}{2}b_n^2} = 1 \ . \tag{10.26}$$

If S is finite then there exists an L such that $|b_n| \leq L$ for all n. Now $Z_{\text{finite action}}$ is the limit when $L \to \infty$ (but $L \neq \infty$) of the expression

$$\prod_n \int_{-L}^{+L} \frac{db_n}{\sqrt{2\pi}}\, e^{-\frac{1}{2}b_n^2} = \prod_n a(L) \ . \tag{10.27}$$

For any finite L we have $a(L) < 1$ so that the above product vanishes. We have seen that the only ϕ's that contribute to the path integral have $\langle S \rangle$ proportional to the number of degrees of freedom. Let us look at a free action in the coordinate representation and then discretize space-time. In $d = 1$ we have

$$S = \frac{1}{2}\int dt\, \dot{\phi}^2 = \frac{1}{2}\sum_{i=0}^{N-1} \frac{(\phi_{i+1} - \phi_i)^2}{\varepsilon} \ . \tag{10.28}$$

And therefore

$$\left\langle \frac{(\phi_{i+1} - \phi_i)^2}{\varepsilon} \right\rangle \sim 1 \ . \tag{10.29}$$

Thus $\dot{\phi}^2 \sim 1/\varepsilon \to \infty$ hence the paths that contribute are *not differentiable.* In fact smooth paths are of measure zero. Equation 10.29 implies that $\delta\phi \propto \sqrt{t}$ which is the central characteristic of a diffusion process. This is not surprising — the random walk, which as we have seen is at the heart of QFT, is just such a process.

It now seems that we have proven that all that we have been doing so far is wrong! How can we justify semi-classical expansion? Remember that this was an expansion around classical solutions, *i.e.* finite action, differentiable fields. There is in fact no problem. While the classical solution is of measure zero, the fields that differ just a bit from it are *not* smooth or with finite action. They not only contribute, but in fact they dominate the path integral just as our naive arguments implied.

10.2 Lagrangian Measure from SD Equations

We next turn to the second topic of this lecture, and present an alternate technique for determinig the Lagrangian path integral measure using the Schwinger-Dyson equations. Let us introduce a convenient generating functional for the Hamiltonian path integral by

$$Z[J,K] = \int [dq\, dp]\, e^{\frac{i}{\hbar}\int (p\dot{q} - H + Jq + Kp)dt} \ . \tag{10.30}$$

It follows that we get

$$\begin{aligned}0 = &\int [dq\,dp]\,\frac{\delta}{\delta p(t)}\,e^{\frac{i}{\hbar}\int (p\dot{q}-H+Jq+Kp)dt} = \\ &= \int [dq\,dp]\,\frac{i}{\hbar}\left(\dot{q}-\frac{\partial H}{\partial p}+K\right)\,e^{\frac{i}{\hbar}\int (p\dot{q}-H+Jq+Kp)dt} = \\ &= \frac{i}{\hbar}\left(\dot{Q}-\frac{\partial H}{\partial p}+K\right)Z[J,K]\ ,\end{aligned} \tag{10.31}$$

where

$$Q = \frac{\hbar}{i}\frac{\delta}{\delta J} \tag{10.32}$$

$$P = \frac{\hbar}{i}\frac{\delta}{\delta K}\ . \tag{10.33}$$

As a consequence we have

$$[Q,J] = [P,K] = \frac{\hbar}{i}\ , \tag{10.34}$$

and all other comutators vanish. Similarly we construct the remaining SD equation. Finaly

$$\left(\dot{Q}-\frac{\partial H}{\partial P}+K\right)Z[J,K] = 0 \tag{10.35}$$

$$\left(\dot{P}+\frac{\partial H}{\partial Q}-J\right)Z[J,K] = 0\ . \tag{10.36}$$

We can use these equations for various things. Let us use them to go from the Hamiltonian to the Lagrangian path integral. To be concrete take

$$L = \frac{1}{2}\,G(q)\dot{q}^2 - V(q)\ . \tag{10.37}$$

This gives $H = \frac{1}{2}\,G^{-1}(q)p^2 + V(q)$, and so our SD equations read

$$\left(\dot{Q}-G^{-1}(Q)P+K\right)Z[J,K] = 0 \tag{10.38}$$

$$\left(\dot{P}-\frac{1}{2}\,G^{-2}(Q)G'(Q)P^2+V'(Q)-J\right)Z[J,K] = 0\ . \tag{10.39}$$

We may write 10.38 as

$$PZ = G(\dot{Q}+K)Z\ . \tag{10.40}$$

Differentiating this with respect to t gives us

$$\begin{aligned}\dot{P}Z &= (G'\dot{Q}(\dot{Q}+K)+G(\ddot{Q}+\dot{K}))Z = \\ &= (G'\dot{Q}^2+G'\dot{Q}K+G\ddot{Q}+g\dot{K})Z\ .\end{aligned} \tag{10.41}$$

At the same time 10.40 gives

$$P^2 Z = G(\dot{Q}+K)P\,Z + G[P,K]\,Z = \\ = \Big(G^2(\dot{Q}+K)^2 - i\hbar G\Big)\,Z\ . \tag{10.42}$$

We can now use equations 10.41 and 10.42 to eliminate P in equation 10.39, and find

$$\Big(G'\dot{Q}^2 + G'\dot{Q}K + G\ddot{Q} + G\dot{K} - \\ -\frac{1}{2}\,G'(\dot{Q}+K)^2 + \frac{1}{2}i\hbar\,G^{-1}G' + V' - J\Big)Z[J,K] = 0\ . \tag{10.43}$$

To get the Lagrangian SD equation from this we simply set $K=0$. The usual generating functional is simply $Z[J] = Z[J, K=0]$. This gives

$$\left(G\ddot{Q} + \frac{1}{2}\,G'\dot{Q}^2 + \frac{1}{2}i\hbar\,G^{-1}G' + V' - J\right)Z[J] = 0\ . \tag{10.44}$$

Note that for

$$I = \int dt\,\frac{1}{2}\,G(q)\dot{q}^2 - V(q)\ , \tag{10.45}$$

we have

$$\frac{\delta I}{\delta q} = \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = -\frac{1}{2}\,G'\dot{q}^2 - G\ddot{q} - V'\ . \tag{10.46}$$

Using this we can write 10.44 as

$$\left(\frac{\delta I}{\delta Q} - \frac{1}{2}i\hbar G^{-1}G' + J\right)Z[J] = 0\ . \tag{10.47}$$

Since $G^{-1}G' = \frac{\delta}{\delta Q}\int dt\,\ln G$ we find

$$\left(\frac{\delta S}{\delta Q} + J\right)Z[J] = 0\ , \tag{10.48}$$

where

$$S = I - \frac{1}{2}\,i\hbar\int dt\,\ln G\ . \tag{10.49}$$

Note that 10.48 is just the standard Lagrangian SD equation, whose solution is

$$Z[J] = \int [dq]\,e^{\frac{i}{\hbar}(S+\int dt\,Jq)}\ , \tag{10.50}$$

while 10.49 determines the measure. We may write the last two equations as

$$Z[J] = \int\prod_t\left(\sqrt{G}\,dq\right)\,e^{\frac{i}{\hbar}(I+\int dt\,Jq)}\ . \tag{10.51}$$

This is just what we would get by direct integration of p's in the phase space path integral. Note that this prescription for going from the Hamiltonian SD equations to their Lagrangian counterparts is just like the derivation of Lagrangian equations of motion from the Hamiltonian equations. The only subtlety here is due to the non-vanishing comutators 10.34. These comutators are the source of the non-trivial measure.

EXERCISES

10.1 Prove that

$$\int_{-\infty}^{+\infty} dx_1 \cdots dx_n \, e^{i\lambda\left((x_1-a)^2+(x_2-x_1)^2+\ldots+(b-x_n)^2\right)} =$$
$$= \left(\frac{i^n \pi^{n+1}}{(n+1)a\lambda^n}\right)^{\frac{1}{2}} \exp\left(\frac{i\lambda}{n+1}(b-a)^2\right) \; .$$

Using this show that the quantum mechanical transition amplitude for free particle is

$$\langle q_f | e^{-i\,\frac{1}{2}\,\hat{p}^2\,t} | q_i \rangle = \frac{1}{\sqrt{2\pi i t}}\, e^{\frac{i}{2t}\,(q_f-q_i)^2} \; .$$

Derive this also in the usual operator formalism.

10.2 A sigma model is given by the Lagrangian

$$\frac{1}{2}\, g_{ab}(\phi)\, \partial_\mu \phi^a \partial^\mu \phi^b \; .$$

Calculate the measure for the coordinate space path integral.

Lecture 11

Classical Symmetry

11.1 Noether Technique

Let us look at the continuous classical symmetries of field theories. Consider the infinitesimal transformation

$$\begin{aligned} x &\rightarrow x' = x + \delta x \quad &(11.1)\\ \phi(x) &\rightarrow \phi'(x') = \phi(x) + \delta\phi(x)\ . \quad &(11.2) \end{aligned}$$

Along with the total variation $\delta\phi(x) = \phi'(x') - \phi(x)$ it is also useful to define the form variation $\delta_0\phi(x) = \phi'(x) - \phi(x)$. These two infinitesimal variations are related according to $\delta\phi(x) = \delta_0\phi(x) + \delta x^\mu \partial_\mu \phi(x)$. The Jacobian for the above coordinate transformation is

$$J = \left| \frac{\partial x'^\mu}{\partial x^\nu} \right| = |\delta^\mu_\nu + \partial_\nu \delta x^\mu| = 1 + \partial_\mu \delta x^\mu\ . \quad (11.3)$$

On the other hand, the action changes according to

$$\begin{aligned} I \rightarrow I' &= \int dx'\, \mathcal{L}'(x') = \int dx\, J\, \mathcal{L}'(x') = \\ &= \int dx\, (1 + \partial_\mu \delta x^\mu)(\mathcal{L}(x) + \delta\mathcal{L}(x)) = \\ &= I + \int dx\, (\delta\mathcal{L} + \mathcal{L}\, \partial_\mu \delta x^\mu)\ . \quad (11.4) \end{aligned}$$

Therefore, we find

$$\delta I = \int dx\, (\delta\mathcal{L} + \mathcal{L}\, \partial_\mu \delta x^\mu)\ . \quad (11.5)$$

Now, since $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi)$, we have $\delta_0\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\, \delta_0\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\, \delta_0\partial_\mu\phi$, and hence

$$\delta I = \int dx\, \left(\frac{\partial\mathcal{L}}{\partial\phi}\, \delta_0\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\, \delta_0\partial_\mu\phi + \delta x^\mu \partial_\mu\mathcal{L} + \mathcal{L}\, \partial_\mu \delta x^\mu \right)\ . \quad (11.6)$$

Using the classical equations of motion $\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0$, as well as the fact that δ_0 and ∂_μ commute, we find

$$\delta I = \int dx\, \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\, \delta_0\phi + \delta x^\mu \mathcal{L} \right) \ . \tag{11.7}$$

Our transformations give us the variations δx and $\delta\phi$ as functions of a set of independent parameters ω^a. In terms of these variations

$$\delta I = \int dx\, \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} (\delta\phi - \delta x^\nu \partial_\nu\phi) + \delta x^\mu \mathcal{L} \right) \ . \tag{11.8}$$

If the transformations that we have considered correspond to a symmetry of the classical theory then $\delta I = 0$. As a consequence we find

$$\partial_\mu {j_a}^\mu = 0 \ , \tag{11.9}$$

where we have introduced the currents

$${j_a}^\mu = T^\mu{}_\nu \frac{\partial \delta x^\nu}{\partial\omega^a} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \frac{\partial\delta\phi}{\partial\omega^a} \ . \tag{11.10}$$

These currents are given in terms of the canonical energy-momentum tensor

$$T^\mu{}_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \partial_\nu\phi - \mathcal{L}\, \delta^\mu_\nu \ . \tag{11.11}$$

As we have shown, for each parameter ω^a we get a conserved current j^μ_a, *i.e.* a current whose divergence vanishes. We now define quantities called charges to be

$$Q_a = \int dV\, {j_a}^0 \ , \tag{11.12}$$

where $dV = dx^1 dx^2 ... dx^{d-1}$ is the volume element of space, just as dx has been shorthand for the volume element of spacetime. We now have

$$\partial_0 Q_a = \int dV\, \partial_0 {j_a}^0 = -\int dV\, \partial_i {j_a}^i = -\oint_{S_\infty} d\vec{S} \cdot \vec{j}_a \ . \tag{11.13}$$

Usualy $\vec{j}$'s vanish fast enough at spatial infinity so that the last integral is zero[1]. We then find that the associated charges[2] are constants of motion. As we see, the Noether technique that we have just presented allows us to construct constants of motion corresponding to continuous symmetries of the classical theory.

[1] Exceptions to this rule can be found in some lower dimensional models. We will see examples of this in later lectures.

[2] It is not difficult to see that these can be put in a manifestly covariant form $Q_a = \int d\Sigma_\mu {j_a}^\mu$, where the integration is over an arbitrary space-like hypersurface extending to infinity.

Symmetries with $\delta x = 0$ are called *internal*. For this case the conserved currents are simply

$$j_a{}^\mu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \delta\phi}{\partial \omega^a} \ . \tag{11.14}$$

For example, for Dirac fermions we have $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$. This is Lagrangian is obviously invariant under the following phase rotations

$$\psi \ \rightarrow \ e^{-i\theta}\,\psi \tag{11.15}$$

$$\bar{\psi} \ \rightarrow \ \bar{\psi}\,e^{i\theta} \ . \tag{11.16}$$

The conserved vector current is

$$j^\mu = -\bar{\psi}\gamma^\mu\psi \ . \tag{11.17}$$

We use γ matrix conventions in which γ^0 is Hermitian, while the γ^i's are anti-Hermitian. It is now easy to show that $j^{\mu\dagger} = j^\mu$. The charge associated with phase rotations is

$$Q = -\int dV\,\psi^\dagger\psi \ . \tag{11.18}$$

This is just the number of positrons minus the number of electrons, *i.e.* Q is simply the electric charge.

All the theories we look at are invariant under spacetime translations. In this case we have $\delta x^\mu = a^\mu$ and $\delta\phi = 0$. The associated conserved current is the canonical energy-momentum tensor 11.11. The appropriate charges are just the components of the total energy-momentum

$$P_\nu = \int dV\,T^0{}_\nu \ . \tag{11.19}$$

The above is an example of *space-time symmetries*. For these we have $\delta x \neq 0$. In all of them the energy-momentum tensor plays a central role. Along with translation invariance there is one further spacetime symmetry that will always be present — invariance under Lorentz transformations

$$x^\mu \ \rightarrow \ x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{11.20}$$

$$\phi(x) \ \rightarrow \ \phi'(x') = D(\Lambda)\phi(x) \ , \tag{11.21}$$

where $D(\Lambda)$ is a given representation of the Lorentz group that acts on the components of ϕ. For infinitesimal transformations we have

$$\Lambda^\mu{}_\nu \ = \ \delta^\mu_\nu + \omega^\mu{}_\nu \tag{11.22}$$

$$D(\Lambda) \ = \ 1 + \frac{1}{2}\omega^{\mu\nu}\Sigma_{\mu\nu} \ , \tag{11.23}$$

where $\omega^{\mu\nu} = -\omega^{\nu\mu}$ are the parameters and $\Sigma_{\mu\nu}$ the corresponding generators. Therefore,

$$\delta x^\mu \ = \ \omega^{\mu\nu}x_\nu \tag{11.24}$$

$$\delta\phi \ = \ \frac{1}{2}\omega^{\mu\nu}\Sigma_{\mu\nu}\,\phi \ , \tag{11.25}$$

so that we have

$$T^{\mu}{}_{\nu}\delta x^{\nu} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi = T^{\mu\nu}\,\omega_{\nu\rho}\,x^{\rho} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\frac{1}{2}\,\Sigma^{\nu\rho}\,\phi\,\omega_{\nu\rho} =$$
$$= \frac{1}{2}\left(T^{\mu\nu}x^{\rho} - T^{\mu\rho}x^{\nu} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Sigma^{\nu\rho}\,\phi\right)\omega_{\nu\rho}\ . \tag{11.26}$$

The conserved currents are thus

$$M^{\mu\nu\rho} = x^{\nu}T^{\mu\rho} - x^{\rho}T^{\mu\nu} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Sigma^{\nu\rho}\,\phi\ . \tag{11.27}$$

This is the angular momentum tensor. The corresponding charges are just the components of the total angular momentum

$$L_{\nu\rho} = \int dV\,M^{0}{}_{\nu\rho}\ . \tag{11.28}$$

11.2 Energy-Momentum Tensors Galore

The canonical energy-momentum tensor $T^{\mu\nu}$ is in general not symmetric — this is only true for scalar fields. We can, however, find another tensor $\theta^{\mu\nu}$ which is equivalent to the canonical tensor and *is* symmetric. By equivalent we mean that both $T^{\mu\nu}$ and $\theta^{\mu\nu}$ give the same charge P^{ν}. For this to be the case we must have

$$\theta^{\mu\nu} = T^{\mu\nu} + \partial_\rho X^{\rho\mu\nu}\ , \tag{11.29}$$

where $X^{\rho\mu\nu}$ is antisymmetric in all indices. It now follows that $\partial_\mu\theta^{\mu\nu} = 0$. The charges are the same since

$$\int dV\,\partial X^{\rho 0\nu} = \int dV\,\partial_i X^{i0\nu} = \oint_{S_\infty} dS_i X^{i0\nu} = 0\ . \tag{11.30}$$

We have again assumed that fields fall off fast enough at infinity to make the above surface integral vanish. We determine X from the requirement $\theta^{\mu\nu} = \theta^{\nu\mu}$. We have $T^{\mu\nu} - T^{\nu\mu} = -2\partial_\rho X^{\rho\mu\nu}$, and hence

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\partial^{\nu}\phi - \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\,\partial^{\mu}\phi = -2\,\partial_\rho X^{\rho\mu\nu}\ . \tag{11.31}$$

If $T^{\mu\nu}$ and $\theta^{\mu\nu}$ are equivalent why not just use $T^{\mu\nu}$? If we do not consider gravity then the only answer is that the symmetry of $\theta^{\mu\nu}$ makes some expressions simpler. For example $x^{\nu}\theta^{\mu\rho} - x^{\rho}\theta^{\mu\nu}$ is equivalent to $M^{\mu\nu\rho}$. If we turn on gravity, however, then we immediately see the difference between $T^{\mu\nu}$ and $\theta^{\mu\nu}$. The Einstein equations read

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\,\theta_{\mu\nu}\ , \tag{11.32}$$

i.e. gravitation couples to the symmetric energy-momentum tensor not the canonical one. To see this just note that the left-hand-side of the above equation is symmetric. We will not consider gravitation in these lectures, however, we can use the above as a way of calculating $\theta_{\mu\nu}$ directly. The Einstein equations follow from the action $I_{\text{total}} = I_G[g] + I[\phi, g]$, where

$$I_G = -\frac{1}{16\pi G} \int dx \sqrt{|g|}\, R \tag{11.33}$$

is the Einstein-Hilbert action, and the matter action $I[\phi, g]$ follows from our starting action $I[\phi]$ by puting it in a gravitational background $g_{\mu\nu}$. To do this we just take $\eta_{\mu\nu} \to g_{\mu\nu}$ and $dx \to \sqrt{|g|}\, dx$. We find

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -8\pi G \, \frac{1}{\sqrt{|g|}} \frac{\delta I_G}{\delta g^{\mu\nu}} \ . \tag{11.34}$$

It follows that

$$\theta^{\mu\nu} = \frac{1}{\sqrt{|g|}} \frac{\delta I[\phi, g]}{\delta g^{\mu\nu}} \ . \tag{11.35}$$

We finaly turn off the gravity background and get our flat space result. We now have a straightforward prescription for constructing $\theta^{\mu\nu}$.

At the end, let us mention one further spacetime symmetry — dilatations. In d dimensions Bose fields have dimensions

$$[\phi] = L^{\frac{2-d}{2}} \ . \tag{11.36}$$

We define dilatations to be the following transformations

$$\begin{aligned} x &\to x' = \Lambda x & (11.37) \\ \phi(x) &\to \phi'(x') = \Lambda^{\frac{2-d}{2}} \phi(x) \ . & (11.38) \end{aligned}$$

To make things more concrete let us look at scalar fields interacting through a quartic coupling. Under dilatations the action changes according to

$$I \to \int dx \left(\frac{1}{2} (\partial\phi)^2 - \Lambda^2 \frac{1}{2} m^2 \phi^2 - \Lambda^{4-d} \frac{1}{4!} g\phi^4 \right) \ . \tag{11.39}$$

We see that $m \neq 0$ always spoils dilatation invariance. Also, $g \neq 0$ spoils the invariance except in $d = 4$. It is easy to see why $d = 4$ is picked out. In $d = 4$ the coupling constant g is dimensionless. The moral is that only theories with dimensionless constants have dilatation invariance. We now look at infinitesimal dilatations $\Lambda = 1 + \lambda$. In this case we have

$$\begin{aligned} \delta x &= \lambda x & (11.40) \\ \delta\phi(x) &= \frac{2-d}{2} \lambda \, \phi(x) \ . & (11.41) \end{aligned}$$

The associated conserved current is

$$j^\mu_D = T^{\mu\nu} x_\nu + \frac{d-2}{2} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \phi \ . \tag{11.42}$$

For Fermi fields we have $[\psi] = L^{\frac{1-d}{2}}$, and so equations are changed accordingly.

For dilatation invariant theories of scalar fields it is possible to define yet another equivalent energy-momentum tensor, the so-called new improved energy-momentum tensor

$$\Theta_{\mu\nu} = T_{\mu\nu} + a\,(\partial_\mu \partial_\nu - \eta_{\mu\nu}\partial^2)\phi^2 \ . \tag{11.43}$$

For a given theory, the dimensionless constant a is determined from the requirement that $\Theta_{\mu\nu}$ be traceless[3]. The new dilatation current is

$$J^\mu_D = \Theta^{\mu\nu}\, x_\nu \ . \tag{11.44}$$

The divergence of this current is proportional to $\Theta^\mu{}_\mu$, *i.e.* it vanishes. It is easy to show that the new dilatation current J^μ_D is equivalent to the canonical current j^μ_D.

EXERCISES

11.1 In even dimensions d there exists a matrix γ that anticommutes with all the γ^μ matrices[4]. It may be chosen to satisfy $\gamma^2 = 1$, $\gamma^\dagger = \gamma$. Show that massless free Dirac fermions are invariant under axial transformations

$$\begin{aligned} \psi &\rightarrow e^{i\alpha\gamma}\,\psi \\ \bar\psi &\rightarrow \bar\psi\, e^{i\alpha\gamma} \ . \end{aligned}$$

Construct the axial current, and show that it is conserved in the classical theory. Show that the current is Hermitian.

11.2 Construct the angular momentum tensor for electrodynamics. Here the Lagrangian is $\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor.

11.3 Construct $\theta^{\mu\nu}$ for electrodynamics.

11.4 Use equation 11.35 to find $\theta^{\mu\nu}$ for a free scalar particle.

11.5 Show that the new improved energy-momentum tensor $\Theta_{\mu\nu}$ is conserved and symmetric. Determine the parameter a for massless free fields, as well as for massless ϕ^4 theory (in $d = 4$). Show that $T_{\mu\nu}$ and $\Theta_{\mu\nu}$ are equivallent. Prove the equivalence of J^μ_D and j^μ_D.

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[3] The new improved energy-momentum tensor is useful in dealing with conformal symmetry.

[4] This matrix is usualy denoted γ_5. This notation is a throwback to earlier times when space-time indices (in $d = 4$) took on values from 1 to 4.

Lecture 12

Symmetry Breaking

12.1 Goldstone Bosons

Spontaneous symmetry breaking comes about when the vacuum is not invariant under the full symmetry group of the dynamics. We'll first look at symmetry breaking at the classical level. Let us look at a scalar field theory

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - U(\phi) \ , \tag{12.1}$$

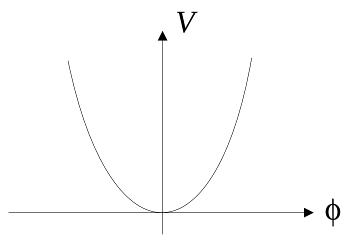
where

$$U(\phi) = \frac{1}{2}\, m^2\phi^2 + \frac{1}{4!}\,\lambda\,\phi^4 \ . \tag{12.2}$$

The energy equals

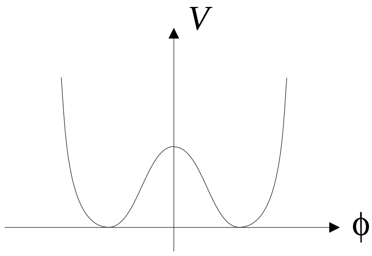
$$E = \int dx_1 \cdots dx_{d-1} \left(\frac{1}{2}\,(\partial_t\phi)^2 + \frac{1}{2}\,(\partial_i\phi)^2 + U(\phi) \right) \ . \tag{12.3}$$

For this to be bounded from bellow we need to have $\lambda > 0$. In the free field case $\lambda = 0$, and we need $m^2 > 0$. As long as $\lambda > 0$ we can have m^2 either positive or negative. In both cases the Lagrangian is invariant under the discrete symmetry $\phi \to -\phi$. In both cases the energy is minimized by constant field configurations. The vacuum (minimum of energy) is thus determined by the minima of $U(\phi)$. For $m^2 > 0$ we have a single minimum at $\phi = 0$.



For $m^2 < 0$ the situation is more interesting. Let us introduce $a^2 = -\frac{6m^2}{\lambda}$. In terms of this we have, up to an additive constant

$$U = -\frac{\lambda}{4!}\,(\phi^2 - a^2)^2 \ . \tag{12.4}$$



$\phi = 0$ is now a local maximum. There are two vacuum states $\phi = a$ and $\phi = -a$. The transformation $\phi \to -\phi$ doesen't leave these vacuums alone, but rather switches one vacuum for the other. We now have symmetry breaking. The case $m^2 < 0$ seems sick — as if we have a tachyon. This is not the case. The only thing that $m^2 < 0$ tells us is that $\phi = 0$ is unstable. The spectrum is gotten by looking at small oscillations about a stable equilibrium. To do this let us expand around one of the vacuums (say a). We then write

$$\phi = a + \eta \ . \tag{12.5}$$

Our Lagrangian is now

$$\mathcal{L} = \frac{1}{2}\,(\partial\eta)^2 - \frac{1}{2}\,(\frac{\lambda a^2}{3})\,\eta^2 - \frac{1}{3!}\,(\lambda a)\eta^3 - \frac{1}{4!}\,\lambda\eta^4 \ . \tag{12.6}$$

Here we se that we are in fact dealing with a particle whose mass squared is $\frac{\lambda a^2}{3} > 0$, *i.e.* not a tachyon. The initial symmetry is now not so easily seen. It lies in the specific relations between the mass and the two coupling constants. In terms of m^2 we have

$$M^2 = -2m^2 \tag{12.7}$$
$$\Lambda_3^2 = -6m^2\Lambda_4 \ . \tag{12.8}$$

Owing to the fact that $a \propto \frac{1}{\sqrt{\lambda}}$ we see that the above shift is a non-perturbative effect. Except for this shift everything else can be treated perturbatively in terms of $\sqrt{\lambda}$ (if in fact $\sqrt{\lambda}$ is small). We would have gotten the same thing had we expanded around $\phi = -a$ instead.

Symmetry breaking is much more interesting if one breaks a continuous symmetry. To illustrate this consider the following vector model

$$\mathcal{L} = \frac{1}{2}\,(\partial\vec{\phi})^2 - U \ , \tag{12.9}$$

where

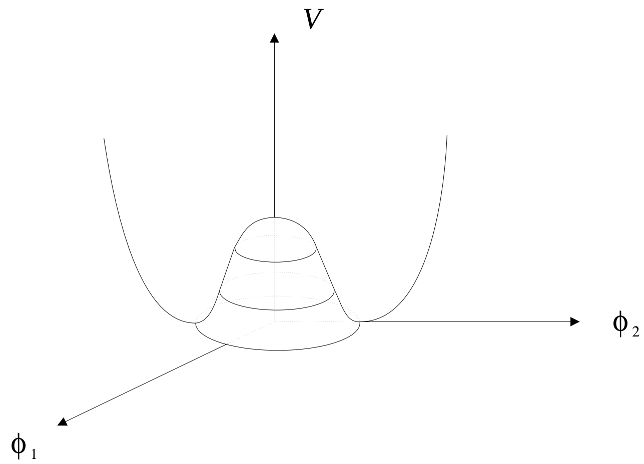
$$U = -\frac{\lambda}{4!}\,(|\vec{\phi}|^2 - a^2)^2 \ , \tag{12.10}$$

and the field is a two component vector

$$\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \ , \tag{12.11}$$

As before, the energy is minimized by constant fields satisfying $\frac{\partial U}{\partial \phi_i} = 0$. The vacuum states thus satisfy

$$\phi_1^2 + \phi_2^2 = a^2 \ . \tag{12.12}$$



The dynamics is invariant under $SO(2)$ rotations in field space

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rightarrow \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \ , \tag{12.13}$$

while this transformation changes one vacuum into another. Again we have symmetry breaking. Again we pick one vacuum and expand about it. Therefore, by writing

$$\vec{\phi} = \begin{pmatrix} a \\ 0 \end{pmatrix} + \vec{\eta} \ , \tag{12.14}$$

we find

$$\mathcal{L} = \frac{1}{2}\,(\partial\eta_1)^2 - \frac{1}{2}\left(\frac{\lambda a^2}{3}\right)\eta_1^2 + \frac{1}{2}\,(\partial\eta_2)^2 + \ldots \tag{12.15}$$

where dots indicate cubic and higher terms, *i.e.* interactions. As in our previous example we have a particle of mass squared $\frac{\lambda a^2}{3}$. What is new is that we now also have a massless particle present. The existence of a massless particle in the spectrum is a direct consequence of the spontaneous symmetry breaking of a continuos symmetry. Such massless particles are called Goldstone bosons. It is easy to see why η_2 is massless. η_1 particles sit at the bottom of a potential that is approximately quadratic, hence η_1 is massive. On the other hand, the η_2 field represent displacements that keep us on the circle of vacuum states — the associated particle feels a flat potential. It follows that the η_2 particles has zero mass.

As a digression, let us make a field redefinition (corresponding to the transformation from Cartesian to polar coordinates). We now have new fields r, φ given by

$$\phi_1 = r\cos\varphi \tag{12.16}$$

$$\phi_2 = r\sin\varphi \; . \tag{12.17}$$

The general $SO(2)$ invariant interaction is given by

$$\mathcal{L} = \frac{1}{2}\,(\partial r)^2 + \frac{1}{2}\,r^2(\partial\varphi)^2 - U(r) \; . \tag{12.18}$$

If the minimum of $U(r)$ is at $r = a \neq 0$ then we have symmetry breaking. We choose a specific vacuum (say $r = a$, $\varphi = 0$) and expand around it

$$r = a + \eta_r \tag{12.19}$$

$$\varphi = \eta_\varphi \; . \tag{12.20}$$

We find

$$\mathcal{L} = \frac{1}{2}\,(\partial\eta_r)^2 - \frac{1}{2}\,U''(a)\eta_r^2 + \frac{1}{2}\,a^2\,(\partial\eta_\varphi)^2 + \ldots \tag{12.21}$$

Again we see that we have a massive particle η_r and a Goldstone boson η_φ. Note that we are certainly free to make a field redefinition since we are dealing with a classical theory. However, the above field redefinition is not good at $r = 0$. This is not a problem because we expand about $r = a$ which is far away from $r = 0$.

We are now ready to treat the symmetry breaking of a general theory. We have some fields ϕ and a symmetry of the action under

$$\phi \to \phi^\omega = T(\omega)\phi \; . \tag{12.22}$$

where $\omega_1, \omega_2, \ldots, \omega_N$ represent a complete set of independent parameters. Let ϕ^0 be a specfic vacuum configuration. We can choose the parameeters ω so that symmetry is broken, *i.e.*

$$\phi^0 \neq T(\omega)\phi^0 \tag{12.23}$$

for $\omega_1, \omega_2, \cdots, \omega_k$ while for $\omega_{k+1}, \omega_{k+2}, \cdots, \omega_N$ we have $\phi^0 = T(\omega)\phi^0$. The spectrum of the theory is determined by the eigenvalues of the mass matrix

$$M^2{}_{ij} = \left.\frac{\partial^2 U}{\partial\phi_i \partial\phi_j}\right|_{\phi^0} \; . \tag{12.24}$$

Goldstone's theorem states that this matrix has exactly k eigenvalues that are zero. Said another way M^2 has k zero modes. Therefore, the theory has k Goldstone bosons present. This theorem is easily proven. Note that

$$\frac{\partial^2 U}{\partial\phi_i^\omega \partial\phi_j^\omega}\,\frac{\partial\phi_j^\omega}{\partial\omega_a} = \frac{\partial}{\partial\phi^\omega}\,\frac{\partial U}{\partial\omega_a} = 0 \; . \tag{12.25}$$

Therefore

$$M^2{}_{ij}\psi_j^{(a)} = 0 \ , \tag{12.26}$$

and

$$\psi_j^{(a)} = \left.\frac{\partial \phi_j^\omega}{\partial \omega_a}\right|_{\phi^0} \qquad a = 1, 2, \ldots, k \tag{12.27}$$

For $a = k+1, \ldots N$ we get $\psi_j = 0$. We thus have k zero modes. For each zero mode we have one Goldstone boson. This proves the theorem. Let us illustrate this on our previous model. The vacuums transform according to

$$\phi^\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \phi^0 \ . \tag{12.28}$$

The zero mode is given by

$$\begin{aligned} \psi = \left.\frac{\partial \phi^\theta}{\partial \theta}\right|_{\phi^0} &= \\ &= \frac{\partial}{\partial\theta} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \phi\Big|_{\phi^0} = \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -a \end{pmatrix} \ . \end{aligned} \tag{12.29}$$

This is indeed a zero mode of the mass matrix which is equal to

$$M^2 = \begin{pmatrix} \frac{\lambda a^2}{3} & 0 \\ 0 & 0 \end{pmatrix} \ . \tag{12.30}$$

12.2 The Higgs Mechanism

We start this section by looking at a $U(1)$ invariant model with symmetry breaking

$$\mathcal{L} = \partial_\mu \bar{\phi} \partial^\mu \phi - \frac{\lambda}{6}\left(\bar{\phi}\phi - \frac{a^2}{2}\right)^2 \ . \tag{12.31}$$

Let us couple this model to electrodynamics. The minimal coupling gives us $\partial_\mu \to \partial_\mu + ieA_\mu$, so that the Lagrangian becomes

$$\mathcal{L} = (\partial_\mu - ieA_\mu)\bar{\phi}\,(\partial^\mu + ieA^\mu)\phi - \frac{\lambda}{6}\left(\bar{\phi}\phi - \frac{a^2}{2}\right)^2 - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \ , \tag{12.32}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The last term is the free action for photons. Now the $U(1)$ invariance is local, *i.e.* we have a gauge symmetry

$$\phi \rightarrow e^{-ie\Lambda(x)}\phi \tag{12.33}$$

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda \ . \tag{12.34}$$

Alternately, we can write ϕ in terms of two real fields

$$\phi = \frac{1}{\sqrt{2}}\,(\phi_1 + i\,\phi_2)\ . \tag{12.35}$$

This turns 12.31 into our $SO(2)$ model, while 12.32 becomes

$$\begin{aligned}\mathcal{L} = {} & \frac{1}{2}\,(\partial_\mu\phi_1 - eA_\mu\phi_2)^2 + \frac{1}{2}\,(\partial_\mu\phi_2 + eA_\mu\phi_1)^2 - \\ & - \frac{\lambda}{4!}\,(\phi_1^2 + \phi_2^2 - a^2)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\ .\end{aligned} \tag{12.36}$$

As before we choose a vacuum and expand around it, therefore

$$\phi_1 = a + \eta_1 \tag{12.37}$$
$$\phi_2 = \eta_2\ . \tag{12.38}$$

In terms of the η fields this becomes

$$\begin{aligned}\mathcal{L} = {} & \frac{1}{2}\,(\partial\eta_1)^2 + \frac{1}{2}\,e^2A^2\eta_2^2 - eA_\mu\eta_2\partial^\mu\eta_1 + \\ & + \frac{1}{2}\,(\partial\eta_2)^2 + \frac{1}{2}\,e^2a^2A^2 + \frac{1}{2}\,e^2A^2\eta_1^2 + \\ & + eaA_\mu\partial^\mu\eta_2 + eA_\mu\eta_1\partial^\mu\eta_2 + e^2aA^2\eta_1 - \\ & - \frac{\lambda}{6}\,a^2\eta_1^2 - \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} + \ldots\end{aligned} \tag{12.39}$$

The infinitesimal form of the gauge transformations is

$$\phi \to \phi - ie\Lambda\,\phi \tag{12.40}$$
$$\bar{\phi} \to \bar{\phi} + ie\Lambda\,\bar{\phi}\ . \tag{12.41}$$

In terms of the real fields ϕ_1, ϕ_2 this becomes

$$\phi_1 \to \phi_1 + e\Lambda\,\phi_2 \tag{12.42}$$
$$\phi_2 \to \phi_2 - e\Lambda\,\phi_1\ . \tag{12.43}$$

Finally, for η_1, η_2 we get

$$\eta_1 \to \eta_1 + e\Lambda\,\eta_2 \tag{12.44}$$
$$\eta_2 \to \eta_2 - e\Lambda\,\eta_1 - e\Lambda a\ . \tag{12.45}$$

Because of the inhomogenous term in the gauge transformation of η_2 it is always possible to choose $\Lambda(x)$ in such a way that $\eta_2 = 0$. The gauge fixed Lagrangian now reads

$$\mathcal{L} = -\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\,e^2a^2A^2 + \frac{1}{2}\,(\partial\eta_1)^2 - \frac{\lambda}{6}\,a^2\eta_1^2 + \ldots \tag{12.46}$$

Reading off the spectrum we see that two seemingly impossible things have happened. First, we don't have a Goldstone boson even though we do have symmetry breaking. Second, the gauge field has acquired a mass. On its own, gauge invariance implies that the gauge fields are massless[1], and the breaking of a continuous symmetry gives rise to Goldstone bosons. However, as we have seen above, together each solves the others problem. This is the Higgs mechanism. In colloquial jargon we say that the photon eats the Goldstone boson and in doing this becomes massive. As a check let us see that degrees of freedom match. We start with $d-2$ degrees of freedom corresponding to the transverse polarizations of the photon. We also have 2 degrees of freedom corresponding to the two scalar particles. All together d degrees of freedom. Afterwords we have $d-1$ degrees of freedom for the massive vector particle (transverse and longitudinal polarizations) and 1 scalar particle. So again we get d degrees of freedom.

The Higgs mechanism is very important phenomenologicaly. First of all, it solves the problem of Goldstone bosons — particles that are not seen experimentaly. Second, it allows us to have massive gauge fields such as the W and Z intermediate vector bosons of the Standard Model. In this way we can explain short range interactions by the exchage of massive particles (Yukawa mechanism) in a gauge invariant theory. As we shall see, gauge invariance is necessary because it gives us renormalizability.

At the end, let us finish this lecture with a few brief comments:

- Symmetry breaking exists for quantum field theories. Everything is the same only instead of the action (or potential) we look at the minima of the effective action (effective potential). We will do this in the following lecture.

- Goldstone bosons remain massless even when we turn on quantum effects. This can be shown perturbatively using Ward identities.

- Perturbatively, once we are near one vacuum we can't reach the other — it doesn't effect us. We know that this is can't always be true. In quantum mechanics (*i.e* QFT in $d = 1$) there is tunneling between different vacua. Even in quantum mechanics we couldn't see tunneling in perturbation theory. We used a non-perturbative approximation — the asymptotic expansion of WKB. We shall see how to do this in field theory when we learn about instantons. In many field theories there are no instantons, and so no tunneling. In this sense quantum field theory is simpler than quantum mechanics.

[1] This is not *realy* true, although it has been part of the field theory folklore for many years. We will look into this in later lectures when we deal with gauge theories.

Lecture 13

Effective Action to One Loop

13.1 The Effective Potential

In a previous lecture we derived a general formula for the generating functional up to one loop. We had

$$Z[J] = \int [d\phi]\, e^{\frac{i}{\hbar}(I[\phi]+J_i\phi_i)} \approx$$
$$\approx e^{\frac{i}{\hbar}(I[\Phi]+J_i\Phi_i)} \left(\det \frac{\partial^2 I}{\partial\Phi_i \partial\Phi_j}\right)^{-1/2} . \tag{13.1}$$

In terms of $W[J]$ this is just

$$W[J] \approx I[\Phi] + J_i\Phi_i + \frac{i\hbar}{2}\ \mathrm{tr}\ \ln \frac{\partial^2 I}{\partial\Phi_i \partial\Phi_j} . \tag{13.2}$$

We now want to write the one loop results in terms of the effective action. As we have seen, the effective action is given by $\Gamma[\varphi] = W[J] - J_i\varphi_i$, where $\varphi_i = \frac{\partial W}{\partial J_i}$. Using this we find

$$\Gamma[\varphi] \approx I[\Phi] + J_i(\Phi_i - \varphi_i) + \frac{i\hbar}{2}\ \mathrm{tr}\ \ln \frac{\partial^2 I}{\partial\Phi_i \partial\Phi_j} \approx$$
$$\approx I[\varphi] + \left(\frac{\partial I}{\partial\varphi_i} + J_i\right)(\Phi_i - \varphi_i) + \frac{i\hbar}{2}\ \mathrm{tr}\ \ln \frac{\partial^2 I}{\partial\varphi_i \partial\varphi_j} . \tag{13.3}$$

By using the fact that both $\frac{\partial I}{\partial\varphi_i} + J_i$, and $\Phi_i - \varphi_i$ are $o(\hbar)$ we finally obtain our one loop result for the effective action

$$\Gamma[\varphi] \approx I[\varphi] + \frac{i\hbar}{2}\ \mathrm{tr}\ \ln \frac{\partial^2 I}{\partial\varphi_i \partial\varphi_j} . \tag{13.4}$$

Now let us put some meat on this result. We will look at scalar field theory in d dimensions. The action is

$$I = \int dx\, \left(\frac{1}{2}(\partial\phi)^2 - V(\phi)\right) , \tag{13.5}$$

so that the one loop effective action becomes

$$\Gamma[\varphi] \approx \int dx\, \left(\frac{1}{2}(\partial\phi)^2 - V(\phi)\right) + \frac{i\hbar}{2}\,\mathrm{tr}\,\ln\left(\partial^2 + V''(\varphi)\right)\ . \tag{13.6}$$

It is very difficult to evaluate the above trace for the case of general $\varphi(x)$. What we will do is calculate the trace for *constant* field configurations. To see what this buys us let us look at the general form of the effective action written as an expansion in derivatives

$$\Gamma[\varphi] = \int dx\, \left(-V_{\text{eff}}(\varphi) + G(\varphi)\,(\partial\varphi)^2 + \ldots\right)\ , \tag{13.7}$$

where dots indicate higher powers in derivatives. For constant fields φ we have

$$\Gamma[\varphi] = -\Omega\, V_{\text{eff}}(\varphi)\ , \tag{13.8}$$

Ω being the volume of space-time. The function $V_{\text{eff}}(\varphi)$ is called the effective potential. Obviously, for free theory we have $V_{\text{eff}}(\varphi) = V(\varphi)$. When there is an interaction our one loop result is

$$V_{\text{eff}}(\varphi) \approx V(\varphi) - \frac{1}{\Omega}\frac{i\hbar}{2}\,\mathrm{tr}\,\ln\,\left(\partial^2 + V''(\varphi)\right)\ . \tag{13.9}$$

We stress again that here φ is a constant. All that we need to do is evaluate the free field determinant

$$\det\left(\partial^2 + m^2\right) = \exp\left(\mathrm{tr}\,\ln\left(\partial^2 + m^2\right)\right)\ . \tag{13.10}$$

To do this we introduce the operator notation $\hat{p}_\mu = i\partial_\mu$, and calculate the trace in the coordinate representation. We find

$$\begin{aligned}\mathrm{tr}\,\ln\left(-\hat{p}^2 + m^2\right) &= \int dx\,\langle x|\ln\left(-\hat{p}^2 + m^2\right)|x\rangle = \\ &= \int dx \int \frac{dk}{(2\pi)^d}\langle x|\ln\left(-\hat{p}^2 + m^2\right)|k\rangle\langle k|x\rangle = \\ &= \int dx \int \frac{dk}{(2\pi)^d}\ln\left(-k^2 + m^2\right)\,\langle x|k\rangle\langle k|x\rangle = \\ &= \Omega \int \frac{dk}{(2\pi)^d}\ln\left(-k^2 + m^2\right)\ .\end{aligned} \tag{13.11}$$

We have used the fact that $\langle x|k\rangle = e^{ik\cdot x}$. To do the remaining integral we first Wick rotate[1] to Euclidian space so that our trace becomes

$$i\Omega \int \frac{d\bar{k}}{(2\pi)^d}\,\ln\left(\bar{k}^2 + m^2\right) = -i\Omega\,\frac{\partial}{\partial\alpha}\int \frac{d\bar{k}}{(2\pi)^d}\,\frac{1}{(\bar{k}^2 + m^2)^\alpha}\Bigg|_{\alpha=0}\ . \tag{13.12}$$

[1]In coordinate space Wick rotation implies $x^0 \to \bar{x}^0 = ix^0$. For momentum space we have $k^0 \to \bar{k}^0 = -ik^0$. As a result of this, x^2 and k^2 go over into the appropriate Euclidian expressions. On the other hand, $k \cdot x$ remains unchanged, which is precisely what we need for plane waves to remain unchanged.

Using standard dimensional regularization formulas this is easily calculated. We get

$$-i\Omega\,\frac{\partial}{\partial\alpha}\left(\frac{1}{(4\pi)^{d/2}}\,\frac{\Gamma(\alpha-\frac{d}{2})}{\Gamma(\alpha)}\,(m^2)^{d/2-\alpha}\right)\Bigg|_{\alpha=0} = -i\Omega\,\frac{\Gamma(-\frac{d}{2})}{(4\pi)^{d/2}}\,(m^2)^{d/2}\ . \quad (13.13)$$

Finally,

$$\text{tr}\ \ln\left(\partial^2+m^2\right) = -i\Omega\,\frac{\Gamma(-\frac{d}{2})}{(4\pi)^{d/2}}\,(m^2)^{d/2}\ . \quad (13.14)$$

Taking $m^2 \to V''(\varphi)$ we get our one loop result for the effective potential

$$V_{\text{eff}}(\varphi) \approx V(\varphi) - \frac{\hbar}{2}\,\frac{\Gamma(-\frac{d}{2})}{(4\pi)^{d/2}}\,\left(V''(\varphi)\right)^{d/2}\ . \quad (13.15)$$

For example, for a model in $d=4$ we have in dimensional regularization $d=4-\epsilon$, and

$$\Gamma(-2+\frac{\epsilon}{2}) = \frac{1}{\epsilon}+\frac{3}{4}-\frac{\gamma}{2}+\dots \quad (13.16)$$

where γ is the Euler-Mascheroni constant[2]. We also have

$$\left(\frac{V''}{4\pi}\right)^{2-\epsilon/2} = \frac{1}{16\pi^2}\,(V'')^2\left(1-\frac{\epsilon}{2}\,\ln\frac{V''}{4\pi}+\dots\right)\ . \quad (13.17)$$

Hence

$$\begin{aligned} V_{\text{eff}}(\varphi) &\approx V(\varphi) - \frac{\hbar}{32\pi^2}\,(V'')^2\left(\frac{1}{\varepsilon}+\frac{3}{4}-\frac{\gamma}{2}-\frac{1}{2}\,\ln\frac{V''}{4\pi}\right) = \\ &= V(\varphi) - \frac{\hbar}{32\pi^2}\,(V'')^2\left(\frac{1}{\varepsilon}+\frac{3}{4}-\frac{\gamma}{2}+\frac{1}{2}\,\ln 4\pi-\frac{1}{2}\,\ln\mu^2\right) + \\ &+ \frac{\hbar}{64\pi^2}(V'')^2\,\ln\left(\frac{V''}{\mu^2}\right)\ . \end{aligned} \quad (13.18)$$

For later convenience we added and subtracted a $\ln\mu^2$ term. We can now write our final one loop answer as

$$V_{\text{eff}}(\varphi) \approx V(\varphi) - \frac{1}{2}\,A\,(V'')^2 + B\,(V'')^2\ln\left(\frac{V''}{\mu^2}\right)\ , \quad (13.19)$$

where we have introduced the constants

$$A = \frac{\hbar}{16\pi^2}\left(\frac{1}{\varepsilon}+\frac{3}{4}-\frac{\gamma}{2}+\frac{1}{2}\,\ln 4\pi-\frac{1}{2}\,\ln\mu^2\right) \quad (13.20)$$

$$B = \frac{\hbar}{64\pi^2}\ . \quad (13.21)$$

[2]The Euler-Mascheroni constant is defined to be $\gamma = \lim_{n\to\infty}(1+1/2+1/3+\ldots+1/n-\ln n)$, and its approximate value is 0.5772.

For ϕ^4 theory we have

$$V(\varphi) = \frac{1}{2}\, m_0^2\varphi^2 + \frac{1}{4!} g_0\, \varphi^4 \ , \tag{13.22}$$

so $V'' = m_0^2 + \frac{1}{2}\, g_0\varphi^2$, and hence

$$\begin{aligned} V_{\text{eff}}(\varphi) = \frac{1}{2}\,(m_0^2 - A m_0^2 g_0)\varphi^2 + \frac{1}{4!}\,(g_0 - 3g_0^2 A)\varphi^4 + \\ + B(m_0^2 + \frac{1}{2}\, g_0\varphi^2)^2 \ln\left(\frac{m_0^2 + \frac{1}{2}\, g_0\varphi^2}{\mu^2}\right) \ . \end{aligned} \tag{13.23}$$

From the effective potential we may immediately read off the physical mass and coupling constant. Their relation to the bare quantities given in the classical potential is

$$m^2 = m_0^2 - A m_0^2 g_0 \tag{13.24}$$

$$g = g_0 - 3g_0^2 A \ . \tag{13.25}$$

We would like to write the effective potential solely in terms of these physical parameters. To do this we invert the above relations. We need to do this only up to linear terms in $\hbar$. We find

$$m_0^2 = m^2 + A m^2 g \tag{13.26}$$

$$g_0 = g_0 + 3A g_0^2 \ . \tag{13.27}$$

It is now a simple matter to plug this in to 13.23. Using the fact that B is proportional to $\hbar$ we get

$$\begin{aligned} V_{\text{eff}}(\varphi) = \frac{1}{2}\, m^2\varphi^2 + \frac{1}{4!}\, g\varphi^4 + \\ + \frac{\hbar}{64\pi^2}\,(m^2 + \frac{1}{2} g\varphi^2)^2 \ln\left(\frac{m^2 + \frac{1}{2}\, g\varphi^2}{\mu^2}\right) \ . \end{aligned} \tag{13.28}$$

Now we see renormalization at work. The input parameters m_0, g_0 are chosen in such a way that the physical parameters m, g living in the effective action are made finite. The fact that the effective action is finite guarantees that all Green's functions are finite. Working with the effective potential was easier than with Γ, and it was good enough to renormalize m and g. What we miss is renormalization of φ, as this corresponds to $Z(\partial\varphi)^2$ terms — terms that vanish for constant φ. The fact that we could shuffle all the infinities into a redefinition of the input parameters is a proof (to one loop) that ϕ^4 theory is renormalizable in $d = 4$.

Looking at our result for $V_{\text{eff}}(\varphi)$ we see the reason for the introduction of μ. In fact, μ is an arbitrary mass scale parameter, needed to make the argument of the above logarithm dimensionless. μ is arbitrary, so it seems that we start with two parameters m_0 and g_0 and end up with three m, g, and μ. A change of μ is a change of the renormalization prescription. In fact, a change in μ can be compensated by a change in m and g such that $V_{\text{eff}}(\varphi)$ stays fixed. To show this we look at the change of V_{eff} when $\mu \to \mu'$. We find

$$\begin{aligned} V_{\text{eff}} \to & \frac{1}{2}\, m^2\varphi^2 + \frac{1}{4!}\, g\varphi^4 + \\ & + \frac{\hbar}{64\pi}\,(m^2 + \frac{1}{2} g\varphi^2)^2 \ln\left(\frac{m^2 + \frac{1}{2}\, g\varphi^2}{\mu^2}\,\frac{\mu^2}{\mu'^2}\right) = \\ & = V_{\text{eff}} - \frac{\hbar}{32\pi^2}\,\ln\frac{\mu'}{\mu}\,(m^2 + \frac{1}{2}\, g\varphi^2)^2 = \\ & = V_{\text{eff}} - \frac{\hbar}{32\pi^2}\,\ln\frac{\mu'}{\mu}\,(m^2 g\varphi^2 + \frac{1}{4} g^2\varphi^4)\ . \end{aligned} \tag{13.29}$$

In the last step we have droped an unimportant constant term. Thus, under $\mu \to \mu'$ the effective potential goes into

$$\frac{1}{2}\left(m^2 - \frac{\hbar}{16\pi^2}\, m^2 g \ln\frac{\mu'}{\mu}\right)\varphi^2 + \frac{1}{4!}\left(g - \frac{3\hbar}{16\pi^2}\, g^2 \ln\frac{\mu'}{\mu}\right)\varphi^4\ . \tag{13.30}$$

Therefore, to one loop, the effective potential $V_{\text{eff}}(\varphi)$ is invariant under

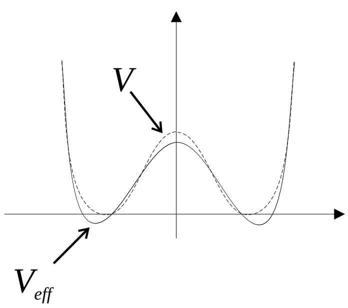
$$\mu \;\to\; \mu' \tag{13.31}$$

$$m^2 \;\to\; m^2 + \frac{\hbar}{16\pi^2}\, m^2 g \ln\frac{\mu'}{\mu} \tag{13.32}$$

$$g \;\to\; g + \frac{3\hbar}{16\pi^2}\, g^2 \ln\frac{\mu'}{\mu} \tag{13.33}$$

Because of this added symmetry we have, in fact, only two independent parameters in our quantum theory, just as in the classical case. We shall look into this in more detail when we investigate renormalization. At this point let us just say that we set μ to a typical energy scale of the phsical process that we are calculating. The above equations then tell us that the mass and coupling depend on the energy scale that we are working on.

Now we can look at the case when $m^2 < 0$. There is, in general, a very small change in going from V to V_{eff}. It is for this reason that we can trust the results of the previous lecture.



13.2 The $O(N)$ Model

The $O(N)$ model is given by the action

$$I = \int dx \left(\frac{1}{2} (\partial \vec{\phi})^2 - \frac{1}{2} m_0^2 \vec{\phi}^2 - \frac{1}{4!} g_0 (\vec{\phi}^2)^2 \right) , \tag{13.34}$$

where

$$\vec{\phi} = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix} . \tag{13.35}$$

The potential is thus

$$V(\vec{\varphi}) = \frac{1}{2} m_0^2 \vec{\varphi}^2 + \frac{1}{4!} g_0 (\vec{\varphi}^2)^2 . \tag{13.36}$$

Differentiating we get

$$\begin{aligned} \frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j} &= m_0^2 \, \delta_{ij} + \frac{1}{4!} g_0 (4 \, \delta_{ij} + 8 \varphi_i \varphi_j) = \\ &= (m_0^2 + \frac{1}{6} g_0 \vec{\varphi}^2) \, \delta_{ij} + \frac{1}{3} g_0 \varphi_i \varphi_j . \end{aligned} \tag{13.37}$$

The fact that $\mathrm{Det} \left(\partial^2 \delta_{ij} + \frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j} \right)$ is $O(N)$ invariant allows us to choose $\vec{\varphi}$ to point in any convenient direction. We choose $\vec{\varphi} = (\phi, 0, \ldots, 0)$. We are going to be calculating the effective potential, so we take φ to be constant. The above determinant is now

$$\begin{vmatrix} \partial^2 + (m_0^2 + \frac{1}{2} g_0 \varphi^2) & & & \\ & \partial^2 + (m_0^2 + \frac{1}{6} g_0 \varphi^2) & & \\ & & \ddots & \\ & & & \partial^2 + (m_0^2 + \frac{1}{6} g_0 \varphi^2) \end{vmatrix} \tag{13.38}$$

This is simply

$$\det(\partial^2 + M_1^2) \, \det(\partial^2 + M_2^2)^{N-1} , \tag{13.39}$$

where

$$M_1^2 = m_0^2 + \frac{1}{2} g_0 \varphi^2 \tag{13.40}$$

$$M_2^2 = m_0^2 + \frac{1}{6} g_0 \varphi^2 . \tag{13.41}$$

Note that Det is a determinant over $O(N)$ and space-time, while det is just over space-time. The corresponding traces will be denoted Tr and tr .

$$\mathrm{Tr} \, \ln I'' = \mathrm{tr} \, \ln(\partial^2 + M_1^2) + (N-1) \, \mathrm{tr} \, \ln(\partial^2 + M_2^2) . \tag{13.42}$$

Now it is a simple matter to calculate $V_{\text{eff}}(\varphi)$. To write this in a manifestly $O(N)$ invariant way we just take $\varphi \to |\vec{\varphi}|$. Finaly, the effective potential of the $O(N)$ model to one loop equals

$$V_{\text{eff}}(\vec{\varphi}) = \frac{1}{2}\, m_0 \vec{\varphi}^2 - \frac{\hbar}{2}\, \frac{\Gamma(-d/2)}{(4\pi)^{d/2}} \cdot$$
$$\cdot \left((m_0^2 + \frac{1}{2}\, g_0 \vec{\varphi}^2)^{d/2} + (N-1)(m_0^2 + \frac{1}{6}\, g_0 \vec{\varphi}^2)^{d/2} \right) . \qquad (13.43)$$

EXERCISES

13.1 Look at ϕ^3 theory and show (to one loop) that it is renormalizable in $d = 6$. Calculate $V_{\text{eff}}(\varphi)$.

13.2 Derive the symmetry transformations of $V_{\text{eff}}(\varphi)$ under $\mu \to \mu'$ for ϕ^3 theory in $d = 6$.

13.3 Consider a massless scalar field theory. Find the minima of $V_{\text{eff}}(\varphi)$. Can we trust our one loop results in this region?

13.4 Complete the $O(N)$ model calculation and obtain $V_{\text{eff}}(\varphi)$ in closed form in terms of m, g, and μ. Determine what transformation of m, g, and μ keeps $V_{\text{eff}}(\varphi)$ unchanged.

13.5 Our one loop result was given in terms of

$$\int \frac{d^4\bar{k}}{(2\pi)^4} \ln(\bar{k}^2 + m^2) \; .$$

In the lecture we calculated this using dimensional regularizations. Do this integral in a different way. Impose a momentum cut-off Λ (*i.e.* integrate over $|\bar{k}| < \Lambda$). Using this, write the effective potential for the ϕ^4 theory in $d = 4$ in terms of the renormalized parameters m, g and the cut-off Λ. Find the transformations of these parameters that do not change the effective potential.

Lecture 14

Solitons

14.1 Perturbative vs. Semi-Classical

We start by taking a look at the relation between asymptotic expansion in $\hbar$, *i.e.* semi-classical or loop expansion, and perturbative expansion in powers of the coupling. To be concrete we will work with ϕ^4 theory

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\,\phi^4 \;. \tag{14.1}$$

Classically, the coupling constant λ is not relevant. To see this we simply rescale the fields according to

$$\phi \to \widetilde{\phi} = \sqrt{\lambda}\,\phi \;. \tag{14.2}$$

The Lagrangian is now

$$\mathcal{L} = \frac{1}{\lambda}\left(\frac{1}{2}(\partial\widetilde{\phi})^2 - \frac{1}{2}m^2\widetilde{\phi}^2 - \frac{1}{4!}\widetilde{\phi}^4\right) \;. \tag{14.3}$$

S has units of action, *i.e.* $[S] = \hbar$. Simple dimensional analysis now gives us $[\mathcal{L}] = \hbar[L]^{-d}$ and $[\lambda] = \hbar^{-1}[L]^{d-4}$. In the classical theory there is no Planck constant, so $\hbar$ stands for any quantity with the dimension of action. Obviously λ doesn't effect the equations of motion — it is just an overall (dimensionfull) constant. In quantum theory we acquire one more universal constant to play with — the Planck constant. As we have seen, quantum theory is given in terms of

$$\frac{1}{\hbar}\mathcal{L} = \frac{1}{\hbar\lambda}\left(\frac{1}{2}(\partial\widetilde{\phi})^2 - \frac{1}{2}m^2\widetilde{\phi}^2 - \frac{1}{4!}\widetilde{\phi}^4\right) \;. \tag{14.4}$$

The relevant parameter is thus $\hbar\lambda$ with dimensions $[\hbar\lambda] = [L]^{d-4}$. In the quantum theory we are not just interested in looking at the solutions of the equations of motion. We sum over *all* field configurations, so the above parameter is important. In $d = 4$ the situation becomes particularly interesting since the theory is then given in terms of a dimensionless parameter. We will look into the significance of this when we talk about renormalization.

For now, the important lesson is that (since $\hbar\lambda$ is the relevant parameter) small λ expansion is the same as small $\hbar$ expansion. It would seem (at least in this model) that perturbation theory is equivalent to loop expansion. That is not the case. Perturbation is a Taylor expansion in powers of λ — in fact in powers of $\hbar\lambda$, only we usually set $\hbar = 1$. As such it doesn't see small $\hbar$ effects like

$$e^{-\frac{1}{\lambda\hbar}\,\text{Something}} \ . \tag{14.5}$$

These effects are smaller than any power of $\lambda\hbar$. Semi-classical expansion, however, does see these terms. From quantum mechanics we know that effects of this type can be very important. For instance tunneling is one such effect. Thus, tunneling is an example of a non-perturbative, though semi-classical effect. In quantum mechanics we treat these effects in the WKB approximation. In field theory we will see how to deal with them when we learn about instantons.

Tunneling is just one of many important non-perturbative effects. We have already seen non-perturbative effects in a simpler setting when we looked at symmetry breaking. For example, for $V = -\frac{1}{2}\,\mu^2\phi^2 + \frac{1}{4!}\,\lambda\phi^4$ the potential has minima at

$$\phi = \pm\sqrt{\frac{6\mu^2}{\lambda}} \ . \tag{14.6}$$

By shifting the field around one of the vacuua, for example

$$\phi = \sqrt{\frac{6\mu^2}{\lambda}} + \eta \ , \tag{14.7}$$

we are left with a theory of the η field. If λ is small then the η theory may be calculated perturbatively (in powers of $\sqrt{\lambda}$). Note, however, that in this case the shift 14.7 is large. In fact, as we have seen, the shift is proportional to $\frac{1}{\sqrt{\lambda}}$ and represents a non-perturbative effect. On the other hand, if λ is large then the shift is small, but all the remaining calculations are given in terms of a large coupling constant, *i.e.* they are immanently non-perturbative.

In the rest of this lecture we will meet another set of non-perturbative effects called solitons. For simplicity we will focus on solitons for models living in $d = 2$.

14.2 Classical Solitons

We look at a set of models in $d = 2$ whose Lagrangian is of the form

$$\mathcal{L} = \frac{1}{2}\,(\partial\phi)^2 - V(\phi) \ . \tag{14.8}$$

The energy is thus

$$E = \int dx\,\left(\frac{1}{2}\,(\partial_0\phi)^2 + \frac{1}{2}\,(\partial_1\phi)^2 + V(\phi)\right) \ . \tag{14.9}$$

The physical requirement that the energy is bounded from bellow implies that, without loss of generality, we may impose $V(\phi) \geq 0$. Vacuum configurations thus have $E = 0$ and are determined by $\partial_0\phi = \partial_1\phi = V(\phi) = 0$. Let us denote the vacuums by ϕ_i. Just as with spontaneous symmetry breaking, interesting results follow if there is more than one vacuum configuration.

We shall look at the space of fields of finite energy. Classically these are the only configurations that could interest us. It turns out that these are the only configurations that will interest us even in the quantum theory. The reason for this will be given later. Anyway, such fields obviously obey

$$\lim_{x\to-\infty} \phi(x,t) = \phi_i \tag{14.10}$$

$$\lim_{x\to+\infty} \phi(x,t) = \phi_j \; . \tag{14.11}$$

We shall say that fields with these particular boundary condition belong to the ε_{ij} sector of our theory. We next introduce a topological charge

$$Q = \phi(x = +\infty, t) - \phi(x = -\infty, t) \; , \tag{14.12}$$

a trivially conserved quantity on the space of finite energy configurations. All fields of finite energy are classified according to their Q value. Fields in the vacuum sectors ε_{ii} have $Q = 0$. We may write this topological charge as

$$Q = \int dx\, \frac{\partial\phi}{\partial x} = \int dx\, \varepsilon^{0\nu}\partial_\nu\phi \; , \tag{14.13}$$

so that $J^\mu = \varepsilon^{\mu\nu}\partial_\nu\phi$ is a conserved current. This is *not* a Noether current — it is not associated with any infinitesimal symmetry of the action. In fact, as we see $\partial_\mu J^\mu = 0$ follows identically, without the use of equations of motion. For this reason it is called a topological current.

The equations of motion for our models are of the form

$$(\partial_0^2 - \partial_1^2)\phi = -U'(\phi) \; . \tag{14.14}$$

We will now look at solutions of 14.14 in the form of traveling waves

$$\phi(x,t) = f(x - vt) = f(\xi) \; . \tag{14.15}$$

It follows that $\partial_0\phi = -v\frac{df}{d\xi}$, and $\partial_1\phi = \frac{df}{d\xi}$ so that f satisfies

$$\frac{d^2 f}{d\xi^2} = \gamma^2 U'(f) \; , \tag{14.16}$$

where $\gamma = (1 - v^2)^{-1/2}$ is the usual relativistic kinematic factor. Multiplying by $\frac{df}{d\xi}$ we easily obtain the first integral of 14.16

$$\frac{d}{d\xi}\frac{1}{2}\left(\frac{df}{d\xi}\right)^2 = \gamma^2 U'(f)\frac{df}{d\xi} = \gamma^2\frac{dU}{d\xi} \; , \tag{14.17}$$

and hence

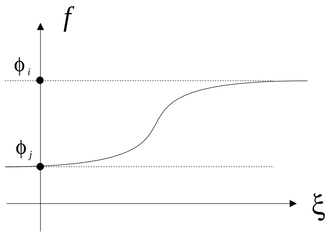
$$\frac{df}{d\xi} = \pm\gamma\sqrt{2V(f)} \ . \tag{14.18}$$

A simple consequence of 14.18 is that $f(\xi)$ can never attain one of the vacuum values ϕ_i unless $f(\xi)$ is identically equal to ϕ_i. We can have non-trivial solutions of 14.18 that satisfy

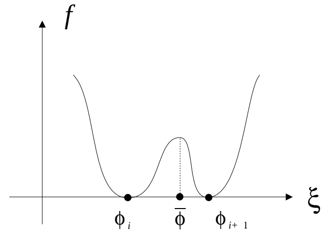
$$\lim_{\xi\to-\infty} f(\xi) = \phi_i \tag{14.19}$$

$$\lim_{\xi\to+\infty} f(\xi) = \phi_j \ , \tag{14.20}$$

where $i \neq j$. From the first integral we see that $f(\xi)$ is monotonic. The sign of $\phi_i - \phi_j$ determines the sign in 14.18. In general, these solutions are of the form



Note that ϕ_i and ϕ_j must be neighboring vacuums (see exercise 14.2). The solution that goes from ϕ_i to ϕ_{i+1} is called the i-th kink. Let us denote by $\bar{\phi}$ the maximum of $V(\phi)$ between ϕ_i and ϕ_{i+1}.



The center of the kink will be at the point ξ_0 at which we have $f(\xi_0) = \bar{\phi}$. Now it is a simple matter to integrate 14.18. For the kink we get

$$\gamma\,(\xi - \xi_0) = \int_{\bar{\phi}}^{f} \frac{df}{\sqrt{2V(f)}} \ . \tag{14.21}$$

The solution that goes from ϕ_{i+1} to ϕ_i is called the i-th anti-kink. Obviously we now use the minus sign in 14.18, and so

$$\gamma\,(\xi - \xi_0) = -\int_{\bar{\phi}}^{f} \frac{df}{\sqrt{2V(f)}} \ . \tag{14.22}$$

We shall soon look at a pair of models where these integrals can be calculated in closed form. Before we do that, however, let us determine the energy of the kink.

$$E = \int dx \left(\frac{1}{2} (\partial_0 \phi)^2 + \frac{1}{2} (\partial_1 \phi)^2 + V(\phi) \right) = \\ = \int_{-\infty}^{+\infty} d\xi \left(\frac{1}{2} (1 + v^2) \left(\frac{df}{d\xi} \right)^2 + V(f) \right) . \tag{14.23}$$

Using equation 14.18 we may write this as

$$E = \gamma^2 \int_{-\infty}^{+\infty} d\xi \, 2V(f) = \int_{-\infty}^{+\infty} d\xi \left(\frac{df}{d\xi} \right)^2 . \tag{14.24}$$

It follows that the kink energy density is given by

$$\rho(\xi) = \left(\frac{df}{d\xi} \right)^2 . \tag{14.25}$$

We could first solve 14.21, plug it into 14.25 to get $\rho(\xi)$, and finally integrate to find E. There is a simpler way. Using 14.18 once again 14.24 may be written as

$$E = \int df \, \frac{df}{d\xi} = \gamma \int_{\phi_i}^{\phi_{i+1}} df \, \sqrt{2V(f)} . \tag{14.26}$$

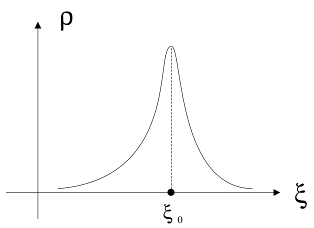
This bypasses the usually difficult task of solving 14.21. The mass of the kink is just the energy at $v = 0$. Thus we see that

$$M_{\text{kink}} = \int_{\phi_i}^{\phi_{i+1}} df \, \sqrt{2V(f)} , \tag{14.27}$$

as well as

$$E_{\text{kink}} = \gamma \, M_{\text{kink}} . \tag{14.28}$$

Note that this is the correct formula for the energy of a relativistic particle of mass M_{kink}. From the general form of the kink solution we conclude that the kink energy density has the bell-shaped form



The kink is a localized energy distribution. It is easy o show that the size of the kink depends on its speed according to the usual relativistic length contraction formula. In fact, the kink looks just like a classical relativistic particle.

14.3 The ϕ^4 Kink

As an illustration of the previous general results let us look at ϕ^4 theory. The potential is now

$$V(\phi) = \frac{\lambda}{4!}\left(\phi^2 - \frac{6\mu^2}{\lambda}\right)^2 = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \text{const} . \tag{14.29}$$

Notice that the mass term has the "wrong" sign, just as in symmetry breaking. This choice gives us two vacuua

$$\phi_\pm = \pm\sqrt{\frac{6\mu^2}{\lambda}} . \tag{14.30}$$

It is convenient to normalize the topological charge so that it takes the values $Q = 0, \pm 1$. The kink solution belongs to the $Q = 1$ sector. In this model we can get the kink in closed form. Using 14.21 we get

$$\gamma(\xi - \xi_0) = \int_0^f \frac{df}{\sqrt{2V(f)}} = $$
$$= \frac{\lambda}{6\mu^2}\sqrt{\frac{12}{\lambda}}\int_0^f \frac{df}{1 - \frac{\lambda}{6\mu^2}f^2} = \frac{\sqrt{2}}{\mu}\text{arc tanh}\left(\sqrt{\frac{\lambda}{6\mu^2}}\,f\right) . \tag{14.31}$$

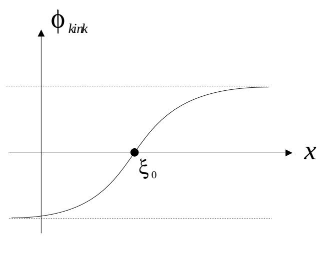
Therefore we have

$$f(\xi) = \sqrt{\frac{6\mu^2}{\lambda}}\tanh\left(\frac{\mu\gamma}{\sqrt{2}}(\xi - \xi_0)\right) . \tag{14.32}$$

The kink solution is thus

$$\phi_{\text{kink}}(\xi) = \sqrt{\frac{6\mu^2}{\lambda}}\tanh\left(\frac{\mu}{\sqrt{2}}\gamma(x - vt - \xi_0)\right) . \tag{14.33}$$

For $t = 0$ the kink is shown in the following figure



For $t \neq 0$ the kink keeps the same shape, only centered at $\xi_0 + vt$. The kink energy density is equal to

$$\rho(\xi) = \left(\frac{df}{d\xi}\right)^2 = \frac{3\mu^4\gamma^2}{\lambda}\frac{1}{\cosh^4\left(\frac{\mu\gamma}{\sqrt{2}}(\xi - \xi_0)\right)} . \tag{14.34}$$

As we can see, the typical size of the kink is $\triangle_{\text{kink}} \sim \frac{1}{\mu\gamma}$. As advertised, this indeed displays the correct relativistic length contraction. The anti-kink belongs to the $Q = -1$ sector, and is just minus the kink solution. The two vacuua are the solutions of the equations of motion corresponding to the $Q = 0$ sector. The mass of the ϕ^4 kink is

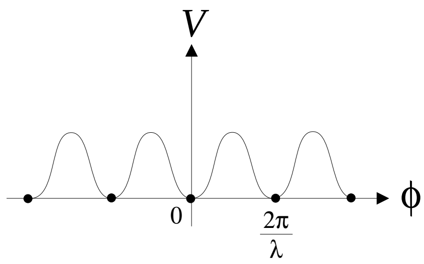
$$M_{\text{kink}} = \frac{12\sqrt{2}}{3}\frac{\mu^3}{\lambda} \ . \tag{14.35}$$

The fact that M_{kink} is proportional to $\frac{1}{\lambda}$ indicates that the kink represents a non-perturbative solution. Remember, we treated this model before when we looked at symmetry breaking. After shifting about a vacuum, we made a perturbative calculation (expanding in $\sqrt{\lambda}$). Note that in this way we could never recover the kink solution. For small λ our perturbative results would agree with low energy experiments. In that case the kink is extremely massive, and can not be produced at low energies.

14.4 The Sine-Gordon Kink

The sine-Gordon model describes a scalar field in $d = 2$ interacting through a periodic potential

$$V(\phi) = \frac{\mu^2}{\lambda^2}(1 - \cos\lambda\phi) \ . \tag{14.36}$$



We now have an infinity of vacuums $\phi_n = n\frac{2\pi}{\lambda}$ for $n \in \mathbb{Z}$. Properly normalized, the topological charge takes on all values in $\mathbb{Z}$. The classical equation of motion for this model is

$$\partial^2\phi + \frac{\mu^2}{\lambda}\sin\lambda\phi = 0 \ . \tag{14.37}$$

For small λ this looks a lot like the Klein-Gordon equation with mass μ, hence the funny name sine-Gordon equation.

Let us calculate the 0^{th} sine-Gordon kink — the one from ϕ_0 to ϕ_1 (the n^{th} kink is obtained from the 0^{th} kink by adding to it $n\frac{2\pi}{\lambda}$). From equation 14.21 we have

$$\begin{aligned}\gamma\,(\xi-\xi_0) &= \int_{\frac{\pi}{\lambda}}^{f}\frac{df}{\sqrt{2V(f)}} = \\ &= \frac{\lambda}{2\mu}\int_{\frac{\pi}{\lambda}}^{f}\frac{df}{\sqrt{\frac{1-\cos\lambda f}{2}}} = \frac{\lambda}{2\mu}\int_{\frac{\pi}{\lambda}}^{f}\frac{df}{\sin\frac{\lambda f}{2}} = \\ &= \frac{1}{\mu}\ln\tan\frac{\lambda f}{4}\;. \end{aligned} \tag{14.38}$$

Therefore

$$f(\xi) = \frac{4}{\lambda}\mathrm{arc\,tan}\left(e^{\mu\gamma\,(\xi-\xi_0)}\right)\;. \tag{14.39}$$

In other words

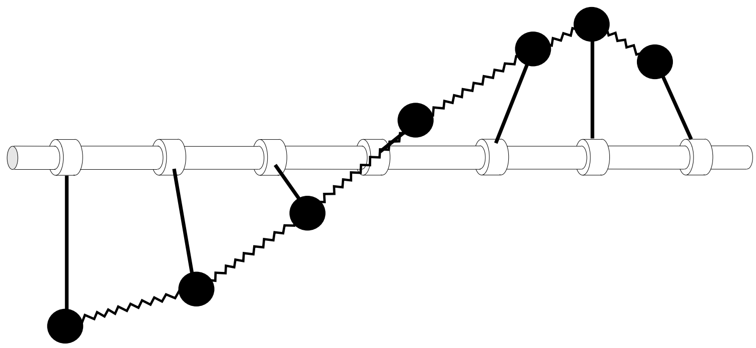
$$\phi_{\mathrm{kink}}(x,t) = \frac{4}{\lambda}\mathrm{arc\,tan}\left(e^{\mu\gamma\,(x-vt-\xi_0)}\right)\;. \tag{14.40}$$

As with the ϕ^4 kink, the typical size of the sine-Gordon kink is $\triangle_{\mathrm{kink}} \sim \frac{1}{\mu\gamma}$. The mass of this kink is

$$M_{\mathrm{kink}} = 8\frac{M}{\lambda^2}\;. \tag{14.41}$$

The kink mass is again large for the case of weak coupling.

As an illustration of this let us look at a mechanical analogue of the sine-Gordon model. We look at an infinite series of pendulums of mass m, length ℓ, at distances a. The pendulums are free to move in the y, z plane only. They are connected by elastic springs of constant k. The whole system is in a gravitational field pointing in the y direction. We measure the angles from the y-axis.



The kinetic and potential energies of the system are

$$T = \sum_n \frac{1}{2} m\ell^2\phi_n^2 \tag{14.42}$$

$$V = \sum_n \frac{1}{2} k\,(\phi_{n+1}-\phi_n)^2 + \sum_n mg\ell(1-\cos\phi_n)\;. \tag{14.43}$$

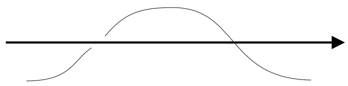
Let us take the continuum limit $a \to 0$. To do this we introduce

$$\frac{m}{a} \rightarrow \rho \tag{14.44}$$
$$ka \rightarrow \sigma \tag{14.45}$$
$$\phi_n(t) \rightarrow \phi(x,t) \tag{14.46}$$

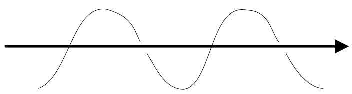
For the Lagrangian we find $L \rightarrow \int dx\mathcal{L}$, where

$$\mathcal{L} = \frac{1}{2}\rho\ell^2(\partial_t\phi)^2 - \frac{1}{2}\sigma(\partial_x\phi)^2 - \rho g\ell(1-\cos\phi) \ . \tag{14.47}$$

If the parameters satisfy $\rho\ell = 1$, $\sigma = 1$, and $\rho g\ell = \frac{\mu^2}{\lambda^2}$ we recover the sine-Gordan model. The kink has $Q = 1$. In this mechanical model it looks like



In the next lecture we will see that the sine-Gordon model has soliton solutions for any value of topological charge. For example, the $Q = -3$ solution corresponds to



Obviously, the topological charge just counts the number of times the system of pendulums winds around the x-axis. Q is the winding number. It takes a finite amount of energy to rotate one pendulum by 2π, however, to change Q, we need to rotate infinitely many pendulums. Therefore, we need an infinite amount of energy to change Q. This is what is at the root of Q conservation.

EXERCISES

14.1 Look at the relation between perturbation and semi-classical expansion for the case of electrodynamics. Do the same for a spin zero theory with both cubic and quartic interactions. In the latter case, show that at least one of the coupling constants must be dimensionfull. Now look at a theory of several vector fields A^a_μ with a cubic and quartic interaction. Can you make such a model solely out of dimensionless coupling constants? Do you know any such model?

14.2 Prove that the kink $f(\xi)$ can never attain one of the vacuum values.

Lecture 15

Solitons Continued

15.1 Bogomolyni Decomposition

We are going to continue with the classical theory of solitons. First of all let us rederive some of the results of the previous lecture by using the technique of Bogomolyni's decomposition. To do this we look at the static energy functional

$$E_{\text{static}}[\phi] = \int dx \left(\frac{1}{2} \left(\frac{d\phi}{dx} \right)^2 + V \right) . \tag{15.1}$$

The extremum of this functional satisfies

$$\frac{d^2\phi}{dx^2} = V'(\phi) . \tag{15.2}$$

The first integral of this equation is $\left(\frac{d\phi}{dx}\right)^2 = 2V(\phi)$, or equivallently

$$\frac{d\phi}{dx} \mp \sqrt{2V(\phi)} = 0 . \tag{15.3}$$

Having this in mind it is useful to write the static energy as

$$E_{\text{static}}[\phi] = \int dx \, \frac{1}{2} \left(\frac{d\phi}{dx} \mp \sqrt{2V(\phi)} \right)^2 \pm \int_{\phi_i}^{\phi_j} d\phi \, \sqrt{2V(\phi)} . \tag{15.4}$$

Following Bogomolyni, we have decomposed the static energy functional into a sum of two terms. The first term vanishes for solutions of (15.3). The second term is topological *i.e.* it is constant in a given sector ε_{ij}. If different vacua are related by symmetry as in the sine-Gordon model then

$$\int_{\phi_i}^{\phi_j} d\phi \, \sqrt{2V(\phi)} = n \int_{\phi^0}^{\phi_1} d\phi \, \sqrt{2V(\phi)} . \tag{15.5}$$

We have assumed that ϕ_j is the n^{th} vacuum to the right of ϕ_i. The Bogomolyni decomposition (15.4) then implies that static fields in the $Q = n$ sector obey

$$E_{\text{static}}[\phi] \geq |n| \int_{\phi^0}^{\phi_1} d\phi \, \sqrt{2V(\phi)} = |n| E_{\text{kink}} \ . \tag{15.6}$$

Let us next consider a superposition of n stationary kinks

$$\tilde{\phi}(x) = \phi_{\text{kink}}(x - x_1) + \phi_{\text{kink}}(x - x_2) + \ldots + \phi_{\text{kink}}(x - x_n) \ . \tag{15.7}$$

The above superpositions are not solutions of the equations of motion. Even very distant kinks overlap a little bit and interact. The kinks either attract or repel and so they can't represent a stationary configuration. This is true, however, if the centers of the kinks are far from each other then $\tilde{\phi}$ is *almost* a solution, *i.e.* $\frac{d\tilde{\phi}}{dx} - \sqrt{2V(\tilde{\phi})}$ is exponentaly small. In fact, if we have $|x_i - x_j| \to \infty$ for all x_i, then

$$\frac{d\tilde{\phi}}{dx} - \sqrt{2V(\tilde{\phi})} \to 0 \ . \tag{15.8}$$

Such a configuration is called a dilute kink gas. A dilute kink gas comes arbitrarily close to being a solution of the static equations of motion. Why should one even consider approximate solutions? Classical physics is only interested in the true solutions of the equations of motion. Classicaly, we look at approximate solutions only as a last resort — when we can't solve the equations of motion exactly. In the $d = 2$ models that we have been looking at we *know* all the static solutions, so why talk about approximate solution? The reason we are interested in such configurations comes from the quantum theory. We have seen in previous lectures that classical solutions play an important role in quantum theory. They are the markers that tell us which regions in the space of all field configurations give dominant contributions to the path integral. In fact, configurations (like the dilute kink gas) that come arbitrarily close to being classical solutions are just as good markers as the classical solutions themselves. Because of this we can now read off the spectrum of a given model. For example, for ϕ^4 theory we have shown the spectrum in Figure 15.1. The dashed lines in this figure correspond to dilute gases consisting of appropriate numbers of kinks (s) and antikinks ($\bar{s}$). Solid lines are the true ground states — the two vacuums and the kink (and anti-kink). The $Q = -1$ part of the spectrum follows from $Q = 1$ by exchanging kinks and anti-kinks. For the sine-Gordon model we find a similar spectrum. This is shown in Figure 15.2. As before, the dashed lines indicate multi-kink configurations. It is obvious how to extend this to all the Q sectors. The above figures tell us how to do semi-classical approximations. The dominant contribution to $Z[J]$ comes from small oscillations about the vacuum. This is what we have been doing so far. If a model has other solutions (kinks) then they contribute as well. To our previous result for $Z[J]$ we need to add the effect of small oscillations about the kink. This is a non-perturbative effect — like the mass, the action of a kink is large for small coupling constants. Because of this, the contribution of oscillations

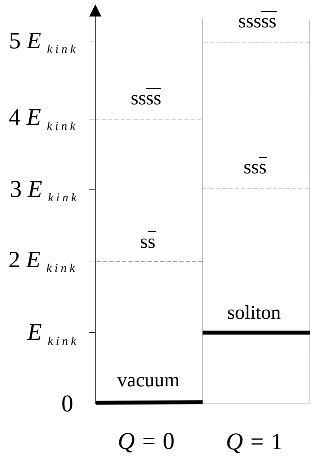


Figure 15.1: The ϕ^4 spectrum. Solid lines depict exact ground states. Approximate ground states are shown in dashed lines.

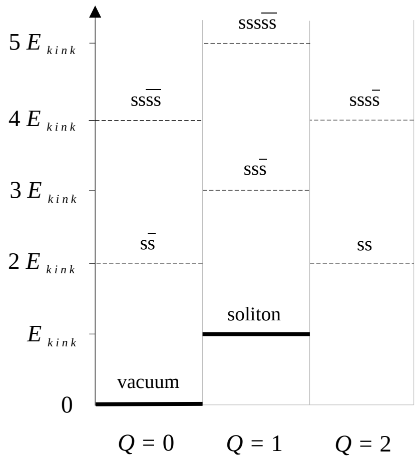


Figure 15.2: sine-Gordon spectrum

about the kink are smaller than of the oscillations about the vacuum. Finally, to be even more precise we must take into account contributions from oscillations about all multi-kink configurations. The lower the energy of the solution about which we expand the greater its contribution to the generating functional. Now we see the true reason why we only looked at finite energy solutions.

Non-trivial finite energy solutions of the classical equations of motions are called kinks or solitons. We have learned to think of kinks as particles. One can then ask what happens when kinks collide. To analyze this we set up a multi-kink configuration at some early time and watch how it evolves. In ϕ^4 theory we can only look at a collision between a kink and anti-kink. After they collide we just get ordinary spreading ripples, i.e excitations over the ordinary vacuum.

Kinks look like ordinary particles until we scatter them. In the more mathematical literature there is a distiction made between kinks and solitons. Solitons are kink solutions that have the added property that they scatter like ordinary particles. In quantum field theory this distinction is not important and so we use the terms kink and soliton interchangeably. Still, it is interesting to look at a theory with *true* solitons. The best example is our sine-Gordon model. In this model we can look at kink kink and kink anti-kink collisions. Initialy things look just as before. The kinks approach and merge forming complicated ripples. Now the interesting thing happens — the ripples further evolve into two kinks that move away from one another. Sine-Gordon kinks scatter like normal particles. The sine-Gordon model has exact multi-soliton solutions. For example, the scattering of two solitons is depicted by the solution $\phi_{\rm ss}$ given by

$$\tan\left(\frac{\lambda\phi_{\rm ss}}{4}\right) = v\frac{\sinh(\mu\gamma x)}{\cosh(\mu\gamma v t)} \ . \tag{15.9}$$

For very early times $\phi_{\rm ss}$ is approximately equal to

$$\phi_k\left(\gamma(x+vt+x_0)\right) + \phi_k\left(\gamma(x-vt-x_0)\right) - \frac{2\pi}{\lambda} \ . \tag{15.10}$$

This is just the superposition of two kinks moving towards each other in the center of mass reference frame. At very late times we find that $\phi_{\rm ss}$ becomes approximately equal to

$$\phi_k\left(\gamma(x+vt-x_0)\right) + \phi_k\left(\gamma(x-vt+x_0)\right) - \frac{2\pi}{\lambda} \ , \tag{15.11}$$

i.e. after the scattering we find the two kinks moving away from each other.

15.2 Derrick's Theorem

All our examples of field theories with solitons have been in $d = 2$ dimensions. We worked in lower dimensions because there we had examples of models in which we could do the calculations exactly. Still, we would like to find models with solitons in more realistic dimensions of spacetime. There is a simple result called Derrick's theorem that limits the number of such models.

Let's look at a theory of (several) scalar fields in d dimensions whose dynamics is given by

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi) \ , \tag{15.12}$$

where, as usual $V(\phi) \geq 0$. Derrick's theorem states that for $d \geq 3$ the only non-singular time independent solutions of finite energy are the vacuums (given by $V(\phi) = 0$. To prove this let us introduce

$$V_1 = \int d^{d-1}x\,\frac{1}{2}\left(\partial_i\phi\right)^2 \tag{15.13}$$

$$V_2 = \int d^{d-1}x\,V(\phi) \ . \tag{15.14}$$

We obviously have $V_1 \geq 0$ as well as $V_2 \geq 0$. These expressions vanish simultaneously only for the ground states. Let us now look at a time independent solution $\phi(\vec{x})$. From it we define a one parameter family $\phi(\vec{x}, \lambda) = \phi(\lambda\vec{x})$, where $\lambda > 0$. For this family the energy equals

$$E(\lambda) = \lambda^{3-d} V_1 + \lambda^{1-d} V_2 \ . \tag{15.15}$$

ϕ is an extremum of the action, and it is time independent. It follows that it extremizes the above energy as well, hence

$$\left. \frac{dE}{d\lambda} \right|_{\lambda=1} = 0 \ . \tag{15.16}$$

This gives

$$(d-3)V_1 + (d-1)V_2 = 0 \ . \tag{15.17}$$

For $d > 3$ we immediately get $V_1 = V_2 = 0$. For $d = 3$ this only gives $V_2 = 0$. Assuming that $\phi(x)$ is not singular we again get that it must be one of the vacuums. For example, for $V(\phi) = \frac{\lambda}{4!}(\phi^2 - a^2)^2$ a possible configuration with $V_2 = 0$ would be $\phi(x) = -a\theta(1-x) + a\theta(x-1)$. Here obviously $V_1 \neq 0$, however this is a singular solution. The only non-singular solutions would be $\phi = \pm a$ *i.e.* the vacuums. Thus Derrick's theorem works for $d = 3$ as well.

There are three ways to get solitons in $d \geq 3$, *i.e.* three ways to get around Derrick's theorem:

1. Derrick's theorem only talks about time-independent solutions. It is posible to find scalar models in $d \geq 3$ which have time dependent non-dissipative solutions. These models need not concern us as they are not relativistically invariant. This is rather obvious — otherwise we could just boost to a reference frame in which the solutions are stationary.

2. We can always look at models with non-vanishing spin where there is no analogue of Derrick's theorem.

3. We can look at scalar models in $d \geq 3$ with constraints.

We shall consider examples of these last two types of models in later lectures.

EXERCISES

15.1 Consider the multi kink configuration of equation (15.7. Show that for distant kinks $\frac{d\tilde{\phi}}{dx} - \sqrt{2V(\tilde{\phi})}$ is exponentialy small.

15.2 Prove that $\phi_{\rm ss}$ given in equation (15.9) represents an exact solution of the sine-Gordon equation of motion. For $t \to -\infty$ this solution goes over into the superposition of two kinks given in (15.10). On the other hand, for $t \to \infty$ it is goes over into the superposition given in equation (15.11). Prove this.

Lecture 16

Quantization of Solitons

16.1 Stability

Let $\phi_{\rm s}(x)$ be a static soliton centered at $x = 0$. We'll look at small fluctuations around this soliton

$$\phi(x,t) \approx \phi_s(x) + \eta(x,t) \ . \tag{16.1}$$

For this to be a solution η must satisfy the linearized equation

$$\partial^2\eta + V''\left(\phi_{\rm s}(x)\right)\eta = 0 \ . \tag{16.2}$$

Therefore the fluctuations η are

$$\eta(x,t) = \sum_n e^{-i\omega_n t}\psi_n(x) \ , \tag{16.3}$$

where

$$\left(-\frac{d^2}{dx^2} + V''\left(\phi_{\rm s}(x)\right)\right)\psi_n(x) = \omega_n^2\psi_n(x) \ . \tag{16.4}$$

The last formula is just a Schrödinger equation for a particle moving in a potential $V''\left(\phi_{\rm s}(x)\right)$. Small fluctuations around $\phi_{\rm s}(x)$ are stable if η always remains small, *i.e.* if $\omega_n^2 \geq 0$. We already know one of the eigenstates ψ_n. In fact

$$\psi_0 = \frac{d}{dx}\phi_{\rm s}(x) \tag{16.5}$$

is a zero mode of the above operator. Therefore

$$\left(-\frac{d^2}{dx^2} + V''\left(\phi_{\rm s}(x)\right)\right)\psi_0 = 0 \ . \tag{16.6}$$

This follows directly from the equation of motion

$$-\frac{d^2\phi_{\rm s}}{dx^2} + V'\left(\phi_{\rm s}(x)\right) = 0 \tag{16.7}$$

by differentiation. Thus $\omega_0 = 0$. We have seeen that $\phi_{\rm s}(x)$ is a monotonic function. It follows that ψ_0 never vanishes, *i.e* has no nodes. An eigenfunction with no nodes is in fact the ground state. We therefore find that $\omega_n^2 \geq 0$, which proves the stability of the static solition solution.

Let us expand the energy

$$E = \int dx \left(\frac{1}{2} (\partial_t \phi)^2 + \frac{1}{2} (\partial_x \phi)^2 + V(\phi) \right) \tag{16.8}$$

about the static soliton. We get

$$\begin{aligned} E[\eta] &= M_{\rm s} + \int dx \left(\frac{1}{2} (\partial_t \eta)^2 + \frac{1}{2} (\partial_x \eta)^2 + V''(\phi_{\rm s}(x))\, \eta^2 \right) + \dots \\ &= M_{\rm s} + \int dx \left(\frac{1}{2} (\partial_t \eta)^2 + \eta[-\frac{1}{2} \partial_x^2 + V''(\phi_{\rm s}(x))]\eta \right) \dots \end{aligned} \tag{16.9}$$

Expanding fluctuations in terms of the modes ψ_n we have

$$\eta(x,t) = \sum_n Q_n(t) \psi_n(x) \ , \tag{16.10}$$

where $\int dx\, \psi_n(x) \psi_m(x) = \delta_{n,m}$. Therefore

$$E[Q_n] = M_{\rm s} + \sum_n (\frac{1}{2} \dot{Q}_n^2 + \omega_n^2 Q_n^2) \ . \tag{16.11}$$

This is just a set of harmonic oscillators indexed by $n > 0$. The $n = 0$ mode represents the free motion of the center of mass. The ground state of this system is thus

$$E_{\rm s} = M_{\rm s} + \sum_{n>0} \frac{1}{2} \omega_n \ . \tag{16.12}$$

$E_{\rm s}$ is the soliton mass (up to one loop). It turns out that the above sum is infinite. Well, it should be. After all the vacuum energy to one loop

$$E_{\rm v} = \sum_k \frac{1}{2} \omega_k^{(0)} \tag{16.13}$$

is also infinite. The $\omega_n^{(0)}$'s are found by expanding around the vacuum. Therefore,

$$\left(-\frac{d^2}{dx^2} + V''(v) \right) \psi_k^{(0)}(x) = \omega_k^{(0)} \psi_k^{(0)}(x) \ . \tag{16.14}$$

What we physically measure is the difference $E_{\rm s} - E_{\rm v}$. Thus, the soliton mass is in fact equal to

$$M_{\rm s} + \frac{1}{2} \sum_{n>0} \omega_n - \frac{1}{2} \sum_k \omega_k^{(0)} \ . \tag{16.15}$$

This is still UV divergent. The finite answer is obtained only after renormalization.

16.2 Path Integral Formalism

The calculations in the previous section are in the operator formalism. We have kept them because of their simplicity. We are now ready to start thinking about quantizing solitons in the path integral formalism. Given a Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi) - V(\phi) \tag{16.16}$$

we want to calculate the generating functional using the semi-classical approximation. Therefore, we expand around classical solutions. Expanding around a soliton $\phi = \phi_{\rm s} + \eta$ we get

$$\mathcal{L} = \mathcal{L}(\phi_{\rm s}) + \frac{1}{2}\eta\left(-\frac{d^2}{dt^2} + \frac{d^2}{dx^2} - V''(\phi_{\rm s})\right)\eta\ . \tag{16.17}$$

We would next need to find the inverse of the above kinetic operator. The problem is that this inverse does not exist. As we have seen, the kinetic operator has a zero mode $\psi_0 = \frac{d}{dx}\phi_{\rm s}$. This zero mode is just a Goldstone boson corresponding to translations. We have broken translation invariance! We did this by expanded around $\phi_{\rm s}(x)$ which is located at $x = 0$. Similarly, if we expanded around $\phi_{\rm s}(x-a)$ (soliton located at $x = a$) we would again break translation invariance. It is easy to see that $\phi_{\rm s}(x) - a\psi_0 = \phi_{\rm s}(x-a)$. The zero mode moves the center of the soliton. Translation invariance is obviously saved by expanding around $\phi_{\rm s}(x-a)$ and adding contributions from all values a. We'll do this in a moment. Before that let us see how translations are broken in the operator formalism. We have

$$\hat{\phi}(x+a) = e^{i\hat{P}a}\hat{\phi}(x)e^{-i\hat{P}a}\ , \tag{16.18}$$

where $\hat{P}$ is the total momentum. If we now shift $\hat{\phi}(x) = \phi_0(x) + \hat{\eta}(x)$ then

$$e^{i\hat{P}a}\left(\phi_0(x) + \hat{\eta}(x)\right)e^{-i\hat{P}a} = \phi_0(x) + \hat{\eta}(x+a) \neq \hat{\phi}(x+a)\ . \tag{16.19}$$

When we calculated the soliton mass in the pervious section we broke translation invariance. This was no problem. We didn't care where we put our soliton, we were just interested in its mass. However, in calculating most things we need to retain translation invariance. How do we do this in the operator formalism? The proceedure is to isolate the center of mass coordinate $\hat{X}$ that is conjugate to $\hat{P}$, *i.e.* satisfies $[\hat{X}(t), \hat{P}(t)] = i$. We know that $\hat{X}$ exists, although it is difficult to construct in terms of ϕ and π. Using it we define the shift about the soliton as

$$\hat{\phi}(x) = \phi_0(x + \hat{X}) + \hat{\eta}(x)\ . \tag{16.20}$$

Translation invariance is no longer broken since $\hat{X}(t)$ takes on all of its possible eigen values. The problems with zero modes also goes away — no symmetry is broken so no Goldstone bosons are present.

Now we turn to path integrals. We want to evaluate

$$\int [d\phi]\, e^{iI[\phi]} \tag{16.21}$$

semi-classicaly by expanding around stationary solitons $\phi(x-X)$. As we have seen we'll need to integrate over the center of mass coordinate $X(t)$. We do this by inserting the following identity into the path integral above:

$$1 = \int [dX]\, \delta\left[\int dx\, \psi_0\left(x - X(t)\right)\left(\phi(x,t) - \phi_{\rm s}(x-X)\right)\right] \triangle\, [\phi]\ . \tag{16.22}$$

ψ_0 is just the zero mode. Obviously

$$\begin{aligned} \triangle[\phi] &= \frac{\delta}{\delta X}\left\{\int dx\, \psi_0(x-X)\left(\phi(x,t) - \phi_{\rm s}(x-X)\right)\right\} = \\ &= \frac{\delta}{\delta X}\left\{\int dy\, \psi_0(y)\left(\phi(y+X) - \phi_{\rm s}(y)\right)\right\} = \\ &= \int dy\, \psi_0(y)\frac{\partial \phi(y+X)}{\partial y} = \int dx\, \psi_0(x-X)\frac{\partial \phi}{\partial x}\ . \end{aligned} \tag{16.23}$$

Note that

$$\triangle[\phi_{\rm s}(x-X)] = \int dx\, \psi_0(x-X)^2 = \triangle_{\rm s} \tag{16.24}$$

does not depend on X. Similarly,

$$I[\phi_{\rm s}(x-X)] = I_{\rm s} \tag{16.25}$$

does not depend on X. We are now ready to look at our path integral. It equals

$$\int [dX] \int [d\phi] e^{iI[\phi]}\, \triangle\, [\phi]\delta\left[\int dx\, \psi_0(x-X)\left(\phi(x,t) - \phi_{\rm s}(x-X)\right)\right]\ . \tag{16.26}$$

We next shift $\phi = \phi_{\rm s}(x-X) + \eta(x-X,t)$, and expand the action to η^2. The part of the integrand that is not in the exponent is expanded to η^0 — this is just the standard stationary phase approximation that we have been using all along. Using (16.24) and (16.25) we find

$$\begin{aligned} &\int [dX] \int [d\eta] e^{iI_{\rm s}} e^{\frac{i}{2}\int dx dt\, \eta(x-X,t) K \eta(x-X,t)} \triangle_{\rm s} \\ &\qquad \delta\left[\int dx\, \psi_0(x-X)\eta(x-X)\right] = \\ &= \triangle_{\rm s} e^{iI_{\rm s}} \int [dX] \int [d\eta] e^{\frac{i}{2}\int dx dt\, \eta(x,t) K \eta(x,t)} \\ &\qquad \delta\left[\int dx\, \psi_0(x)\eta(x)\right]\ . \end{aligned} \tag{16.27}$$

In the last step we shifted $x - X \to x$. The X dependence now factorizes. Now if we write $\eta(x,t) = \sum_n Q_n(t)\psi_n(x)$ we see that the delta function above gets rid of the zero mode Q_0. We have exchanged the zero mode integration for an integration over the collective coordinate X representing the the center of mass.

Our final result is

$$\triangle_{\rm s}\, e^{iI_{\rm s}} \det{}'(K)^{-1/2} \ , \tag{16.28}$$

where the prime indicates that the zero mode should be dropped when calculating the determinant.

EXERCISES

16.1 Consider the ϕ^4 model, as well as the sine-Gordon model. Show that in both cases the potential in the Schrödinger equation (16.4) is

$$V''\left(\phi_{\rm s}(x)\right) = -\frac{\alpha^2 s(s+1)}{\cosh^2(\alpha x)} \ ,$$

for some α and integer s. This is one of the few potentials for which we can exactly solve the Schrödinger equation. Without doing the exact calculation show that for $s = 2$ (the ϕ^4 model) there is a bound state. In fact there is only one. Try to argue why for $s = 1$ (the sine-Gordon model) there is no bound state. How many bound states do you expect for general s?

16.2 Look at the vacuum energy $E_{\rm v}$ to one loop for the ϕ^4 model. Show how it diverges with L (the range of x) and Λ (the momentum cut-off).

16.3 Find the total momentum P in terms of ϕ and π (fields and conjugate momenta) in the classical theory. Show that $\{aP, \phi\}_{P.B.} = a\frac{d\phi}{dx}$, *i.e.* that P generates translations.

Lecture 17

Instanton Preliminaries

17.1 Classical Solutions

In this lecture we are going to present a method for extracting information about the ground state of a quatnum field theory. For simplicity we will work in $d = 1$ dimension, however the method will work for general d. As we have seen, QFT in $d = 1$ is equivallent to non-relativistic quantum mechanics. Working in $d = 1$ will enable us to compare our field theory results with standard quantum mechanical calculations.

Let us consider a theory with Hamiltonian

$$H = \frac{1}{2}p^2 + V(x) \ . \tag{17.1}$$

The potential must be bounded from bellow. Without loss of generality we will assume that $V(x) \geq 0$. We now look at the Wick rotated version of our basic path integral formula

$$\langle x_f | e^{-\frac{1}{\hbar}T\hat{H}} | x_i \rangle = N \int [dx] \, e^{-\frac{1}{\hbar}\bar{I}[x]} \ . \tag{17.2}$$

N is just a normalization constant. The above integration is over all trajectories from $x(-T/2) = x_i$ to $x(T/2) = x_f$, while the Euclidian action is simply

$$\bar{I}[x] = \int_{-T/2}^{T/2} d\tau \, \left(\frac{1}{2}\dot{x}^2 + V(x) \right) \ . \tag{17.3}$$

Equation (17.2) simplifies for $T \to \infty$. In this limit the left hand side of (17.2) is given solely in terms of the ground state, since

$$\langle x_f | e^{-\frac{1}{\hbar}T\hat{H}} | x_i \rangle = \sum_n \langle x_f | n \rangle \langle n | x_i \rangle \, e^{-\frac{1}{\hbar}TE_n} \to \langle x_f | 0 \rangle \langle 0 | x_i \rangle \, e^{-\frac{1}{\hbar}TE_0} \ . \tag{17.4}$$

On the right hand side of (17.2) we must integrate over all paths from $x(-\infty) = x_i$ to $x(+\infty) = x_f$. We will evaluate this path integral semi-classicaly. We find

$$N \int [dx] \, e^{-\frac{1}{\hbar}\bar{I}[x]} \approx N \, e^{-\frac{1}{\hbar}\bar{I}[x]} \ \det\left(-\partial_\tau^2 + V''(x_{\text{class}})\right)^{-1/2} \ . \tag{17.5}$$

Today's lecture consists of mathematical preliminaries that will be needed in order to be able to explicitly evaluate the above semi-classical result. The reason for doing is obvious — it will enable us to learn about the ground state of our theory. Before we start, let us emphasize that we will be evaluating expressions corresponding to an Euclidian field theory, however, the ground state we are finaly interested in is of the starting, Minkowskian theory. Having derived (17.4) and (17.5) we can now set $\hbar = 1$.

For semi-classical results to be aplicable we must have

$$\bar{I}[x_{\text{class}}] = \int_{-\infty}^{+\infty} d\tau \left(\frac{1}{2}\dot{x}_{\text{class}}^2 + V(x_{\text{class}})\right) < \infty \ . \tag{17.6}$$

This implies that we have to choose both x_i and x_f to be absolute minima of $V(x)$. The classical solutions x_{class} satisfy

$$-\partial_\tau^2 \, x_{\text{class}} + V'(x_{\text{class}}) = 0 \ . \tag{17.7}$$

Even without explicitly solving this equation we know what its solutions are going to look like qualitativly. The above is just the equation of motion for a non-relativistic particle of unit mass moving in a potential $-V$.

Let us, for example, look at the potential given in Figure 17.1. The only possible boundary conditions that lead to solutions with finite Euclidian action are $x_i = x_f = 0$. From the inverted potential in Figure 17.2 we see that the only possible motion is $x_{\text{class}}(\tau) = 0$, *i.e.* the trivial vacuum solution.

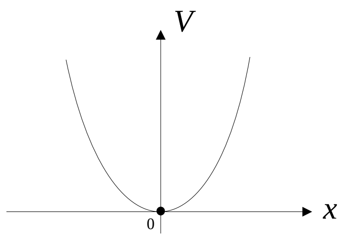


Figure 17.1: Single well potential

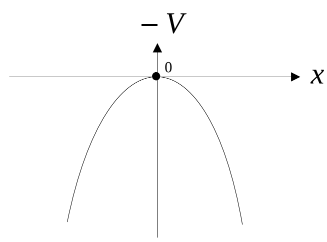


Figure 17.2: Associated "particle" moves in this inverted potential

As a more interesting example let us look at the double well potential of Figure 17.3. The associated particle moves in the inverted potential given in Figure 17.4.

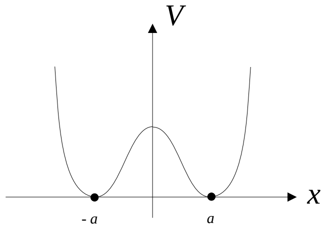


Figure 17.3: Double well potential

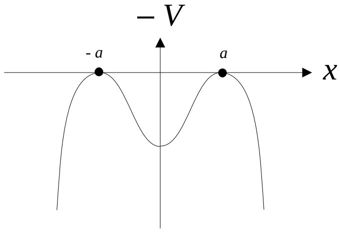


Figure 17.4: Associated "particle" moving in inverted double well potential

As we have seen, we have to consider only motions with $\pm a$ as initial and final states. There are four possible motions. Two are just vacuum solutions $x_{\text{class}}(\tau) = a$ and $x_{\text{class}}(\tau) = -a$. However, we now have two non-trivial motions. For one $x_i = -a$ and $x_f = a$ — represented by the rolling of the particle from $-a$ to a. The last solution corresponds to a rolling in the opposite direction. Non-trivial, finite action solutions of the classical Euclidian theory are called instantons.

17.2 The Determinant

As we have seen in equation (17.5) we will have to learn how to evaluate determinants of the form

$$\det\left(-\partial_\tau^2 + W(\tau)\right) \ , \tag{17.8}$$

where $W(\tau) = V''\left(x_{\text{class}}(\tau)\right)$. Obviously W is a bounded function of τ. The beauty of working in $d = 1$ is that we can give an exact formula for these determinants. Before we do this it is instructive to derive equation (17.5) a bit more precisely. In the path integral we need to integrate over all functions satisfying

$$x(-T/2) = x_i \tag{17.9}$$
$$x(+T/2) = x_f \ . \tag{17.10}$$

If $x_{\text{part}}(\tau)$ is any such function we may write the general one as

$$x(\tau) = x_{\text{part}}(\tau) + \sum_n c_n x_n(\tau) \ , \tag{17.11}$$

where the x_n's represent an orthonormal basis on the space of functions that vanish at the end points $\pm T/2$. In the semi-classical approximation it is best if we choose $x_{\rm class}(\tau)$ for our particular solution. Diagonalizing the Hermitian differential operator $-\partial_\tau^2 + W(\tau)$ we obtain a convenient basis of x_n's. Thus

$$\left(-\partial_\tau^2 + W(\tau)\right) x_n(\tau) = \lambda_n x_n(\tau) \ , \tag{17.12}$$

where

$$\int_{-T/2}^{T/2} d\tau \, x_n(\tau) x_m(\tau) = \delta_{nm} \ . \tag{17.13}$$

We are now in a position to give a precise meaning to the path integral measure. We just define it to be

$$[dx] = \prod_n (2\pi)^{-1/2} dc_n \ . \tag{17.14}$$

The semi-classical approximation to our path integral is now a product of simple Gaussian integrals, and we directly obtain (17.5) given in terms of the determinant (17.8). Well, it is high time to evaluate this determinant. To do this we consider the differential equation

$$\left(-\partial_\tau^2 + W\right) \psi(\tau) = \lambda \psi(\tau) \ , \tag{17.15}$$

subject to the boundary conditions

$$\psi(-T/2) \ = \ 0 \tag{17.16}$$

$$\partial_\tau \psi(-T/2) \ = \ 1 \ . \tag{17.17}$$

The first requirement is obvious, while the second just gives us a normalization for ψ. For every λ we now get a unique solution $\psi_\lambda(\tau)$. In general, $\psi_\lambda(T/2)$ is not zero. In fact, this only happens when λ equals one of the eigenvalues λ_n given in equation (17.12). Now it follows that for any two functions $W^{(1)}(\tau)$ and $W^{(2)}(\tau)$ we have

$$\frac{\det\left(-\partial_\tau^2 + W^{(1)}(\tau) - \lambda\right)}{\det\left(-\partial_\tau^2 + W^{(2)}(\tau) - \lambda\right)} = \frac{\psi_\lambda^{(1)}(T/2)}{\psi_\lambda^{(2)}(T/2)} \ . \tag{17.18}$$

The proof of this important result goes as follows: The left hand side is a meromorphic function[1] of λ. It has simple zeros at $\lambda_n^{(1)}$ and simple poles at $\lambda_n^{(2)}$. The same is true for the right hand side. It now follows that the ratio of the two sides of this equation is an *analytic* function. For λ going to infinity in any direction (except along the positive real axis) this analytic function goes to 1. There is only one such function, and it is just 1. This completes the proof of the above relation. As a consequence of this we see that

$$\frac{\det\left(\partial_\tau^2 + W(\tau)\right)}{\psi_0(T/2)} = \text{const} \ , \tag{17.19}$$

[1] A complex function is meromorphic if it only has simple poles and zeros.

i.e. it is independent of $W(\tau)$. Now all that is left is to evaluate this constant by working with the simplest $W(\tau)$ — that of the harmonic oscillator. Now $V = \frac{1}{2}\omega^2 x^2$ and we are dealing with a potential like the one in Figure 17.1. As we have seen, the only solution is the vacuum $x_{\text{class}}(\tau) = 0$. Thus

$$W(\tau) = V''(0) = \omega^2 \ . \tag{17.20}$$

For the harmonic oscillator ψ_0 thus satisfies

$$\begin{aligned} \left(-\partial_\tau^2 + \omega^2\right)\psi_0(\tau) &= 0 \qquad (17.21)\\ \psi_0(-T/2) &= 0 \qquad (17.22)\\ \partial_\tau \psi_0(-T/2) &= 1 \ . \qquad (17.23) \end{aligned}$$

This is easily solved and we find

$$\psi_0(\tau) = \frac{1}{\omega}\sinh\omega(\tau + \frac{T}{2}) \ . \tag{17.24}$$

Therefore, using (17.2), (17.5) and (17.19) we get

$$\langle x_f = 0 | e^{-T\hat{H}} | x_i = 0 \rangle = N\left(\frac{\text{const}}{\omega}\sinh\omega T\right)^{-1/2} \ . \tag{17.25}$$

Next we evaluate both sides for $T \to \infty$. The left hand side gives

$$\langle x_f = 0 | e^{-T\hat{H}} | x_i = 0 \rangle \to e^{-TE_0}\,|\chi_0(0)|^2 \ , \tag{17.26}$$

where $\chi_0(x)$ is the ground state wave function in the coordinate representation. On the other hand, the right hand side becomes

$$N\left(\frac{\text{const}}{\omega}\sinh\omega T\right)^{-1/2} \to N\,e^{-\frac{1}{2}\omega T}\left(\frac{2\omega}{\text{const}}\right)^{1/2} \ . \tag{17.27}$$

Comparing the last two equations we get the well known result for the ground state energy of the harmonic oscillator

$$E_0 = \frac{1}{2}\omega \ . \tag{17.28}$$

We also find

$$|\chi_0(0)|^2 = N\sqrt{\frac{2\omega}{\text{const}}} \ . \tag{17.29}$$

A simple quantum mechanics calculation for the harmonic oscillator gives us

$$|\chi_0(0)|^2 = \sqrt{\frac{\omega}{\pi}} \ , \tag{17.30}$$

and thus const $= 2\pi N^2$. Our final result for a generic determinant is thus

$$\det\left(-\partial_\tau^2 + W(\tau)\right) = 2\pi N^2\,\psi_0(T/2) \ . \tag{17.31}$$

As expected, the right hand side of (17.5) no longer depends on N.

EXERCISES

17.1 Derive (17.30) in the usual quantum mechanics formalism.

17.2 Calculate the determinant for the harmonic oscillator from first principles, *i.e.* as a product of eigenvalues.

Lecture 18

Instantons

18.1 Double Well Potential

We continue our investigation of instantons. The static energy of a $d+1$ dimensional Minkowski theory is equal to the Euclidian action of the same theory in d dimensions[1]. Static solitons of the $d+1$ model are extremums of the static energy for which $E_{\text{stat}} < \infty$. It follows that these static solitons are in fact the instantons of the d dimensional model, *i.e.* extremums of the Euclidian action satisfying $\bar{I} < \infty$.

In our previous lectures we have constructed static solitons for scalar theories in $d = 2$ dimensions. Now we see that these are just our instantons in $d = 1$. To be concrete let us look at the model with quartic potential

$$V(x) = -\frac{1}{2}\mu^2 x^2 + \frac{\lambda}{4!}x^4 + \text{const} . \tag{18.1}$$

The constant is chosen so that $V(x) \geq 0$. We may write this potential as

$$V(x) = \frac{\lambda}{4!}(x^2 - a^2) , \tag{18.2}$$

where $a = \sqrt{\frac{6\mu^2}{\lambda}}$. As we have seen the instanton equals

$$x_{\text{inst}}(\tau) = \sqrt{\frac{6\mu^2}{\lambda}} \tanh\left(\frac{\mu}{\sqrt{2}}\tau\right) . \tag{18.3}$$

The above instanton corresponds to a "particle" rolling from $-a$ to a in the inverted potential $-U$ The anti-instanton is the same solution with opposite sign corresponding to a roll from a to $-a$. (18.3) is the instanton centered at $\tau = 0$. $x_{\text{inst}}(\tau - \tau_0)$ is just as good a solution and if we plot $x_{\text{inst}}(\tau - \tau_0)$ and the Lagrangian evaluated at $x_{\text{inst}}(\tau - \tau_0)$ we get The name instanton is chosen to indicate the

[1]This is true for all models in which the Lagrangian is of the form $T - V$, where T depends only on the time derivatives of the fields and V only on the fields and their spatial derivatives.

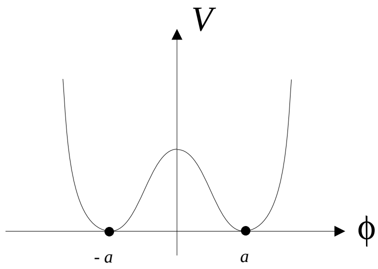


Figure 18.1: Double well potential.

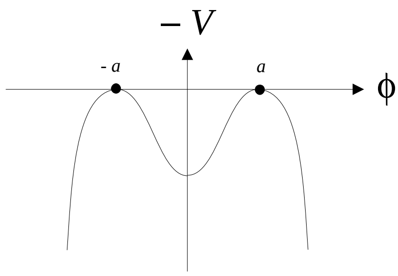


Figure 18.2: "Particle" moves in inverse double well potential.

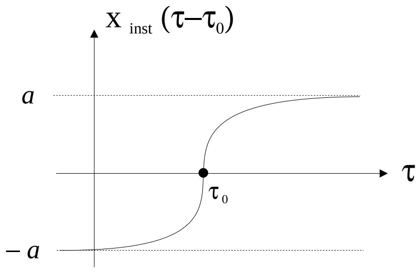


Figure 18.3: Instanton centered at τ_0.

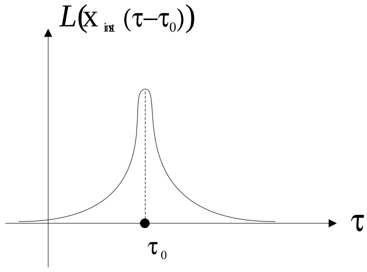
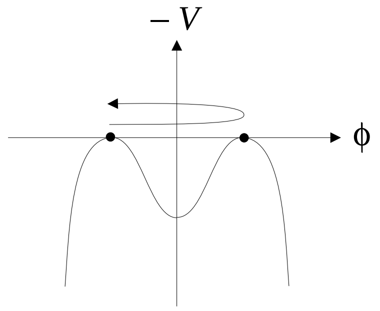


Figure 18.4: Lagrangian evaluated at the instanton solution.

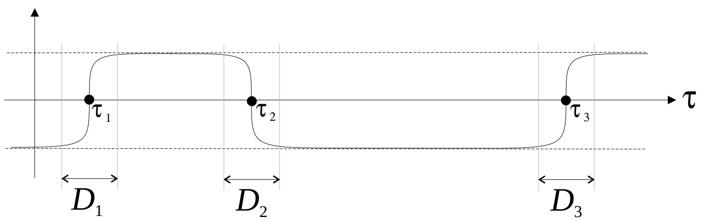
fact that this solution is very localized at one specific instant of Euclidian time τ. As we have seen, (18.3) is an asymptotic solution — it is exact only for $T \to \infty$. We have argued previously why asymptotic solutions are just as important to us as exact solutions. Well, if we allow solutions like (18.3) then we must also allow a host of other asymptotic solutions corresponding to multiple rolls. For instance we have the following solution



belonging to the $-a \to -a$ sector. Instantons are well localized in time so the above approximate solution is simply

$$x_2 \approx x_{\text{inst}}(\tau - \tau_1) - x_{\text{inst}}(\tau - \tau_2) \ . \tag{18.4}$$

This is again an asymptotic solution if $\tau_1 - \tau_2$ is large. Repeating this reasoning we see the need for including all multi-instanton contributions. One such approximate solution in the $-a \to a$ sector is



The regions D_a are only ones in which the multi-instanton deviates appreciably from the vacuum. Let us now go to the path integral. We found that the transition amplitude

$$\langle x_f | e^{-\frac{1}{\hbar} T \hat{H}} | x_{\text{inst}} \rangle \tag{18.5}$$

is semi-clasically just a sum of contributions

$$e^{-\frac{1}{\hbar}\bar{I}[x_{\text{class}}]} \int [d\eta] \, \exp\left(-\frac{1}{2\hbar} \int d\tau \, \eta \left(\partial_\tau{}^2 + V''(x_{\text{class}}) \right) \eta \right) \ . \tag{18.6}$$

Let us look at the asymptotic solution consisting of a dilute gas of n instantons and anti-instantons

$$x_{\tau_1,\ldots,\tau_n}(\tau) \approx x_{\text{inst}}(\tau - \tau_1) \pm x_{\text{inst}}(\tau - \tau_2) + \ldots \tag{18.7}$$

Its contribution to (18.6) is

$$e^{-\frac{1}{\hbar} n \bar{I}_I} \int \prod_\tau d\eta(\tau) \, \exp\left(-\frac{1}{2\hbar} \int d\tau \, \eta \left(-\partial_\tau^2 + V''(x_{\tau_1,\ldots,\tau_n}) \right) \eta \right) \approx$$

$$\approx Z_0\, e^{-\frac{1}{\hbar}n\bar{I}_I} \prod_{a=1}^{n} \int \prod_{\tau\epsilon D_a} d\eta(\tau)\, \exp\left(-\frac{1}{2\hbar}\int_{D_a} d\tau\, \eta\left(-\partial_\tau{}^2 + V''(x_{\tau_a})\right)\eta\right) \tag{18.8}$$

$$\approx Z_0\, e^{-\frac{1}{\hbar}n\bar{I}_I} \prod_{a=1}^{n} K\ , \tag{18.9}$$

where we have introduce

$$K = \frac{\int [d\eta]\, \exp\left(-\frac{1}{2\hbar}\int d\tau\, \eta\left(-\partial_\tau^2 + V''(x_{\text{inst}})\right)\eta\right)}{\int [d\eta]\, \exp\left(-\frac{1}{2\hbar}\int d\tau\, \eta\left(-\partial_\tau^2 + V''(-a)\right)\eta\right)}\ . \tag{18.10}$$

Obviously K and $\bar{I}_I$ do not depend on the location of the instanton, hence the contribution of (18.7) is

$$Z_0\left(e^{-\frac{1}{\hbar}\bar{I}_I}K\right)^n\ . \tag{18.11}$$

The complete n-instanton contribution is thus

$$\begin{aligned} Z_n &= \int_{-T/2}^{T/2} d\tau_1 \int_{\tau_1}^{T/2} d\tau_2 \cdots \int_{\tau_{n-1}}^{T/2} d\tau_n\, Z_0\left(e^{-\frac{1}{\hbar}\bar{I}_I}K\right)^n = \\ &= Z_0\left(e^{-\frac{1}{\hbar}\bar{I}_I}K\right)^n \int_{-T/2}^{T/2} d\tau_1 \int_{\tau_1}^{T/2} d\tau_2 \cdots \int_{\tau_{n-1}}^{T/2} d\tau_n = \\ &= \frac{1}{n!} Z_0\left(e^{-\frac{1}{\hbar}\bar{I}_I}KT\right)^n\ . \end{aligned} \tag{18.12}$$

In our previous lecture we already evaluated the vacuum, or zero instanton, contribution Z_0. This was just the result for the harmonic oscillator potential with $\omega^2 = V''(-a) = 2\mu^2$. We had

$$\psi_0(T/2) = \frac{1}{\omega}\sinh\omega T\ , \tag{18.13}$$

so that for large T we have

$$Z_0 = (2\pi\hbar 1\omega\sinh\omega T)^{-1/2} \approx \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{1}{2}\omega T}\ . \tag{18.14}$$

Obviously we now have

$$\begin{aligned} \langle -a|e^{-\frac{1}{\hbar}T\hat{H}}|-a\rangle &= Z_0 + Z_2 + Z_4 + \ldots = \\ &= Z_0\cosh\left(KTe^{-\frac{1}{\hbar}\bar{I}_I}\right)\ . \end{aligned} \tag{18.15}$$

Similarly,

$$\begin{aligned} \langle a|e^{-\frac{1}{\hbar}T\hat{H}}|-a\rangle &= Z_1 + Z_3 + Z_5 + \ldots = \\ &= Z_0\sinh\left(KTe^{-\frac{1}{\hbar}\bar{I}_I}\right)\ . \end{aligned} \tag{18.16}$$

Combining the last three expressions we get

$$\begin{aligned}
\langle \pm a| e^{-\frac{1}{\hbar}T\hat{H}}|-a\rangle &= \\
&= \frac{1}{2}\left(\frac{\omega}{\pi\hbar}\right)^{1/2} \exp -\frac{1}{\hbar}T\left(\frac{1}{2}\hbar\omega + \hbar K e^{-\frac{1}{\hbar}\bar{I}_I}\right) \mp \\
&\mp \frac{1}{2}\left(\frac{\omega}{\pi\hbar}\right)^{1/2} \exp -\frac{1}{\hbar}T\left(\frac{1}{2}\hbar\omega - \hbar K e^{-\frac{1}{\hbar}\bar{I}_I}\right) \ . \qquad (18.17)
\end{aligned}$$

Hence, te degenerate state of energy $\frac{1}{2}\hbar\omega$ splits into two states with energies

$$E_\pm = \frac{1}{2}\hbar\omega \pm \hbar K e^{-\frac{1}{\hbar}\bar{I}_I} \ . \qquad (18.18)$$

We also have

$$\langle \pm a|+\rangle\langle +|-a\rangle = \frac{1}{2}\left(\frac{\omega}{\pi\hbar}\right)^{1/2} \qquad (18.19)$$

$$\langle \pm a|-\rangle\langle -|-a\rangle = \mp\frac{1}{2}\left(\frac{\omega}{\pi\hbar}\right)^{1/2} \ , \qquad (18.20)$$

from which we see that the true vacuum $|-\rangle$ is given by the antisymmetric combination of the two single well eigenstates, while the state $|+\rangle$ corresponds to the symmetric combination and has a slightly higher energy. This is depicted in Figure 18.5. From the figure we see that $|-\rangle$ obviously has the lower energy since it "spends less time" under the potential well. These results are exactly what one gets from standard quantum mechanics when one takes tunneling into consideration.

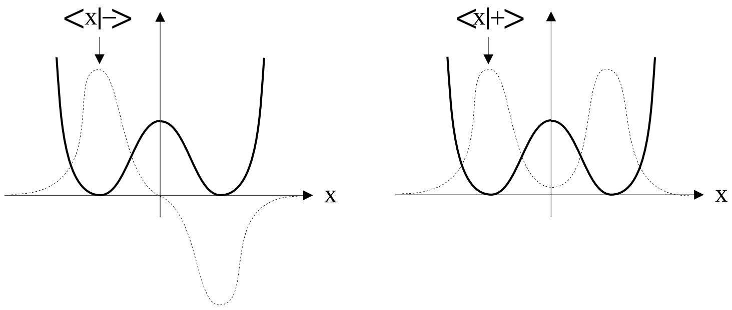


Figure 18.5: The double well and its two lowest eigenstates

We still have to calculate $\bar{I}_I$ and K. The first is no problem, the second needs a bit more work. Remember, we had a Schrödinger problem

$$\left(-\partial_\tau^2 + V''(x_{\text{inst}})\right) x_n(\tau) = \lambda_n x_n(\tau) \qquad (18.21)$$

$$\int_{-T/2}^{T/2} d\tau\, x_n(\tau) x_m(\tau) = \delta_{nm} \ . \qquad (18.22)$$

We can immediately see that we have

$$x_0(\tau) = \frac{1}{\sqrt{\bar{I}_I}}\,\dot{x}_{\text{inst}} \tag{18.23}$$

for the eigenstate with $\lambda_0 = 0$. Since $x_0(\tau)$ has no nodes it follows that it is the ground state, hence all the other λ_n's are positive.

We previously defined our measure to be

$$[dx] = \prod_n (2\pi\hbar)^{-1/2} dc_n = \frac{1}{\sqrt{2\pi\hbar}}\, dc_0 \prod_{n>0} (2\pi\hbar)^{-1/2} dc_n \ . \tag{18.24}$$

Our semi-classical approximation of the path integral thus realy gives

$$\begin{aligned} Ne^{-\frac{1}{\hbar}\bar{I}_I} \int \frac{1}{\sqrt{2\pi\hbar}}\, dc_0 \int \prod_{n>0} (2\pi\hbar)^{-1/2} \exp\left(-\frac{1}{2\hbar}\sum_{n>0} \lambda_n {c_n}^2\right) = \\ = e^{-\frac{1}{\hbar}\bar{I}_I} \int \frac{dc_0}{\sqrt{2\pi\hbar}}\, \det{}' \left(-\partial_\tau^2 + V''(x_{\text{inst}})\right)^{-1/2} . \end{aligned} \tag{18.25}$$

As before, the prime on the determinant implies that we take a product over all the eigenvalues excluding the zero mode. The deformation of x in the c_0 direction is just a translation. We had

$$x = x_{\text{inst}} + \sum c_n x_n \ , \tag{18.26}$$

hence if $c_0 \to c_0 + dc_0$ we get

$$dx = dc_0\, x_0 = \frac{1}{\sqrt{\bar{I}_I}}\, dc_0\, \frac{dx_{\text{inst}}}{d\tau} \ . \tag{18.27}$$

On the other hand, a translation of the center of an instanton gives

$$dx = d\tau_0\, \frac{dx_{\text{inst}}}{d\tau_0} \ , \tag{18.28}$$

hence we see that

$$\frac{dc_0}{\sqrt{2\pi\hbar}} = \sqrt{\frac{\bar{I}_I}{2\pi\hbar}}\, d\tau_0 \ . \tag{18.29}$$

We have *already* integrated over τ_0, so finally we find

$$K = \sqrt{\frac{\bar{I}_I}{2\pi\hbar}}\, \det{}' \left(-\partial_\tau^2 + V''(x_{\text{inst}})\right)^{-1/2} \ . \tag{18.30}$$

We shall not calculate $\det'$, rather let us show *a posteriori* that our dilute gas approximation makes sense. In summing the contributions of multi-instantons we used the Taylor expansion formula for the exponential function

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \ . \tag{18.31}$$

Using the leading term in Stirling's asymtotic formula we have $n! \sim n^n$. It now follows that only the terms satisfying $n \lesssim t$ contribute to the above sum. In our instanton formulas we had $t = KTe^{-\frac{1}{\hbar}\bar{I}_I}$. In terms of instanton density n/T we get

$$\frac{n}{T} \lesssim Ke^{-\frac{1}{\hbar}\bar{I}_I} \; . \tag{18.32}$$

Therefore, the instanton density is indeed small in the semi-classical limit that we have been investigating.

18.2 Periodic Potential

Let us next consider a periodic potential of unit period. The vacuums are indexed by $N \in \mathbb{Z}$).

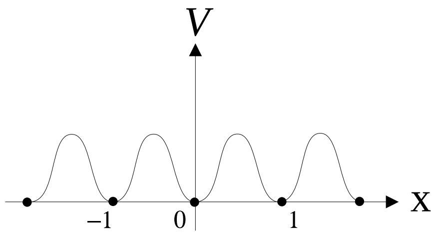


Figure 18.6: Periodic potential

Let us look at the $N \to M$ transition amplitude

$$\begin{aligned} &\langle M|e^{-\frac{1}{\hbar}T\hat{H}} + |N\rangle = \\ &= \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{1}{2}T\omega} \sum_{n=0}^{\infty}\sum_{\bar{n}=0}^{\infty} \frac{1}{n!}\frac{1}{\bar{n}!}\left(KTe^{-\frac{1}{\hbar}\bar{I}_I}\right)^{n+\bar{n}} \delta_{n-\bar{n},M-N} \; . \end{aligned} \tag{18.33}$$

This is even simpler than the double well result. Now there is no restriction that instanton should follow anti-instanton. The only restriction is that we stay in the same sector $M - N$. Using

$$\delta_{n-\bar{n},M-N} = \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{i\theta(n-\bar{n}-M+N)} \; , \tag{18.34}$$

we get

$$\begin{aligned} \langle M|e^{-\frac{1}{\hbar}T\hat{H}}|N\rangle &= \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{1}{2}T\omega} \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{i\theta(N-M)} \\ &\sum_{n=0}^{\infty} \frac{1}{n!}\left(KTe^{-\frac{1}{\hbar}\bar{I}_I+i\theta}\right)^n \sum_{\bar{n}=0}^{\infty} \frac{1}{n!}\left(KTe^{-\frac{1}{\hbar}\bar{I}_I-i\theta}\right)^{\bar{n}} = \\ &= \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{1}{2}T\omega} \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{i\theta(N-M)} \exp\left(KTe^{-\frac{1}{\hbar}\bar{I}_I}e^{i\theta}\right) \exp\left(KTe^{-\frac{1}{\hbar}\bar{I}_I}e^{-i\theta}\right) = \\ &= \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{1}{2}T\omega} \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{i\theta(N-M)} \exp\left(2KTe^{-\frac{1}{\hbar}\bar{I}_I}\cos\theta\right) \; . \end{aligned} \tag{18.35}$$

We have thus found a continuum of energy eigenstates that correspond to the splitting of the infinitely degenerate ground state. The energy spectrum now consists of bands. The lowest energy band is thus

$$E(\theta) = \frac{1}{2}\hbar\omega + 2\hbar K \cos\theta e^{-\frac{1}{\hbar}\bar{I}_I} \ , \tag{18.36}$$

which is just what we get from doing quantum mechanics in a periodic potential.

18.3 Decay of the False Vacuum

As a final, and most interesting application of instantons let us consider a potential of the form shown in Figure 18.7. The point $x = 0$ is now only a local minimum.

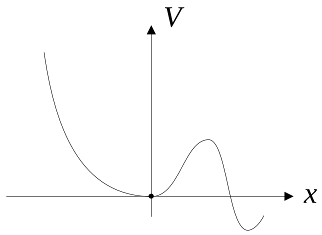


Figure 18.7: Potential with a metastable state.

Assuming that the well is deep enough we will have (neglecting tunneling) a lowest energy state centered at $x = 0$ — this state doesn't "see" the right side of the potential. Well, we know that this can't be right. This state can be very near to an eigenstate, but after enough time it leaks, *i.e.* tunnels, to the true vacuum. Phenomenologicaly we can treat these metastable states in a very simple way. We assume that they are represented by energy eigenstates with complex energy $E - i\gamma$. Then we have

$$\psi(t) = e^{-iEt-\gamma t} \ , \tag{18.37}$$

so that the probability to be in this state decreases with time according to

$$|\psi(t)|^2 = e^{-2\gamma t} \ . \tag{18.38}$$

The metastable state, or false vacuum, thus decays with a half-life $\mathcal{T} = \frac{1}{2\gamma}$. To look at metastable states in our instanton formalism, as always, we need to consider the motion of a particle in the inverse potential $-V$ shown in Figure 18.8. We look at the stability of the false vacuum, *i.e.* the $0 \to 0$ path integral.

The dominant solution is $x = 0$, however, we also have the bounce corresponding to starting at $x = 0$ at $\tau = -T/2$, rolling to the barrier, recoiling and coming back at $\tau = T/2$ to $x = 0$. There is also a solution with two bounces, three bounces, and so on. We would thus expect

$$\langle 0|e^{-\frac{1}{\hbar}T\hat{H}}|0\rangle = Z_0 + Z_1 + Z_2 + \ldots =$$

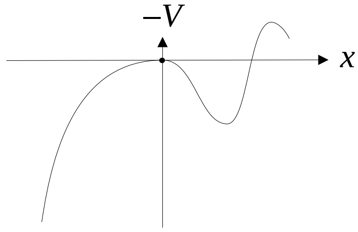


Figure 18.8: "Particle" moves in the inverse potential.

$$
\begin{aligned}
&= Z_0 + Z_0 \left(KTe^{-\frac{1}{\hbar}\bar{I}_B}\right) + \frac{1}{2!} Z_0 \left(KTe^{-\frac{1}{\hbar}\bar{I}_B}\right)^2 + \ldots = \\
&= Z_0 \exp\left(KTe^{-\frac{1}{\hbar}\bar{I}_B}\right) = \\
&= \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega\frac{T}{2}} \exp\left(KTe^{-\frac{1}{\hbar}\bar{I}_B}\right) \ , \qquad (18.39)
\end{aligned}
$$

which gives the energy to be

$$E_0 = \frac{1}{2}\,\hbar\omega + \hbar K e^{-\frac{1}{\hbar}\bar{I}_B} \ . \qquad (18.40)$$

This not quite correct, but we shall see that the important ingredient is present. To do this we look at $K \propto \det{}' \left(-\partial_\tau{}^2 + V''(x_B)\right)^{-1/2}$. Remember that the operator

$$-\partial_\tau{}^2 + V''(x_B) \ . \qquad (18.41)$$

has a zero mode corresponding to the time derivative of the bounce. Unlike the kink, the bounce is not a monotonic function but has a maximum. The bounce solution is shown in Figure 18.9. This implies that the zero mode x_B has a single

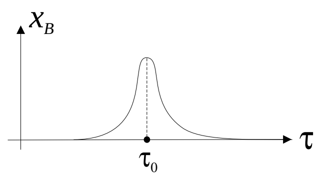


Figure 18.9: The bounce solution

zero — a node. It follows then that it is not the lowest eigenstate. Therefore, without knowing anything further about the potential we have established that (18.41) has precisely one negative eigenstate. As a consequence of this we see that K is pure imaginary. This is precisely what we want. Ofcourse, now we know why (18.39) and (18.41) can't realy be true. To get them we integrated over all the non-zero modes. One of those integrations was of the form

$$\int_{-\infty}^{+\infty} dx\, e^{ax^2} \ , \qquad (18.42)$$

for $a > 0$. There is a way to deal with this kind of integral — by carefully performing an analytic continuation. If we do this then the actual result that one gets is

$$E_0 = \frac{1}{2}\hbar\omega + \frac{1}{2}\hbar K e^{-\frac{1}{\hbar}\bar{I}_B} \ , \tag{18.43}$$

where $K = -i|K|$. Thus $\gamma = \frac{1}{2}\hbar|K|e^{-\frac{1}{\hbar}\bar{I}_B}$, and so

$$\mathcal{T} = \frac{1}{\hbar|K|}e^{-\frac{1}{\hbar}\bar{I}_B} \ . \tag{18.44}$$

This is the correct result, yet the analytical continuation that is needed seems to leave a bitter taste. We shall not do this analytical contination at this moment. We'll return later to this problem and we shall see how one can use the Schwinger-Dyson equations to establish the above result without any tricks.

We end this lecture where we began — with the identity between static solitons in $d+1$ dimensions and instantons in d dimensions. As a consequence of this Derrick's theorem tells us that there are no instantons, and so no tunneling, in scalar field theories in $d > 1$. As we have seen, there are ways to bypass Derrick's theorem, however it is true that most field theories do not poses instanton solutions, *i.e.* do not have tunneling. This is why we can have symmetry breaking.

Lecture 19

Gauge Theories

19.1 Gauge Theories on a Lattice

We look at a theory of fields $\phi(na)$ on an Euclidian lattice of spacing a, invariant under a global symmetry

$$\phi(na) \to G\phi(na) \ , \tag{19.1}$$

where $G^\dagger = G^{-1}$. If we look at the action we see that mass term and self interactions are given by products of fields at the same point, while the kinetic term is of the form

$$\phi^\dagger(ma)\phi(na) \ , \tag{19.2}$$

with $m \neq n$. Because of this the mass term and interactions are invariant under the much larger local symmetry

$$\phi(na) \to G(na)\phi(na) \ , \tag{19.3}$$

Local symmetries are also called gauge symmetries. Note, however, that the kinetic term is *not* invariant under (19.3) since $G^\dagger(ma)G(na) \neq 1$ when $m \neq n$. We will now modify the action so as to make it gauge invariant. In the kinetic term we just substitute

$$\phi^\dagger(ma)\phi(na) \to \phi^\dagger(ma)U(ma, na)\phi(na) \ , \tag{19.4}$$

where we have introduced a unitary operator $U(ma, na)$ which transports $\phi(na)$ to $U(ma, na)\phi(na)$ that transforms like $\phi(ma)$. This is just the definition of $U(ma, na)$, hence under gauge transformations (19.3) we have

$$U(ma, na) \to G(ma)U(ma, na)G^\dagger(na) \ . \tag{19.5}$$

Obviously the general U is generated by transports along the links $U(na+e_\mu a, na)$.

Let us now introduce some basic lattice terminology. Lattice points na are called vertices. The links are vectors connecting a vertex with its nearest neighbors (along the positive direction). This is illustrated on the example of a hyper-cubic

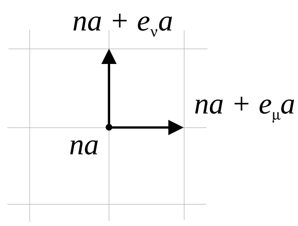


Figure 19.1: The vertex na and links to two neighboring vertices.

lattice in Figure 19.1. For transports of a unit step in the negative direction just note that we have

$$U(na - e_\mu a, na) = U^\dagger(na, na - e_\mu a)\ . \tag{19.6}$$

It is easy to see that a general U depends not only on the end points, but also on the path γ connecting them. Two inequivalent U's corresponding to the same end points are shown in Figure 19.2 What is common to U_{γ_1} and U_{γ_2} is that they

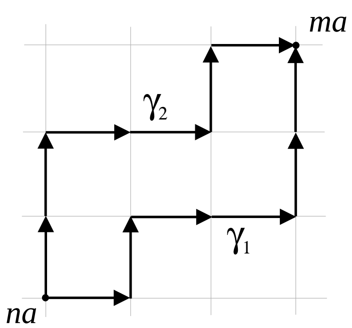


Figure 19.2: Two inequivalent paths from na to ma.

transform the same way. Of special interest are transports along closed loops. We will denote these by λ. Transports along closed curves transform according to

$$U_\lambda \to G(na) U_\lambda G^\dagger(na)\ . \tag{19.7}$$

Because of this it follows that for each closed loop λ we may define a Wilson loop variable

$$W_\lambda = \text{ tr } U_\lambda\ , \tag{19.8}$$

that is gauge invariant. The smallest loops on a lattice are called plaquettes and will be denoted by P. A typical plaquette is shown in Figure 19.3 In terms of transports along the links the associated loop variable is

$$\begin{aligned} W_P = \text{ tr } \Big(& U^\dagger(na + e_\nu a, na)\, U^\dagger(na + e_\nu a + e_\mu a, na + e_\nu a) \cdot \\ & \cdot U(na + e_\nu a + e_\mu a, na + e_\mu a)\, U(na + e_\mu a, na) \Big)\ . \end{aligned} \tag{19.9}$$

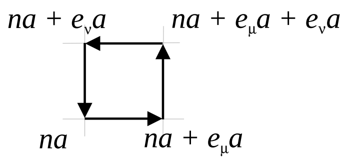


Figure 19.3: A plaquette represents the smallest loop on the lattice.

We can now look at two types of theories. For one U is just given, *i.e.* all the link transports are known. This corresponds to ϕ coupled to an external gauge field. In the other type of theory we make the gauge degree of freedom dynamical as well. In this case we need to construct an action purely for the gauge field. This action must be gauge invariant, real, and it must vanish when we turn off the gauge field (*i.e.* when we set $U = 1$). Our final requirement will be the lattice version of our quest for dynamics in terms of local field theories. On a lattice this implies that the action should consist of a sum of nearest neighbor terms.

Wilson has proposed the following action for $SU(N)$ gauge theory

$$I = -\frac{N}{g^2}\sum_P (1 - \frac{1}{N}\,\mathrm{tr}\,U_P)\ . \tag{19.10}$$

For the $U(1)$ gauge group it is customary to normalize the action differently. We then have

$$I = -\frac{1}{2e^2}\sum_P (1 - U_P)\ . \tag{19.11}$$

These choice are obviously gauge invariant and given in terms of nearest neighbor coupling since they are given as functions of the Wilson loop variables over plaquettes. Obviously if $U_P = 1$ then $I = 0$. The last thing to check is reality. Note that for each plaquette P there is another one $-P$ which traverses the same points in the opposite order. This is shown in Figure 19.4. Obviously we have

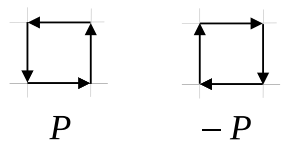


Figure 19.4: The plaquettes P and $-P$.

$U_{-P} = U_P^\dagger$. Because of this the above expressions for Wilson's action are indeed real. Normalizations are chosen to agree with the standard conventions in the continuum limit. For this reason the $U(1)$ coupling constant is written as e and not g — as we shall see $U(1)$ gauge theory is just electrodynamics.

19.2 The Continuum Limit

Let us now look at the continuum limit of our expression. We write

$$U(na+e_\mu a, na) = \exp\left(igaA_\mu^b(na)t_b\right) \ , \tag{19.12}$$

where the generators t_a satisfy the Lie algebra corresponding to our gauge group. Therefore

$$[t_a, t_b] = if_{abc}t_c \tag{19.13}$$
$$t_a^\dagger = t_a \ . \tag{19.14}$$

For non-Abelian, compact and semi-simple groups we can normalize the generators according to $\text{tr}\ (t_a t_b) = \frac{1}{2}\delta_{ab}$. For example for $SU(2)$ we have $t_a = \frac{1}{2}\sigma_a$. A simple calculation for $a \to 0$ gives us

$$W_P \to \text{tr}\ \exp\left(iga^2 F_{\mu\nu}^b t_b\right) \ , \tag{19.15}$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf_{abc}A_\mu^b A_\nu^c \ . \tag{19.16}$$

From now on we cut down on indices by introducing so-called matrix valued fields

$$A_\mu = A_\mu{}^a t_a \tag{19.17}$$
$$F_{\mu\nu} = F_{\mu\nu}{}^a t_a \ . \tag{19.18}$$

In the continuum theory (19.12) may be written as

$$U(x+dx, x) = \exp\left(igA_\mu dx^\mu\right) \ . \tag{19.19}$$

Because of this one often writes the transport along γ as

$$U_\gamma = P\exp\left(ig\int_\gamma A_\mu dx^\mu\right) \ . \tag{19.20}$$

We should stress that this is a formal expression. The path ordering P just means that we are to multiply the contributions given by (19.19) in the order from the start to the end of γ. Only in the case of the $U(1)$ group (QED) do we have commuting of the A's and hence

$$U_\gamma = \exp\left(ie\int_\gamma A_\mu dx^\mu\right) \ . \tag{19.21}$$

By Stokes' theorem [1] we find

$$U_\lambda = \exp\left(ie\oint_\lambda A_\mu dx^\mu\right) = \exp\left(ie\int_S F_{\mu\nu}dx^\mu dx^v\right) \ , \tag{19.22}$$

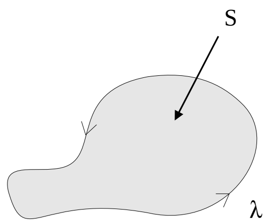


Figure 19.5: The closed path λ and a hyper-surface S that it bounds.

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. For small contours λ this is precisely what we got in (19.15). We now calculate the continuum limit of Wilson's action. First look at QED. We have

$$U_P = \exp(iea^2 F_{\mu v}) \approx 1 - \frac{1}{2}\, e^2 a^4 (F_{\mu\nu})^2 \ , \tag{19.23}$$

where we have disregarded the imaginary term since it will cancel in the action. Therefore

$$I = -\frac{1}{2e^2} \sum_P (1 - U_P) = -\frac{1}{4} \sum_P a^4 (F_{\mu\nu})^2 = -\frac{1}{4} \int dx\, F_{\mu\nu} F_{\mu\nu} \ , \tag{19.24}$$

which is the correct action for QED. Similarly, for $SU(N)$ gauge theory, we have

$$\text{tr}\ U_P = \ \text{tr}\ \exp(iga^2 F_{\mu\nu}) \approx N - \frac{1}{2}\, g^2 a^4 F_{\mu\nu}{}^a F_{\mu\nu}{}^b \ \text{tr}\ (t_a t_b) \ . \tag{19.25}$$

Using the fact that $\text{tr}\ (t_a t_b) = \frac{1}{2}\, \delta_{ab}$, we find the continuum limit of the Yang-Mills action to be

$$I = -\frac{N}{g^2} \sum_P \left(1 - \frac{1}{N}\, \text{tr}\ U_P\right) = -\frac{1}{4} \sum_P a^4 (F^a_{\mu\nu})^2 = -\frac{1}{4} \int dx\, F^a_{\mu\nu} F^a_{\mu\nu} \ . \tag{19.26}$$

Before we leave the lattice and concentrate on the continuum field theories given by (19.24) and (19.26) let us just note that Wilson's action is not the unique lattice action that has this continuum limit. It is however the simplest.

Wick rotating (19.26) and (19.16) we get the corresponding Minkowski expressions

$$I = -\frac{1}{4} \int dx\, F^a_{\mu\nu} F^a_{\mu\nu} = -\frac{1}{2} \int dx\ \text{tr}\ (F_{\mu\nu} F^{\mu\nu}) \ . \tag{19.27}$$

and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu \ , \tag{19.28}$$

or more compactly

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \ . \tag{19.29}$$

[1] This is the generalization to d dimensions of the well known integral theorem derived by Stokes: $\oint_C \vec{A} \cdot \vec{dl} = \int_S \text{curl}\vec{A} \cdot \vec{dS}$.

Yang-Mills theories are often written in terms of anti-Hermitian fields. This corresponds to using anti-Hermitian generators for our Lie algebra. The new generators satisfy

$$[T_a, T_b] = f_{abc} T_c \qquad (19.30)$$

$$T_a{}^\dagger = -T_a \ , \qquad (19.31)$$

and are related to the Hermitian generators according to $T_a = -it_a$. The new fields are

$$\hat{A}_\mu = g A_\mu^a T_a = -ig A_\mu^a t_a = -ig A_\mu \ , \qquad (19.32)$$

and similarly for all other matrix valued fields. In terms of the new fields we have

$$I = \frac{1}{2g^2} \int dx \ \mathrm{tr} \ (F_{\mu\nu} F^{\mu\nu}) \qquad (19.33)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \ , \qquad (19.34)$$

where we have dropped the hat over the fields. These expressions are somewhat nicer because the coupling constant is now an overall constant multiplying the action. We have also gotten rid of a factor of i in the expression for the field strength $F_{\mu\nu}$. We will work with Yang-Mills theories in the following lectures. At this point let us go back to the simpler and more familiar case of electrodynamics — Abelian gauge theory.

19.3 Electrodynamics

As we have seen

$$I = -\frac{1}{4} \int dx \, F_{\mu\nu} F^{\mu\nu} \qquad (19.35)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \qquad (19.36)$$

The coupling constant e falls out of the picture. This is in fact a free theory. Using the above expression for the field strength $F_{\mu\nu}$ the action written in terms of A becomes

$$I = \int dx \, \frac{1}{2} \, A^\mu (g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu) A^\nu \ . \qquad (19.37)$$

This action is invariant under the gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \omega \ , \qquad (19.38)$$

Hence, all physical quantities, *i.e.* those that can in principle be measured, will also be invariant under (19.38). Figure 19.6 denotes two inequivalent gauge fields and the sets of all gauge variations called gauge orbits We lose nothing from the predictive power of our theory if we fix the gauge. We do this by imposing an extra condition χ that the fields must satisfy. This condition must be such that

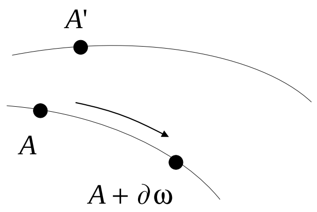


Figure 19.6: Two inequivalent gauge fields and their orbits.

it picks out one representative along each gauge orbit. One of the simplest gauge condition is the Lorentz gauge

$$\chi \equiv \partial_{\mu} A^{\mu} = 0 \ , \tag{19.39}$$

which has the nice property that it doesn't break manifest covariance. How do we know that this is a good gauge condition? To see this suppose A^{μ} is a field that doesen't satisfy the gauge condition. If the above is a good gauge fixing then there exists one and only one field $A^{\mu} = A^{\mu} + \partial^{\mu}\omega$ that satisfies the gauge condition. Thus

$$-\partial_{\mu} A^{\mu} = \partial^{2}\omega \ . \tag{19.40}$$

The equation $\partial^{2}\omega = f$ can always be solved (consistent with the appropriate boundary conditions), so the Lorentz condition is indeed a good gauge. The Lorentz condition doesen't completely fix the gauge, since if $\partial_{\mu} A^{\mu} = 0$ then also $\partial_{\mu} A'^{\mu} = 0$ if we have $\partial^{2}\omega = 0$. This residual symmetry will not bother us however.

In the Lorentz gauge we have

$$I = -\int dx \, \frac{1}{2} \, \partial_{\mu} A_{\nu} \partial^{\mu} A^{\nu} \ . \tag{19.41}$$

The sign of the action is such that the space components A^{i} have the same sign as the scalar field. Let us now make contact with ordinary QED in $d = 4$ written in a more familiar fashion. We have

$$A^{\mu} = (\phi, \vec{A}) \tag{19.42}$$
$$\partial_{\mu} = (\partial_{0}, \bigtriangledown) \ , \tag{19.43}$$

so that

$$F_{0i} = \left(-\frac{\partial}{\partial t} \vec{A} - \bigtriangledown \phi \right)_{i} = E_{i} \ . \tag{19.44}$$

We also have

$$F_{12} = \partial_{1} A_{2} - \partial_{2} A_{1} = -(\mathrm{curl} \vec{A})_{3} = -B_{3} \ . \tag{19.45}$$

In fact, the space-space components of the field strength are related to the magnetic field according to

$$F_{ij} = -\varepsilon_{ijk} B_{k} \ . \tag{19.46}$$

The field strenght is thus given in terms of electric and magnetic fields as

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix} . \tag{19.47}$$

We now define the dual tensor to be

$$^*F^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} , \tag{19.48}$$

so that

$$^*F^{01} = \frac{1}{2}(F_{23} - F_{32}) = -B_1 \tag{19.49}$$

$$^*F^{12} = -\frac{1}{2}(F_{30} - F_{03}) = +E_3 , \tag{19.50}$$

etc. Thus

$$^*F^{\mu\nu} = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & E_1 \\ B_3 & E_2 & -E_1 & 0 \end{pmatrix} , \tag{19.51}$$

Which gives us

$$^*F_{\mu\nu} = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & E_3 & -E_2 \\ -B_2 & -E_3 & 0 & E_1 \\ -B_3 & E_2 & -E_1 & 0 \end{pmatrix} . \tag{19.52}$$

Note that

$$\partial_\mu {}^*F^{\mu\nu} = 0 \tag{19.53}$$

is a direct consequence of the definition of the field strength in terms of gauge fields. On the other hand, the equation of motion following from QED action is

$$\partial_\mu F^{\mu\nu} = 0 . \tag{19.54}$$

These last two equations are the (source free) Maxwell equations. Although similar looking, as we have seen, they carry totaly different information.

Equations (19.53) and (19.54) give us a further symmetry of sourceless QED in $d = 4$ dimensions

$$F_{\mu\nu} \to F_{\mu\nu}\cos\theta + {}^*F_{\mu\nu}\sin\theta . \tag{19.55}$$

In terms of the electric and magnetic fields this gives

$$\begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} \to \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} , \tag{19.56}$$

Therefore this symmetry exchanges the roles of the electric and magnetic fields.

The Maxwell equations with sources are

$$\begin{aligned} \mathrm{div}\vec{B} &= 0 && (19.57) \\ \mathrm{curl}\vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 && (19.58) \\ \mathrm{div}\vec{E} &= \rho && (19.59) \\ \mathrm{curl}\vec{B} - \frac{\partial \vec{E}}{\partial t} &= \vec{J} \, . && (19.60) \end{aligned}$$

The first two are as in the vacuum case, the last two have in addition a source term. Writing $J^\mu = (\rho, \vec{J})$, we cast the above equations in a covariant form. We find

$$\begin{aligned} \partial_\mu {}^*F^{\mu\nu} &= 0 && (19.61) \\ \partial_\mu F^{\mu\nu} &= J^\nu \, . && (19.62) \end{aligned}$$

As we immediately see, the second equation gives the consistency condition

$$\partial_\mu J^\mu = 0 \, . \tag{19.63}$$

This is just the equation of continuity corresponding to the conservation of charge. In the old notation this is just $\frac{\partial \rho}{\partial t} + \mathrm{div}\vec{J} = 0$. The action in external field that gives the above equations is

$$I = \int dx \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu \right) \, . \tag{19.64}$$

As we see, the addition of the source term has distinguished between the $\vec{E}$ and $\vec{B}$ fields. The equations of motion the duality transformation (19.55). At the same time the Lorentz force law

$$\vec{F} = e\vec{E} + e\vec{v} \times \vec{B} \, . \tag{19.65}$$

distinguishes between $\vec{E}$ and $\vec{B}$.

EXERCISES

19.1 Why don't we consider expressions such as $\mathrm{tr}\,({U_\lambda}^2)$ in the action? After all they are also gauge invariant.

19.2 Derive equations (19.15) and (19.16) for the continuum limit of W_P.

19.3 Consider if $\chi \equiv A^0 = 0$ is a good gauge fixing condition for QED. Is there a residual symmetry?

19.4 Show that the pure QED action may be written in terms of the dual tensor ${}^*F_{\mu\nu}$.

19.5 Show that (19.48) is a dual trnasformation, *i.e.* that $^{**}F_{\mu\nu} = -F_{\mu\nu}$.

19.6 Calculate the field strength and its dual in terms of electric and magnetic fields in $d = 2$ and $d = 3$.

19.7 Look at the gauge transformation of the QED action I given in equation (19.64). When is I gauge invariant?

19.8 Modify the Lorentz force law as well as the Maxwell equations by adding the possibility of a magnetic charge g (in addition to the electric charge e) in such a way that the theory is now invariant under (19.55).

19.9 Show that the addition of a mass term to the QED action spoils the gauge invariance. Show that the massive vector field has 3 independent components (consistent with the fact that it is a spin 1 field). Show that the massless vector field has 2 independent components (two states of helicity).

Lecture 20

Differential Geometry and Gauge Fields

20.1 Differential Forms

We begin this lecture with an introduction to a differential forms. Forms represent a compact index-free notation that is often quite usefull in physics. Their gratest utility comes in the treatment of gauge fields. Let x^μ be the coordinates on a d dimensional manifold $\mathcal{M}$ (most often this will be our space time, and d will be the dimension of space time). The differentials dx^μ represent a basis of objects called 1-forms. A general 1-form is

$$\Omega_1 = \omega_\mu(x) dx^\mu \ . \tag{20.1}$$

At every point on $\mathcal{M}$ the 1-forms belongs to a d dimensional vector space. An associative and antisymmetric product of forms (the wedge product) is defined as

$$dx^\mu \wedge dx^\nu = - \, dx^\nu \wedge dx^\mu \ . \tag{20.2}$$

The objects $dx^\mu \wedge dx^\nu$ represent a basis of 2-forms, a general 2-form being

$$\Omega_2 = \frac{1}{2!} \, \omega_{\mu\nu}(x) dx^\mu \wedge dx^\nu \ . \tag{20.3}$$

Obviously, at each point of $\mathcal{M}$, 2-forms belong to a $\binom{d}{2}$ dimensional vector space. Similarly, one defines 3-forms, 4-forms, etc. For convenience 1 is the basis of 0-forms, which are therefore just functions. Forms are basis-independent (*i.e.* geometrical) object. This determines how the components $\omega_{\mu\nu\cdots}(x)$ transform. So far we have been dealing with algebra. Now let us introduce the notion of a derivative. The exterior derivative of an n-form Ω_n is defined to be

$$d\,\Omega_n = \partial_\mu \left(\frac{1}{n!} \omega_{\nu_1 \cdots \nu_n}(x) \right) dx^\mu \wedge dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_n} \ . \tag{20.4}$$

As we can see,the exterior derivative d takes n-forms into an $(n+1)$-forms. From the above definition it follows that d is nilpotent, *i.e.*

$$d^2 = 0 \ . \tag{20.5}$$

One can easily show that d satisfies the graded Leibnitz rule for the derivative of a product

$$d(\Omega \wedge \Phi) = d\,\Omega \wedge \Phi + (-)^{\Omega}\Omega \wedge d\,\Phi \ , \tag{20.6}$$

where the grading $(-)^{\Omega}$ associates +1 to even forms and -1 to odd forms. From the definition of the wedge product (20.2) it follows that

$$\Omega \wedge \Phi = (-)^{\Omega\Phi}\Phi \wedge \Omega \ . \tag{20.7}$$

Let us now introduce some relevant termiology. If $d\,\chi = 0$ then χ is called a *closed* form. χ is an *exact* form if there exists a form Φ such that $\chi = d\,\Phi$ is true on the whole manifold. An exact form is always closed, the converse is not true in general. The cohomology of d is the search for all closed forms that are not exact. This slices the space of all forms into disjoint pieces called cohomology classes. All elements of a given cohomology class are equal modulo an exact form. Many important problems in physics can be stated in terms of the cohomology of a certain nilpotent operator. We shall return to this briefly a bit later in this lecture. For now let us continue to add some more pieces to the mathematical structure of forms. Before we go on we will simplify our notation by dropping the symbol $\wedge$ for the wedge product. From the context it is easy to see if something is a form or not, and forms can be multiplied in only one way.

As we have seen, spaces of n-forms and $(d-n)$-forms have the same dimensionality. In fact we can map one into the other by the Hodge duality transformation. This is most simply given by its action on the basis differentials

$$^*dx^\mu = \frac{1}{(d-1)!}\sqrt{|g|}\,\varepsilon^\mu{}_{\nu\rho\cdots\tau}\,dx^\nu\,dx^\rho\cdots dx^\tau \tag{20.8}$$

$$^*dx^\mu\,dx^\nu = \frac{1}{(d-2)!}\sqrt{|g|}\,\varepsilon^{\mu\nu}{}_{\rho\cdots\tau}\,dx^\rho\cdots dx^\tau \ , \tag{20.9}$$

and so on. The picture is completed by writing

$$^*1 = \frac{1}{d!}\sqrt{|g|}\,\varepsilon_{\mu\nu\cdots\tau}dx^\mu\,dx^\nu\cdots dx^\tau = \sqrt{|g|}dx \ . \tag{20.10}$$

The dual of 1 is thus the invariant volume element on $\mathcal{M}$. Remember that while $\varepsilon_{\mu\nu\cdots\tau}$ is not a tensor $\sqrt{|g|}\varepsilon_{\mu\nu\cdots\tau}$ is. As written the above formulas are valid for manifolds of both Euclidian and Minkowski signature. In this course we mostly work in flat spacetime in coordinates for which $\sqrt{|g|} = 1$. We shall asume this in the rest of the lecture. Still it is important to note that the Hodge dual is the first and only place in the mathematics of forms where the metric appears.

This definition of duality is consistent with what we had in the previous lecture. For a 2-form in $d = 4$ we have

$$\begin{aligned} {}^*F &= {}^*\left(\frac{1}{2}\,F_{\mu\nu}dx^\mu\, dx^\nu\right) = \frac{1}{2}\,F_{\mu\nu} * dx^\mu\, dx^\nu = \\ &= \frac{1}{2}\,F_{\mu\nu}\frac{1}{2}\,\varepsilon^{\mu\nu}{}_{\alpha\beta}dx^\alpha\, dx^\beta = \frac{1}{2}\,{}^*F_{\alpha\beta}dx^\alpha\, dx^\beta\ . \end{aligned} \tag{20.11}$$

The dual operation allows us to construct two more important differential operators. The co-differential is defined to be

$$\delta = {}^*d\,{}^* \tag{20.12}$$

It lowers form number by one. Like the exterior derivative d, the co-differential δ is also nilpotent. The Laplacian is defined to be

$$\triangle = d\delta + \delta d\ . \tag{20.13}$$

Obviously it is a second order derivation operatior with the important property that it doesen't change form number. In analogy with closed and exact forms we now have *co-closed* and *co-exact* forms. A form is *harmonic* if its Laplacian is zero.

Integration of forms is a map from d-forms to 0-forms. It is easily constructed once we recognize that the d-form basis element $dx^0\, dx^1 \cdots dx^{d-1}$ changes under a coordinate transformation just like the usual d-dimensional volume element dx. Therefore, we define integration in the following way

$$\begin{aligned} \int \Omega_d &= \int \frac{1}{d!}\,\omega_{\mu_1\cdots\mu d}\, dx^{\mu_1}\cdots dx^{\mu d} = \\ &= \int \omega_{0\cdots(d-1)}\, dx^0\cdots dx^{(d-1)} = \int \omega_{0\cdots(d-1)}\, dx\ . \end{aligned} \tag{20.14}$$

In the last step we just perform an ordinary integration.

The boundary of a d-dimensional manifold $\mathcal{M}$ (denoted $\partial\mathcal{M}$) is itself a manifold and it is $(d-1)$-dimensional. The central result of integral calculus is the Stokes formula

$$\int_{\mathcal{M}} d\,\Omega_{d-1} = \int_{\partial\mathcal{M}} \Omega_{d-1}\ . \tag{20.15}$$

This represents a generalization of the integration theorems of vector calculus. The boundary operation ∂ is nilpotent. This is easy to prove. We cast integration as an inner product

$$\int_{\mathcal{M}} \Omega = \langle\mathcal{M}|\Omega\rangle\ . \tag{20.16}$$

Stokes' theorem then states that

$$\langle\mathcal{M}|d\,\Omega\rangle = \langle\partial\mathcal{M}|\Omega\rangle\ . \tag{20.17}$$

It follows that ∂ is the adjoint of d with respect to the above inner product. The nilpotence of ∂ now follows immediately from the nilpotency of d. A piece of a manifold is called a *chain*. If a chain C has no boundary, *i.e.* $\partial C = 0$ it is called a *cycle*. If a chain satisfies $C = \partial B$ then C is a boundary. To illustrate this we look at Figure 20.1 A and B are two 1-cycles on our manifold $\mathcal{M}$ (a torus). B is

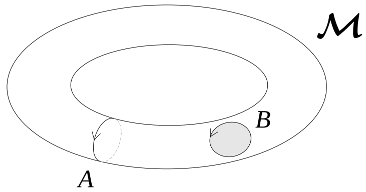


Figure 20.1: Two 1-cycles A and B on the torus.

a boundary of the 2-chain represented by the hatched area. A is not a boundary of any chain on $\mathcal{M}$.

20.2 Gauge Fields as Forms

As we have seen, gauge theories are much simpler if one uses covariant notation. This can be further simplified by going to a form notation. The gauge field is now a 1-form in spacetime

$$A = A_\mu dx^\mu = A_\mu{}^a dx^\mu \, T_a \ . \tag{20.18}$$

From this we get the field strength 2-form

$$F = dA + A^2 \ . \tag{20.19}$$

Note that $A^2 \neq 0$ even though we are using a wedge product. This is because A is matrix valued. Let us show that this agrees with our previous definition in terms of components

$$\begin{aligned} \frac{1}{2} F_{\mu\nu} \, dx^\mu \, dx^\nu &= \partial_\nu A_\mu dx^\nu \, dx^\mu + A_\mu A_\nu dx^\mu \, dx^\nu = \\ &= \frac{1}{2} (\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \, dx^\nu + \frac{1}{2} [A_\mu, A_\nu] dx^\mu \, dx^\nu = \\ &= \frac{1}{2} \left(\partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + f_{abc} \, A_\mu^b A_\nu^c \right) dx^\mu \, dx^\nu \, T_a \ . \end{aligned} \tag{20.20}$$

From the definition (20.19) it follows that the field strength satisfies the Bianchi identity

$$dF + [A, F] = 0 \ . \tag{20.21}$$

The gauge transformation law reads

$$A \to A' = G(A + d)G^{-1} \ . \tag{20.22}$$

If we write

$$V = G^{-1}dG \ , \tag{20.23}$$

then $A' = G(A - V)G^{-1}$, so that the field strength transforms according to

$$\begin{aligned} F \to F' &= dA' + A'^{\,2} = \\ &= d\left(G(A-V)G^{-1}\right) + G(A-V)(A-V)G^{-1} = \\ &= G\left(V(A-V) + dA - dV + (A-V)V + (A-V)^2\right)G^{-1} \ . \end{aligned} \tag{20.24}$$

From the definition of V we have

$$dV = -\,G^{-1}dGG^{-1}dG = -\,V^2 \ . \tag{20.25}$$

Using this we finally find

$$F \to F' = GFG^{-1} \ , \tag{20.26}$$

so, even though the field strength is in general not invariant as was the case in electrodynamics, still it transforms very simply under the gauge variation. The dual of F also transforms in this way

$$^*F \to {}^*F' = {}^*(GFG^{-1}) = G^*FG^{-1} \ . \tag{20.27}$$

For a matrix valued form M we define the action of the covariant derivative as

$$DM = dM + [A, M\} \ . \tag{20.28}$$

The bracket $[\ .\ \}$ just denotes the graded commutator, *i.e.* for even M it is a commutator, while for odd M it is an anti-commutator. Note that the Bianchi identity is just $DF = 0$. It is easy to show that

$$D^2M = [F, M\} \ . \tag{20.29}$$

The covariant derivative is not nilpotent.

If we write the gauge transformation as $G = e^{-\lambda}$, then for infinitesimal transformations one gets

$$\delta A \ = \ D\lambda \tag{20.30}$$

$$\delta F \ = \ [F, \lambda] \ , \tag{20.31}$$

which may be trivially shown.

The only candidates for a Yang-Mills action in $d = 4$ are $\int$ tr (F^2) and $\int$ tr (F^*F), since we must integrate over a gauge invariant 4-form. The two other possibilities $\int$ tr $(^*F^*F)$ and $\int$ tr $(^*FF)$ are the same as the ones above. Note that $\int$ tr F^*F works as an action in *any* number of dimensions. Indeed, this is just our previously derived gauge action

$$I = \frac{1}{2g^2}\int \text{tr}\ (F^*F) \ . \tag{20.32}$$

As we see, in $d = 4$ we have another candidate, and that is $\int$ tr F^2. In $d = 6$ we could take $\int$ tr F^3, and so on . The expressions $\chi_n =$ tr F^n are called Chern characters, and will play an important role later.

EXERCISES

20.1 Show that n-forms form a $\binom{d}{n}$ dimensional vector space at each point of $\mathcal{M}$. As a corrolary we see that there are as many n-forms as $(d-n)$-forms. Also, there are no forms with $n > d$.

20.2 Prove the graded Leibniz rule for exterior derivatives.

20.3 Show that the Hodge operation is truly a duality, *i.e.* that $^{**}\omega_n$ is, up to a possible sign, just ω_n. Try to determine this sign dependence. Note that the sign depends on n and d. It also depends on weather the manifold $\mathcal{M}$ is Euclidian of Minkowskian.

20.4 Show that the co-differential δ lowers form number by one. Prove that it is nilpotent. Also, show how it acts on a product of two forms.

20.5 Write the Maxwell equations in terms of forms in 3 dimensional space treating time separately. Derive the equation of continuty. Give the relation between E and B fields and potentials. Unlike the familiar vector formulas these work on spatial manifolds of any dimension and any metric.

20.6 Take the E 1-form from the previous problem and write out the following Stokes theorems in more familiar vector notation

$$\int_V d^*E = \int_{\partial V} {}^*E$$

$$\int_S dE = \int_{\partial S} E \ .$$

Do the same thing for the B 2-form for

$$\int_V dB = \int_{\partial V} B$$

$$\int_S d^*B = \int_{\partial S} {}^*B \ .$$

In the above V is a volume and S is some surface. Look at a 0-form ω and integrate it over a line

$$\int_L d\,\omega = \int_{\partial L} \omega \ .$$

What does this give when space is one dimensional?

20.7 Using the results of problem (20.5) write down the Maxwell equations in components for the case of a two dimensional space $x^\mu = (\theta, \varphi)$ with metric

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \ .$$

This is S_2, a sphere of unit radius. Spacetime is thus $\mathbb{R} \times S_2$.

20.8 Prove equation (20.29). Show how D acts on a product of matrix valued forms.

20.9 Show that equation (20.32) is indeed our previously derived Yang-Mills action. Show that

$$\int_{\mathcal{M}} \mathrm{tr}\ F^2 = \int_{\mathcal{M}} \mathrm{tr}\ d\left(AdA + \frac{2}{3}\,A^3\right) = \int_{\partial\mathcal{M}} \mathrm{tr}\ \left(AdA + \frac{2}{3}\,A^3\right)\ . \quad (20.33)$$

The expression in parenthesis is called the Chern-Simons 3-form, while tr F^2 is often called the Pontryagin density. As a result of this homework assignment, adding a term like $\int trF^2$ to our action (20.32) doesn't change the dynamics since it is just a surface term. In later lectures we shall see that this additional term *will* have a profound influence on our quantum theory — on its ground state.

20.10 Show that the Chern characters are gauge invariant and closed.

20.11 Prove that the Chern-Simons 3-form is invariant under infinitesimal gauge transformations.

Lecture 21

Euclidian Yang-Mills and Topology

21.1 The Pontryagin Index

In this lecture we will spend a little time looking at some formal aspects of pure Euclidian Yang-Mills theories in various dimensions. We start with $d = 4$. As we have seen, the Yang-Mills action is

$$I = -\frac{1}{2g^2}\int_{E_4} \text{tr}\ (F^* F) = -\frac{1}{2g^2}\int_{E_4} dx\ \text{tr}\ (F_{\mu\nu}F_{\mu\nu})\ . \tag{21.1}$$

In Euclidian space we have

$${}^*F_{\mu\nu}{}^*F_{\mu\nu} = F_{\mu\nu}F_{\mu\nu}\ . \tag{21.2}$$

Note that the Minkowski space version of this formula is ${}^*F_{\mu\nu}{}^*F^{\mu\nu} = -F_{\mu\nu}F^{\mu\nu}$. Let us now look at the following inequality

$$-\int dx\ \text{tr}\ (F_{\mu\nu} \pm {}^*F_{\mu\nu})^2 \geq 0\ . \tag{21.3}$$

The minus sign comes in because we are using anti-Hermitian fields. Expanding the square and using (21.2) we find

$$I \geq \frac{8\pi^2}{g^2}|Q|\ , \tag{21.4}$$

where we have defined the Pontryagin index to be

$$Q = -\frac{1}{8\pi^2}\int_{E_4} \text{tr}\ F^2 = -\frac{1}{16\pi^2}\int_{E_4} dx\ \text{tr}\ (F_{\mu\nu}{}^*F_{\mu\nu})\ . \tag{21.5}$$

We shall soon see that $Q \in \mathbb{Z}$. The Pontryagin index, therefore, classifies gauge fields of finite action. In each sector, *i.e.* for each value of Q, the action is minimized by self dual and anti-self dual fields

$$F_{\mu\nu} = \pm^* F_{\mu\nu}\ . \tag{21.6}$$

This immediately follows from (21.4). Such fields exist in the Euclidian theory by virtue of the relation

$$^{**}F_{\mu\nu} = F_{\mu\nu} \ . \tag{21.7}$$

Thus $** = 1$, and the eigenvalues of $*$ are ± 1.

We shall return to (21.6) when we look at Yang-Mills instantons. At this point let us spend some time on the Pontryagin index. As we have seen in a previous exercise

$$Q = -\frac{1}{8\pi^2}\int_{S_3} \text{tr}\ \left(AdA + \frac{2}{3}A^3\right)\ , \tag{21.8}$$

where $S_3 = \partial E_4$ is the sphere at infinity. Q thus depends on the gauge fields on the boundary S_3. Fields of finite action obviously have $F = 0$ on S_3. Using this we may write the Pontryagin index as

$$Q = \frac{1}{24\pi^2}\int_{S_3} \text{tr}\ A^3\ . \tag{21.9}$$

In fact, the vanishing of the field strength F implies that A is pure gauge, *i.e.* $A = GdG^{-1}$. Differentiating $GG^{-1} = 1$ we find $dGG^{-1} + GdG^{-1} = 0$, so we get

$$Q = -\frac{1}{24\pi^2}\int_{S_3} \text{tr}\ (G^{-1}dG\, G^{-1}dG\, G^{-1}dG) = -\frac{1}{24\pi^2}\int_{S_3} \text{tr}\ V^3\ , \tag{21.10}$$

where we have defined $V = G^{-1}dG$ as in the previous lecture. Obviously we have $Q = Q(G)$ where G is a map from S_3 to the gauge group. We shall use (21.10) to derive some properties of $Q(G)$. It immediately follows that

$$Q(G^{-1}) = -Q(G)\ , \tag{21.11}$$

and hence $Q(I) = 0$. Now we will write $G = G_1G_2^{-1}$. As a consequence we get

$$V = (G_1G_2^{-1})^{-1}d(G_1G_2^{-1}) = G_2(G_1^{-1}dG_1)G_2^{-1} + G_2dG_2^{-1}\ , \tag{21.12}$$

which gives the simpler relation

$$G_2^{-1}VG_2 = V_1 - V_2\ . \tag{21.13}$$

Using this we find

$$\begin{aligned} \text{tr}\ V^3 &= \text{tr}\ (G_2^{-1}VG_2)^3 = \text{tr}\ (V_1 - V_2)^3 = \\ &= \text{tr}\ (V_1^3 - 3V_1^2V_2 + 3V_1V_2^2 - V_2^3) = \\ &= \text{tr}\ (V_1^3 - V_2^3) + 3d\,\text{tr}\ (V_1V_2)\ , \end{aligned} \tag{21.14}$$

where the last step follows from $dV = -V^2$. Integrating this over S_3, and using the fact that $\partial S_3 = 0$ (since S_3 is itself a boundary of E_4) we find

$$Q(G_1G_2^{-1}) = Q(G_1) - Q(G_2)\ . \tag{21.15}$$

This relation along with (21.11) gives us

$$Q(G_1G_2) = Q(G_1) + Q(G_2) \ . \tag{21.16}$$

A simple consequence of this is that

$$Q(G^n) = n\, Q(G) \ . \tag{21.17}$$

We have shown that $Q(I)$ vanishes, but in fact this is true for any G connected to the identity. To prove this just look at infinitesimal elements $G = I + g + o(g^2)$. This gives $V = G^{-1}dG = (I - g)d(I + g) + o(g^2) = dg + o(g^2)$. The Pontryagin index Q obviously vanishes in this case since it is $o(g^3)$. Therefore $Q(G) = 0$ implies that G is homotopicaly equivalent to the identity $G \sim I$, and vice versa. Now we see that $G \sim GH$ for all $H \sim I$.

We have seen that gauge transformations G fall into disjoint classes. Each class consists of homotopicaly equivalent gauge transformations. The classes are labeled by their Pontryagin index. The normalization of the Pontryagin index in (21.5) was chosen so that Q takes on integer values.

21.2 The Chern-Simons Action

We first met the integral of the Pontryagin density as a possible addition to the action of a gauge theory in $d = 4$ dimensions. As we saw, this addition represents a topological term. It does not effect the classical dynamics. However, it does effect the ground state of the quantum theory. Similarly, in $d = 2n$ dimensions one may add the integral of the n^{th} Chern character $\chi_n = \text{ tr } F^n$ to the Yang-Mills action.

One can also construct a related term in odd dimensional theories. While we were manipulating the Pontryagin density we discovered the Chern-Simons 3-form. Let's take this as a possible action in $d = 3$ dimensions. Thus

$$I_{\text{cs}} = \frac{k}{4\pi}\int_B \text{tr } \left(AdA + \frac{2}{3}\,A^3\right) \ , \tag{21.18}$$

where B is our $d = 3$ manifold and $\partial B = 0$. In one of the previous exercises we have seen that this is gauge invariant under infinitesimal transformations. Let us now look at how it transforms under large gauge transformations

$$A \to G(A + d)G^{-1} = G(A - V)G^{-1} \ . \tag{21.19}$$

Now we have

$$\begin{aligned} \text{tr } \left(AdA + \frac{2}{3}\,A^3\right) &= \text{ tr } \left(AF - \frac{1}{3}\,A^3\right) \to \\ &\to \text{ tr } \left((A - V)F - \frac{1}{3}\,(A - V)^3\right) = \\ &= \text{ tr } \left(AF - \frac{1}{3}\,A^3\right) + \text{ tr } \left(-VF + \frac{1}{3}\,A^3 - \frac{1}{3}(A - V)^3\right) \ . \end{aligned} \tag{21.20}$$

Note that

$$VF = V(dA + A^2) = dVA + VA^2 = -V^2A + VA^2 \ , \tag{21.21}$$

where we have thrown away $d(VA)$ since we will integrate over B, and B has no boundary. We thus get

$$I_{\text{cs}} \to I_{\text{cs}} - 2\pi k Q \ . \tag{21.22}$$

If the gauge group has a trivial homotopy group, then I_{cs} is gauge invariant. If however, $Q \in \mathbb{Z}$ then we no longer have gauge invariance. Still, it is not the action that is important, but rather the phase $\exp iI_{cs}$. This will be gauge invariant but only for specific values of coupling constant — namely if $k \in \mathbb{Z}$. This is a very interesting example of how gauge invariance and topology conspire to make the coupling constant an integer. The equations of motion following from I_{cs} are $F = 0$, *i.e.* locally A can be gauged away, so there are no local degrees of freedom. Globaly, however, the A's are not trivial.

In the case of electrodynamics we are delaing with the $U(1)$ gauge group. $U(1)$ has a trivial homotopy group, so in this case I_{cs} is always gauge invariant. When added to the standard electrodynamics action in $d = 3$ the Chern-Simons term gives us a gauge invariant way of giving the gauge field a mass.

21.3 The Wess-Zumino Functional

There is another interesting thing that we can do. Lets start from a $d = 2$ dimensional theory on a manifold Σ such that $\Sigma = \partial B$. This is illustrated in Figure 21.1 Note that we also have $\Sigma = -\partial B'$ [1], so

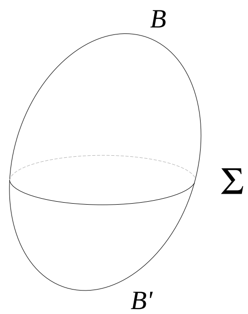


Figure 21.1: The $d = 2$ dimensional manifold Σ is spanned by the $d = 3$ dimensional manifolds B and $-B'$. Thus, $\partial(B \cup B') = 0$.

$$2\pi Q = \frac{1}{12\pi}\int_{B\cup B'} \text{tr}\ V^3 = \frac{1}{12\pi}\int_B \text{tr}\ V^3 - \frac{1}{12\pi}\int_{-B'} \text{tr}\ V^3 \ . \tag{21.23}$$

[1] The normals to B and B' both point outward. The boundary of B is Σ, while the boundary of B' is Σ traversed in the negative direction, *i.e.* $-\Sigma$.

This is true for any two manifolds whose boundary is Σ. It follows that for $k \in \mathbb{Z}$ the phase $\exp ik\Gamma$, where Γ is the Wess-Zumino functional defined as

$$\Gamma = \frac{1}{12\pi} \int_B \mathrm{tr}\ V^3 , \tag{21.24}$$

does not depend on B, but only on its boundary Σ. In fact, although $k\Gamma$ is an integral over a 3-form it is an addition to a $d = 2$ dimensional action (the Wess-Zumino-Witten model). We shall look into the Wess-Zumino functional in more detail when we deal with gauge anomalies.

EXERCISES

21.1 Why is it not possible to derive an inequality like (21.4) in the Minkowski theory?

21.2 Write equations (20.21), (20.22), (20.26), (20.28), (20.29), (20.30), (20.31) from the previous lecture in terms of the fields $A_\mu, F_{\mu\nu}$ as well as in terms of the components $A^a_\mu, F^a_{\mu\nu}$. Re-derive the result of exercise 20.9 in terms of the component fields.

Lecture 22

The Axial Anomaly

22.1 Schwinger Model

Electrodynamics in two dimensions is called the Schwinger model and is exactly solvable. The Lagranian in Euclidian space is

$$\mathcal{L} = \mathcal{L}_A + \mathcal{L}_m = \frac{1}{4e^2} F_{\mu\nu}F_{\mu\nu} + \bar{\psi}\,\gamma_\mu(\partial_\mu + iA_\mu)\,\psi \ . \tag{22.1}$$

The partition function is

$$Z = \int [dA]\, e^{-W[A] - \int dx \mathcal{L}_A} \ , \tag{22.2}$$

where the Fermi fields have been integrated out to yield $W[A]$, *i.e.*

$$\begin{aligned} e^{-W[A]} &= \int [d\psi d\bar{\psi}]\, e^{-\int dx\, \bar{\psi}\,\gamma_\mu(\partial_\mu + iA_\mu)\,\psi} = \\ &= \det(\partial\!\!\!/ + iA\!\!\!/) \ . \end{aligned} \tag{22.3}$$

Note that $W[A]$ is just the generating functional of connected graphs for Dirac Fermions in an external field A_μ. We next proceed to derive the vector and axial-vector Ward identities for $W[A]$. The vector transformations (gauge transformations) are given by

$$\begin{aligned} \psi(x) &\rightarrow \psi'(x) = e^{i\lambda(x)}\,\psi(x) \\ \bar{\psi}(x) &\rightarrow \bar{\psi}'(x) = \bar{\psi}(x)\, e^{-i\lambda(x)} \ . \end{aligned} \tag{22.4}$$

The change of $W[A]$ under these transformations depends on the change of the matter Lagrangian $\mathcal{L}_m$ as well as change of the Fermionic measure $[d\psi d\bar{\psi}]$ in the path integral. Naively, the measure seems gauge invariant. In deriving the axial-vector Ward identity, we will see that it is possible to define the measure precisely so that it *is* in fact invariant with respect to (22.4). On the other hand, the change of the Lagrangian for infinitestimal $\lambda(x)$ is

$$\delta\,\mathcal{L}_m = i\,\bar{\psi}\gamma_\mu\psi\,\partial_\mu\lambda \ . \tag{22.5}$$

Equation (22.3) gives us

$$\begin{aligned} e^{-W[A]} &= \int [d\psi d\bar{\psi}]\, e^{-\int dx\, \mathcal{L}_m[\psi,A]} = \\ &= \int [d\psi d\bar{\psi}]'\, e^{-\int dx\, \mathcal{L}_m[\psi',A]} = \\ &= e^{-W[A]} - \int dx\, [d\psi d\bar{\psi}]\, \delta\, \mathcal{L}_m(x)\, e^{-\int dy \mathcal{L}_m(y)} \ . \end{aligned} \tag{22.6}$$

This can be written in terms of $W[A]$ as

$$\int dx\, \lambda(x) \partial_\mu \left(\frac{\delta W[A]}{\delta A_\mu(x)} \right) e^{-W[A]} = 0 \ . \tag{22.7}$$

Since $\lambda(x)$ is arbitrary we find

$$\partial_\mu \frac{\delta W[A]}{\delta A_\mu(x)} = 0 \ . \tag{22.8}$$

This is the vector Ward identity. This derivation holds true in any dimension. Equation (22.8) just tells us that there is no anomaly associated with the gauge symmetry.

The axial-vector Ward identity is calculated in the same way. We now look at the local axial-vector transformations

$$\begin{aligned} \psi(x) &\rightarrow \psi'(x) = e^{i\alpha(x)\gamma_5}\psi(x) \\ \bar{\psi}(x) &\rightarrow \bar{\psi}'(x) = \bar{\psi}(x) e^{i\alpha(x)\gamma_5} \ . \end{aligned} \tag{22.9}$$

where the 2-dimensional γ-matrix algebra is given in Appendix A. The infinitestimal change in the Lagrangian is now

$$\delta\, \mathcal{L}_m = i(\partial_\mu \alpha)\, \bar{\psi}\gamma_\mu\gamma_5\psi \ . \tag{22.10}$$

Here, however, we will have to be more careful when dealing with the measure. We will find that imposing the gauge invariance of the measure automatically makes the measure not invariant under the axial-vector transformations. In regularizing the Fermionic measure, we will use the operator $\not{D}$. In Euclidian space the operator $\not{D} = \gamma_\mu(\partial_\mu + iA_\mu)$ is Hermitian, so the eigenvalue equation

$$\not{D}\, \varphi_n(x) = \lambda_n\, \varphi_n(x) \ , \tag{22.11}$$

has solutions $\varphi_n(x)$ which form an ortho-normal basis. Therefore

$$\int dx\, \varphi_n^\dagger(x) \varphi_m(x) = \delta_{nm} \ . \tag{22.12}$$

Due to Hermiticity, the eigenvalues λ_n are real. An important property of this operator is that its eigenvalues are gauge invariant, *i.e.* equation (22.11) implies

$$\begin{aligned} \not{D}\,'\varphi_n'(x) &= \gamma_\mu(\partial_\mu + iA_\mu + i\partial_\mu\lambda) e^{-i\lambda(x)}\varphi_n(x) = \\ &= e^{-i\lambda(x)} \not{D}\ \varphi_n(x) = \lambda_n\, \varphi_n'(x) \ . \end{aligned} \tag{22.13}$$

Expanding the spinors ψ, $\bar{\psi}$ in this basis we find

$$\psi(x) = \sum_n a_n \varphi_n(x) \tag{22.14}$$

$$\bar{\psi}(x) = \sum_n \varphi_n^\dagger(x) \bar{b}_n \ , \tag{22.15}$$

where a and $\bar{b}$ are independent Grassmann numbers. We define the Fermionic measure to be

$$[d\psi d\bar{\psi}] = [da][d\bar{b}] = \prod_n da_n \prod_m d\bar{b}_m \ . \tag{22.16}$$

Under (22.9) the a's change in the following way

$$a'_n = \sum_k C_{nk} a_k \ , \tag{22.17}$$

where the matrix C is defined to be

$$C_{nk} = \delta_{nk} + i \int dx\, \varphi_n^\dagger(x)\alpha(x)\gamma_5\varphi_k(x) \ . \tag{22.18}$$

Thus, since the a's are Grassmann we get

$$[da]' = [\det C]^{-1}[da] \ . \tag{22.19}$$

Similarly for the $\bar{b}$'s we find

$$[d\bar{b}]' = [\det C]^{-1}[d\bar{b}] \ . \tag{22.20}$$

Therefore, for infinitesimal axial-vector transformations, the measure changes according to

$$[d\psi d\bar{\psi}]' = e^{-2i\int dx\, \alpha(x) A(x)}\, [d\psi d\bar{\psi}] \ , \tag{22.21}$$

where we have defined

$$A(x) = \sum_n \varphi_n^\dagger(x)\gamma_5\varphi_n(x) \ . \tag{22.22}$$

The non-invariance of the measure, *i.e.* $A(x) \neq 0$, will be the cause of the anomaly.

To calculate $A(x)$ we need to regularize the above infinite sum. We need to do this in such a way as to preserve gauge invariance. Following Fujikawa, we regularize the sum by smoothly cutting-of the modes n corresponding to large eigenvalues of $\not{D}$ according to

$$A(x) = \lim_{M\to\infty} \sum_n \varphi_n^\dagger(x)\gamma_5 \exp\left(-{\lambda_n}^2/M^2\right)\varphi_n(x) =$$
$$= \lim_{M\to\infty} \sum_n \varphi_n^\dagger(x)\gamma_5 \exp\left(-\not{D}^{\,2}/M^2\right)\varphi_n(x) \ . \tag{22.23}$$

Note that this is a trace, so we can choose to evaluate it in a different basis. Plane waves represent the simplest choice, thus

$$A(x) = \lim_{M\to\infty} \text{tr} \int \frac{dk}{(2\pi)^2}\, e^{-ik\cdot x}\gamma_5 \exp\left(-\not{D}^{\,2}/M^2\right)\, e^{ik\cdot x} \ . \tag{22.24}$$

Now tr denotes a trace over spinor indices. An easy calculation gives us

$$\begin{aligned} e^{-ik\cdot x}\, \not{D}^{\,2}\, e^{ik\cdot x} &= \\ &= \frac{i}{2}\, \gamma_\mu\gamma_\nu F_{\mu\nu} - i(\partial_\mu A_\mu) - (ik_\mu + iA_\mu)^2 \ . \end{aligned} \tag{22.25}$$

Using this, the anomaly can be written as

$$A(x) = \lim_{M\to\infty} \text{tr} \left(\gamma_5\, e^{-\frac{i}{2M^2}\gamma_\mu\gamma_\nu F_{\mu\nu}}\right) e^{\frac{i}{M^2}(\partial_\mu A_\mu)} \int \frac{dk}{(2\pi)^2}\, e^{-\frac{k_\mu k_\mu}{M^2}} \ . \tag{22.26}$$

We next perform the Gaussian integration in the above formula. The $\partial_\mu A_\mu$ term vanishes in the $M \to \infty$ limit. The only piece which can survive is

$$\begin{aligned} A(x) &= \lim_{M\to\infty} \frac{M^2}{4\pi}\ \text{tr}\ \left(\gamma_5 \left(-i/2M^2\, \gamma_\mu\gamma_\nu F_{\mu\nu}\right)\right) = \\ &= -\frac{i}{8\pi}\ \text{tr}\ (\gamma_5\gamma_\mu\gamma_\nu) F_{\mu\nu} = -\frac{1}{4\pi}\epsilon_{\mu\nu}F_{\mu\nu} \ . \end{aligned} \tag{22.27}$$

The change in the measure is therefore given by

$$[d\psi d\bar{\psi}]' = e^{i/2\pi \int dx\, \alpha(x)\epsilon_{\mu\nu}F_{\mu\nu}}\, [d\psi d\bar{\psi}] \ . \tag{22.28}$$

Although this final result is specific to $d = 2$ dimensions, the calculations of anomalous Ward identities in d dimensions are just as simple. We simply substitute the corresponding d-dimensional Gaussian integral into equation (22.26).

To derive the Ward identity, as before, we inspect the change in $W[A]$

$$\begin{aligned} e^{-W[A]} &= \int [d\psi d\bar{\psi}]\, e^{-\int dx\, \mathcal{L}_m[\psi,A]} = \\ &= \int [d\psi d\bar{\psi}]'\, e^{-\int dx\, \mathcal{L}_m[\psi',A]} = e^{-W[A]} - \\ &- \int dx\, [d\psi d\bar{\psi}] \left(\delta\mathcal{L}_m(x) - \frac{i}{2\pi}\, \alpha(x)\epsilon_{\mu\nu}F_{\mu\nu}(x)\right) e^{-\int dy \mathcal{L}_m(y)} \ . \end{aligned} \tag{22.29}$$

The change in the matter Lagrangian $\delta\mathcal{L}_m$ is given in (22.10). A crucial property of $d = 2$ dimensional γ-matrix algebra is the relation

$$i\gamma_\mu\gamma_5 = \epsilon_{\mu\nu}\gamma_\nu \ , \tag{22.30}$$

which enables us to transform the above expression into

$$0 = \int dx\, \alpha(x) \left(\partial_\mu \epsilon_{\mu\nu} \frac{\delta W[A]}{\delta A_\nu(x)} - \frac{1}{2\pi}\, \epsilon_{\mu\nu}F_{\mu\nu}\right) e^{-W[A]} \ . \tag{22.31}$$

The axial-vector Ward identity thus reads

$$\epsilon_{\mu\nu}\partial_\mu \frac{\delta W[A]}{\delta A_\nu(x)} = \frac{1}{2\pi}\,\epsilon_{\mu\nu}F_{\mu\nu}(x)\ . \tag{22.32}$$

The two Ward identities that we have just derived allow us to explicitly calculate $W[A]$ and thus solve the Schwinger model. The axial-vector Ward identity tells us that $W[A]$ is quadratic in A_μ. Let us take the simplest choice, the local expression

$$W[A] = a\int dx\, A_\mu(x)A_\mu(x)\ . \tag{22.33}$$

By using equation (22.8) we see that this choice is good if we are in the Lorentz gauge $\partial\cdot A = 0$. For (22.32) to hold we need to fix the parameter a to be $\frac{1}{2\pi}$. We have thus validated our guess and have found that in the Lorentz gauge we have

$$e^{-W[A]} = e^{-\frac{1}{2\pi}\int dx\, A_\mu(x)A_\mu(x)}\ . \tag{22.34}$$

Going back to the total partition function Z, we may now write it in terms of an effective theory of self-interacting photons, whose dynamics follows from the effective Lagrangian

$$\mathcal{L}_{\rm eff} = \frac{1}{4e^2}\,F_{\mu\nu}F_{\mu\nu} + \frac{1}{2\pi}\,A_\mu A_\mu\ , \tag{22.35}$$

which can easily be put into a more suggestive form

$$\mathcal{L}_{\rm eff} = -\frac{1}{2e^2}\,A_\mu\left(\partial^2 - \frac{e^2}{\pi}\right)A_\mu\ . \tag{22.36}$$

The Schwinger model is revealed to be just a theory of two free scalar fields. The mass of these is $m^2 = \frac{e^2}{\pi}$ and is purely dynamically generated, being simply a consequence of the anomaly in the axial-vector symmetry. This model is also instructive in that it shows the equivalence of two simple though different looking Lagrangians, one of which has fermions in it and the other only bosons.

Let us recapitulate what we have found. We have shown that in $d = 2$ dimensions we have

$$\det(\partial\!\!\!/ + iA\!\!\!/) = e^{-W[A]}\ , \tag{22.37}$$

where $W[A] = \frac{1}{2\pi}\int dx\, A_\mu(x)A_\mu(x)$ for A_μ in the Lorentz gauge. The unique way to write a gauge invariant result is

$$\begin{aligned} W[A] &= \frac{1}{2\pi}\int dx\, A_\mu(x)\left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right)A_\nu(x) = \\ &= -\frac{1}{4\pi}\int dx\, F_{\mu\nu}\frac{1}{\partial^2}F_{\mu\nu}\ . \end{aligned} \tag{22.38}$$

Note that the above gauge invariant expression for $W[A]$ is non-local in A. This is not surprising. What is surprising is that in $d = 2$ there exists a gauge in which the effective action is local.

22.2 Current Correlators in $d = 2$

Having familiarized ourselves with the Schwinger model, we now prove a general theorem for $d = 2$ dimensional models that couple Fermions to an Abelian gauge field. The general Poincaré invariant[1] form for a current correlation function reads

$$\langle J_\mu(x)J_\nu(y)\rangle = \delta_{\mu\nu}\,\Pi_1(x-y) - \frac{\partial_\mu\partial_\nu}{\partial^2}\,\Pi_2(x-y)+ \\ + \left(\frac{\partial_\mu\epsilon_{\nu\alpha}\partial_\alpha}{\partial^2} + \frac{\partial_\nu\epsilon_{\mu\alpha}\partial_\alpha}{\partial^2}\right)\Pi_3(x-y)\ . \tag{22.39}$$

Different models give different funcitions Π_1, Π_2, Π_3. A theory is trivial if it has $\Pi_1 = \Pi_2 = \Pi_3 = 0$. As we have seen, the unique property of spinors in $d = 2$ dimensions is that the axial-vector current is dual to the vector current

$$J_{5\,\mu} = \bar\psi\gamma_5\gamma_\mu\psi = i\epsilon_{\mu\nu}\bar\psi\gamma_\nu\psi = i\epsilon_{\mu\nu}J_\nu\ . \tag{22.40}$$

This determines the mixed current correlation function to be

$$\langle J_{5\,\mu}(x)J_\nu(y)\rangle = i\epsilon_{\mu\nu}\,\Pi_1 - i\epsilon_{\mu\nu}\frac{\partial_\sigma\partial_\nu}{\partial^2}\,\Pi_2+ \\ + i\left(\delta_{\mu\nu} - 2\frac{\partial_\mu\partial_\nu}{\partial^2}\right)\Pi_3\ . \tag{22.41}$$

Gauge invariance imposes $\Pi_1 = \Pi_2$, as well as $\Pi_3 = 0$, while axial-vector conservation implies

$$\partial_\mu\langle J_{5\,\mu}(x)J_\nu(y)\rangle = 0\ , \tag{22.42}$$

which holds if $\Pi_1 = \Pi_3 = 0$. As we see, it is impossible for a non-trivial theory to have both vector and axial-vector currents conserved[2]. As we have seen, in the path integral formalism this manifested itself in the fact that it was impossible to regularize the Fermionic measure in a gauge invariant and axial-vector invariant way at the same time.

22.3 Axial Anomaly via Point-Splitting

In this section we will re-derive the axial-vector anomaly, this time working in the operator formalism. We start with Dirac Fermions coupled to a $U(1)$ gauge field. Dynamics follows from the Lagrangian

$$\mathcal{L} = \bar\psi(i\partial\!\!\!/ - m - A\!\!\!/)\psi\ . \tag{22.43}$$

The axial-vector current $J_5^\mu(x) = i\bar\psi(x)\gamma_5\gamma^\mu\psi(x)$ is classically conserved for $m = 0$. Upon quantization this symmetry is lost. The expression for $J_5^\mu(x)$ contains both

[1] Actually Galilei invariant, since we are working in the Euclidian theory.

[2] This conclusion can be bypassed with the introducton of further degrees of freedom in such a way that the vector and axial-vector currents are not dual to each other.

the operators ψ and $\bar{\psi}$ located at the same point in space time. This is ill defined, as the fields are operator distributions. We regularize currents by point splitting in a way that preserves gauge invariance. For example, the axial-vector current for $\varepsilon^0 > 0$ becomes

$$J_5^\mu(x|\varepsilon) = i\bar{\psi}(x+\varepsilon/2)e^{-i\int_{x-\varepsilon/2}^{x+\varepsilon/2} A(z)\cdot dz}\gamma_5\gamma^\mu\psi(x-\varepsilon/2) \ . \tag{22.44}$$

Using the Heisenberg equations $(i\partial\!\!\!/ - A\!\!\!/)\psi = 0$ we find that the regularized axial-vector current satisfies

$$\partial_\mu J_5^\mu(x|\varepsilon) = i\varepsilon^\nu F_{\nu\mu} J_5^\mu(x|\varepsilon) \ , \tag{22.45}$$

where $F_{\mu\nu}$ is the usual $U(1)$ field strength. The dominant singularity in bringing two fields at the same point is a c-number term. Therefore, we have

$$A(x)B(y) = \langle 0|A(x)B(y)|0\rangle + \ldots \tag{22.46}$$

where dots indicate less singular terms in $(x-y)$. Applying this operator product expansion to the expression for $J_5^\mu(x|\varepsilon)$ leads to

$$J_5^\mu(x|\varepsilon) = \text{ tr } (\gamma_5\gamma^\mu S(x-\varepsilon/2, x+\varepsilon/2)) + \ldots \tag{22.47}$$

where $S_{\alpha\beta}(x,y) = \frac{1}{2}\langle 0|T\left(\psi_\alpha(x)\bar{\psi}_\beta(y)\right)|0\rangle$ is the massless Fermion propagator in the presence of an external field. Therefore, it satisfies

$$(i\partial\!\!\!/ - A\!\!\!/)S(x,y) = \delta(x-y) \ . \tag{22.48}$$

Introducing the obvious notation $S(x,y) = \langle x|\hat{S}|y\rangle$ and $i\partial\!\!\!/ = \hat{p\!\!\!/}$ we can write the above equation as

$$(\hat{p\!\!\!/} - \hat{A\!\!\!/})\,\hat{S} = 1 \ . \tag{22.49}$$

Solving for $\hat{S}$ we find

$$\hat{S} = \left((1-\hat{A\!\!\!/}\hat{S}_0)\,\hat{S}_0^{-1}\right)^{-1} \ , \tag{22.50}$$

where $\hat{S}_0$ represent the free propagator. Expanding in A and inserting resolutions of the identity $\int |z\rangle dz\langle z| = 1$ we get

$$S(x,y) = S_0(x-y) + \int dz\, S_0(x-z)\,(-A\!\!\!/(z))\,S_0(z-y) + \ldots \tag{22.51}$$

This is simply the perturbative expansion of the propagator in an external field. Putting this into our expression for the regularized axial-vector current we find

$$\begin{aligned} J_5^\mu(x|\varepsilon) = &\text{ tr } (\gamma_5\gamma^\mu S_0(-\varepsilon)) - \\ &- \int dz\, A_\sigma(z)\,\text{tr }(\gamma_5\gamma^\mu S_0(x-\varepsilon/2-z)\gamma^\sigma S_0(z-x-\varepsilon/2)) + \ldots \end{aligned} \tag{22.52}$$

The free propagator can be explicitly calculated in closed form. We find

$$S_0(x) = \int \frac{dp}{(2\pi)^2}\, e^{-ip\cdot x}\, \frac{\not{p}}{p^2 + i\epsilon} = \frac{i}{2\pi}\, \frac{\gamma^\mu x_\mu}{x \cdot x}\ . \tag{22.53}$$

Only the first piece of (22.51) is of order $\frac{1}{\varepsilon}$ and can give a finite contribution to the right hand side of equation (22.45). Using the $d = 2$ dimensional trace identity

$$\text{tr}\ (\gamma_5 \gamma^\mu \gamma^\nu) = 2\epsilon^{\mu\nu}\ , \tag{22.54}$$

our final result becomes

$$\partial_\mu J_5^\mu(x) = -\frac{1}{2\pi}\, \epsilon^{\mu\nu} F_{\mu\nu}(x)\ . \tag{22.55}$$

The axial-vector current is not conserved. If we went back to normal units in which $\hbar \neq 1$ we would see that the right hand side of the above equation is proportional to $\hbar$. The anomaly is a byproduct of the quantization. This is precisely what we derived earlier in the functional formalism. The only difference is that the above calculation was done in the Minkowski formalism.

EXERCISES

22.1 Show that the path integral measure defined in the lecture is indeed invariant under vector transformations.

22.2 Modify the Fujikawa procedure in such a way that the path integral measure is invariant under axial-vector transformations. How does the measure now behave under vector transformations?

22.3 Define a point splitting regularization that preserves axial-vector symmetry. Calculate $\partial_\mu J^\mu$.

22.4 Derive the anomalous axial-vector Ward identity in $d = 4$ dimensions in both the functonal and operator formalisms.

Lecture 23

Gauge Anomalies

23.1 Cochains, Cocycles and the Coboundary Operator

We look at a theory consisting of gauge fields A and matter fields Φ, and desribed by an action $I[A, \Phi]$. Integrating out matter we get

$$e^{\frac{i}{\hbar}W[A]} = \int [d\Phi]\, e^{\frac{i}{\hbar}I[A,\Phi]} \ . \tag{23.1}$$

If the gauge symmetry is anomalous then the action is invariant while the measure is not, *i.e.* we have

$$I[A^G, \Phi^G] = I[A, \Phi] \tag{23.2}$$
$$[d\Phi]^G = [d\Phi]\, e^{2\pi i\, \alpha[A\,;G]} \ . \tag{23.3}$$

We now find that

$$W[A^G] - W[A] = 2\pi\hbar\, \alpha[A\,;G] \ . \tag{23.4}$$

For obvious reasons α is called the anomaly. From the above we see that the anomaly must satisfy the following consistency condition

$$\alpha[A^G\,;G'] - \alpha[A\,;GG'] + \alpha[A\,;G] = 0 \ , \tag{23.5}$$

for all G, G' belonging to the gauge group. In general α is non-local in A. As we can see in (23.4), the anomaly is a one loop effect.

Any function of a gauge field A and p gauge group elements $G_1, G_2, \dots, G_p$ is called a p-cochain and will be denoted as $\omega_p = \omega_p[A\,;G_1, G_2, \dots, G_p]$. We next introduce an operator Δ that acts as a gauge variation on 0-cochains

$$\Delta\omega_0[A] = \omega_0[A^G] - \omega_0[A] \ . \tag{23.6}$$

The action on a general p-cochain is determined by imposing the following nilpotence property $\Delta_2\Delta_1\omega_p = 0$, valid for all p-cochains. Δ is called the coboundary operator. Let us see how Δ acts on 1-cochains. We start by mimicking the action on 0-cochains as much as possible, and write $\Delta_2\omega_1[A\,;G_1] = \omega_1[A^{G_2}\,;G_1] + \ldots$ The missing terms follow from nilpotence. We find

$$\Delta\omega_1[A\,;G_1] = \omega_1[A^G\,;G_1] - \omega_1[A\,;GG_1] + \omega_1[A\,;G] \ . \tag{23.7}$$

Similarly, for p-cochains, we find

$$\begin{aligned}\Delta\omega_p &= \omega_p[A^G\,;G_1,G_2,\ldots,G_p] - \omega_p[A\,;GG_1,G_2,\ldots,G_p]+ \\ &\quad + \omega_p[A\,;G,G_1G_2,\ldots,G_p] - \ldots + (-)^p\omega_p[A\,;G,G_1,G_2,\ldots,G_{p-1}G_p] - \\ &\quad - (-)^p\omega_p[A\,;G,G_1,G_2,\ldots,G_{p-1}] \ .\end{aligned} \tag{23.8}$$

As we see, Δ increases cochain number by one. We end this mathematical asside by introducing some standard nomenclature. If a cochain ω_p satisfies $\Delta\omega_p = 0$ (modulo $\mathbb{Z}$) then it is called a cocycle. A p-cochain is a coboundary if $\omega_p = \Delta\omega_{p-1}$. From the nilpotence property of Δ it follows that coboundaries are automatically cocycles. Cocycles that are not coboundaries form the cohomology group of Δ.

Armed with the above mathematical preliminaries we may now recast equation (23.4) as

$$\Delta W = 2\pi\hbar\,\alpha \ , \tag{23.9}$$

while the consistency condition (23.5) immediately follows from nilpotency of Δ, and simply states that the anomaly is a 1-cocycle, *i.e.* $\Delta\alpha = 0$. From our experience with nilpotent operators we should expect that the cohomology group plays a central role. Indeed, if we have

$$\alpha' = \alpha + \Delta\omega \ , \tag{23.10}$$

where ω is a 0-cochain local in A, then obviously one is just dealing with a trivial redefinition of the measure according to

$$[d\Phi]' = [d\Phi]\, e^{2\pi i\,\omega[A]} \ . \tag{23.11}$$

To check this all we need do is gauge transform both sides. In this sense α and α' are equivallent. As a corollary we find that $\alpha = \Delta\omega$ is equivallent to no anomaly, *i.e.* $\alpha = 0$. If an anomaly is realy present in a theory then it belongs to the cohomology group of Δ. Said another way, the cohomology group of Δ contains all the possible anomaly terms for gauge theories based on a given gauge group.

In order to construct a theory that is classically equivalent to the original one but gauge invariant at the quantum level we define our functional integral with a new measure. Instead of $[d\Phi]$ we define the measure

$$[D\Phi] = [d\Phi]\,[dH]\, e^{2\pi i\,\alpha[A\,;H]} \ , \tag{23.12}$$

where we have introduced an extra field H that takes values in the gauge group. As we shall soon see, there exists a measure for H such that $[dH]^G = [dH]$. Now, in order for the full measure $[D\Phi]$ to be gauge invariant we must have

$$\alpha[A^G\,;H^G] - \alpha[A\,;H] + \alpha[A\,;G] = 0 \ . \tag{23.13}$$

With the substitution $G' = G^{-1}H$ the cocycle condition for the anomaly may be written as

$$\alpha[A^G\,; G^{-1}H] - \alpha[A\,; H] + \alpha[A\,; G] = 0 \ , \tag{23.14}$$

and so the invariance of the new measure is satisfied if we have

$$H^G = G^{-1}H \ . \tag{23.15}$$

Note now that for $[dH]$ we just take the Haar measure $\prod_i dH_{ii} \prod_{i<j} dH_{ij} dH_{ji}$ which is indeed invariant under the above gauge transformation.

The new measure allows us to define the anomaly free model in the following way

$$\begin{aligned} e^{\frac{i}{\hbar}\,\overline{W}\,[A]} &= \int [D\Phi]\, e^{\frac{i}{\hbar}\, I[A,\Phi]} = \\ &= \int [d\Phi]\,[dH]\, e^{\frac{i}{\hbar}\,\overline{I}\,[A,\Phi,H]} \ , \end{aligned} \tag{23.16}$$

where

$$\overline{I}\,[A,\Phi,H] = I[A,\Phi] + 2\pi\hbar\,\alpha[A\,; H] \ , \tag{23.17}$$

is the new action. Here we explicitly see that the two models are classicaly equivallent, *i.e.* have the same $\hbar \to 0$ limit.

If we integrate out Φ we get

$$e^{\frac{i}{\hbar}\,\overline{W}\,[A]} = e^{\frac{i}{\hbar}\,W[A]} \int [dH]\, e^{2\pi i\,\alpha[A\,;H]} \ . \tag{23.18}$$

Using the defining relation for the anomaly this becomes

$$e^{\frac{i}{\hbar}\,\overline{W}\,[A]} = \int [dH]\, e^{\frac{i}{\hbar}\,W[A^H]} \ , \tag{23.19}$$

which is manifestly gauge invariant.

In the new model the action is no longer gauge invariant, in fact

$$\overline{I}^{\,G} - \overline{I} = -\,2\pi\hbar\,\alpha[A\,; G] \ . \tag{23.20}$$

The non-invariance of $\overline{I}$ is such that it precisely cancells the anomaly in the measure and so leads to an anomaly free theory. Note that, became of α, $\overline{I}\,[A,\Phi,H]$ is non-local, and hence represents an effective action. This can also be seen from the fact that $\overline{I}$ has explicit $\hbar$ dependence. From its definition we see that $\alpha[A\,; 1] = 0$. Gauge fixing the new theory by setting $H = 1$ we return to our starting theory.

23.2 Chern Forms and the Descent Equations

Let us work with gauge field A written as anti-Hermitian matrix valued 1-forms. The gauge variation of A is given by

$$A^G = G(A+d)G^{-1} \ . \tag{23.21}$$

The associated field strength $F = dA + A^2$ transforms homogenously under the above variation, *i.e.* we have

$$F^G = GFG^{-1} \ . \tag{23.22}$$

The field strength also satisfies the Bianchi identity $DF = dF + [A,F] = 0$, where D is the covariant derivative. We have previously define the Chern forms to be $\chi_n = \text{Tr } F^n$. They are obviously gauge invariant objects. We may write this as

$$\Delta \, \text{Tr } F^n = 0 \ . \tag{23.23}$$

The Chern forms are also closed. Using $\text{Tr } (AF^n) = \text{Tr } (F^n A)$ we get

$$d \, \text{Tr } F^n = D \, \text{Tr } F^n = n \, \text{Tr } (DF \, F^{n-1}) = 0 \ , \tag{23.24}$$

where the last step follows from the Bianchi identity. From the above we see that *localy* we may write

$$\text{Tr } F^n = d\omega^0_{2n-1} \ . \tag{23.25}$$

The superscript and subscript respectively denote cochain and form number. By using the fact that d and Δ commute we directly get from the above equation that $d\Delta\omega^0_{2n-1} = 0$. Again, localy, this implies that

$$\Delta\omega^0_{2n-1} = -d\omega^1_{2n-2} \ . \tag{23.26}$$

Similarly, by nilpotence of Δ, we get

$$\Delta\omega^1_{2n-2} = -d\omega^2_{2n-3} \tag{23.27}$$
$$\Delta\omega^2_{2n-3} = -d\omega^3_{2n-4} \ , \tag{23.28}$$

etc. These are the descent equations. As we shall see, each of the ω forms plays an important role in quantum field theory.

If we have a closed form χ, *i.e.* one that satisfies $d\chi = 0$, then *localy* $\chi = d\omega$. If ω exists on the whole manifold then χ is called exact. We would like to solve the above equation for ω. To do this we might first look to find an operator d^{-1} that satisfies $d^{-1}d = 1$. It is impossible to do this, since by acting on any closed form this would give an inconsistency. We can, however, find an operator k, called the homotopy operator, that satisfies $kd + dk = 1$. Acting with this on χ gives

$$\chi = (kd + dk)\chi = dk\chi \ , \tag{23.29}$$

where the last equality follows since χ is closed. We immediately see that we have

$$\omega = k\chi \ . \tag{23.30}$$

Now we shall present an explicit construction of the homotopy operator. We begin by writting the definition of the field strength F and its Bianchi identity as

$$\begin{aligned} dA &= F - A^2 &(23.31)\\ dF &= FA - AF \ . &(23.32)\end{aligned}$$

We now see that on the space of polynomials $P(F,A)$ the exterior derivative acts algebraicaly. As an added restriction we shall only consider polynomials that vanish at $A = F = 0$. Let us look at the following one parameter family of gauge potentials and field strengths

$$\begin{aligned} A_t &= tA &(23.33)\\ F_t &= tF + (t^2 - t)A^2 \ . &(23.34)\end{aligned}$$

We next introduce the operator ℓ_t such that

$$\begin{aligned} \ell_t A_t &= 0 &(23.35)\\ \ell_t F_t &= dtA \ . &(23.36)\end{aligned}$$

It is easy to see that ℓ_t satisfies the Leibnitz rule. Finally, we define the homotopy operator k according to

$$kP(F,A) = \int_0^1 \ell_t P(F_t, A_t) \ . \tag{23.37}$$

As may be easily checked the homotopy operator satisfies the following two properties

$$\begin{aligned} kd + dk &= 1 &(23.38)\\ k^2 &= 0 \ . &(23.39)\end{aligned}$$

From the above definition we easily find, for example, that

$$\begin{aligned} kA &= 0 &(23.40)\\ kF &= A &(23.41)\\ k(AF) &= -\frac{1}{2}A^2 &(23.42)\\ k(FA) &= \frac{1}{2}A^2 \ , &(23.43)\end{aligned}$$

and so on. Even from these simple examples we see that k does not satisfy the Leibnitz rule, and hence is not a derivative.

Now we shall use the homotopy operator to calculate the ω's from the descent equations. We find

$$\begin{aligned}\omega^0_{2n-1} &= k\,\mathrm{Tr}\ F^n = \\ &= \mathrm{Tr}\ \int_0^1 \ell_t F_t^n = n\,\mathrm{Tr}\ \int_0^1 \ell_t F_t F_t^{n-1} = n\,\mathrm{Tr}\ \int_0^1 dt A F_t^{n-1}\ . \end{aligned} \tag{23.44}$$

Of particular interest is the result for $n = 2$, where

$$\omega^0_3 = k\,\mathrm{Tr}\ F^2 = \mathrm{Tr}\ (AF - \frac{1}{3}A^3) = \mathrm{Tr}\ (AdA + \frac{2}{3}A^3)\ . \tag{23.45}$$

Note that $\mathrm{Tr}\ F^2$ and ω^0_3 are the Pontryagin and Chern-Simons densities respectively.

Let us next analyze the gauge transformation properties of the forms ω^0_{2n-1}, remembering that $d\omega^0_{2n-1}$ is gauge invariant. Using the fact that ω^0_{2n-1} is a trace we have

$$\begin{aligned}&\omega^0_{2n-1}(F^G, A^G) = \\ &= \omega^0_{2n-1}(GFG^{-1}, G(d+A)G^{-1}) = \omega^0_{2n-1}(F, A-V)\ ,\end{aligned} \tag{23.46}$$

where $V = G^{-1}dG$. We can now easily show that the form

$$\xi_{2n-1} = \omega^0_{2n-1}(F, A-V) - \omega^0_{2n-1}(F, A) - \omega^0_{2n-1}(0, -V)\ , \tag{23.47}$$

is closed, and is given by a polynomial in A and F that vanishes at $A = F = 0$. We thus have $\xi_{2n-1} = d\beta_{2n-2}$ where $\beta_{2n-1} = k\xi_{2n-1}$. Therefore, we have found

$$\omega^0_{2n-1}(F^G, A^G) = \omega^0_{2n-1}(F, A) + \omega^0_{2n-1}(0, -V) + d\beta_{2n-2}\ , \tag{23.48}$$

as well as

$$\beta_{2n-2} = k\ \left(\omega^0_{2n-1}(F, A-V) - \omega^0_{2n-1}(F, A) - \omega^0_{2n-1}(0, -V)\right)\ . \tag{23.49}$$

For $n = 2$ this gives

$$\Delta\omega^0_3 = \frac{1}{3}\,\mathrm{Tr}\ V^3 + d\beta_2 \tag{23.50}$$

$$\beta_2 = \mathrm{Tr}\ VA\ . \tag{23.51}$$

23.3 Atiyah-Singer Index Theorem

Let us look at the eigenvalue problem for the Dirac operator $i\,\not{D} = i\,\gamma_\mu D_\mu$ on an Euclidian $d = 2n$ dimensional manifold $\mathcal{M}$. We have

$$i\,\not{D}\,\varphi_i = \lambda_i\,\varphi_i\ . \tag{23.52}$$

In the Euclidian theory $i\not{D}$ is Hermitian, therefore, the λ's are real, while the eigenstates may be chosen orthonormal $\int dx\, \varphi_i^* \varphi_j = \delta_{ij}$. Acting with the chirality projectors $P_\pm = \frac{1\pm\gamma_5}{2}$ we get

$$i\not{D}\,\varphi_\pm = \lambda\,\varphi_\mp \;, \tag{23.53}$$

so that we may distinguish two cases. For $\lambda \neq 0$ the eigenstates can't be chiral fields. Further, if φ is an eigenstate with eigenvalue λ, then $\gamma_5\varphi$ is an eigenstate with eigenvalue $-\lambda$. For these eigenstates we thus have $\int dx\, \varphi_i^* \gamma_5 \varphi_i = 0$. On the other hand, the zero modes (eigenstates withj $\lambda = 0$) can have definite chirality. The index of the Dirac operator is defined to be

$$\mathrm{ind}(\not{D}\,) = N_+ - N_- \;, \tag{23.54}$$

where $N_\pm$ denotes the number of zero modes of positive and negative chirality respectively. For zero modes $\int d\xi\, \varphi_i^* \gamma_5 \varphi_i$ is equal to the chirality of φ_i. We may write the index as

$$\int d\xi \;\mathrm{Tr}\, \left(\gamma_5 \,\exp(\frac{i\not{D}}{M})^2 \right) \;. \tag{23.55}$$

The above formula is easily calculated if we work in the φ_i basis. It is valid for any value of the regulator M. Let us now calculate the above expression in the plane wave basis. To do this we need $\gamma_\mu\gamma_\nu F_{\mu\nu} = \gamma_\mu\gamma_\nu[D_\mu, D_\nu] = 2\not{D}^{\,2} - 2D^2$. We now easily find $e^{-k\cdot\xi}\not{D}^{\,2} e^{k\cdot\xi} = \frac{1}{2}\gamma_\mu\gamma_\nu F_{\mu\nu} + (\partial\cdot A) + (ik+A)^2$. Using this we may write the index as

$$\begin{aligned} \mathrm{ind}(\not{D}\,) &= \int d\xi \,\frac{dk}{(2\pi)^{2n}}\,\mathrm{Tr}\, \left(\gamma_5 \,\exp\left(\frac{1}{2M^2}\gamma_\mu\gamma_\nu F_{\mu\nu} + \frac{(\partial\cdot A)}{M^2} + \frac{(ik+A)^2}{M^2} \right)\right) = \\ &= \int d\xi \left(\frac{M^2}{4\pi}\right)^n \mathrm{Tr}\, \left(\gamma_5 \,\exp\left(\frac{1}{2M^2}\gamma_\mu\gamma_\nu F_{\mu\nu} + \frac{(\partial\cdot A)}{M^2} \right)\right) \;, \end{aligned} \tag{23.56}$$

the remaining trace is over spinor and gauge indices, and is easily calculated for $M \to \infty$. To do this we use the fact that in $d = 2n$ spinors have 2^n components, as well as the γ-matrix identity $\mathrm{Tr}\,(\gamma_5\gamma_{\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_{2n}}) = (-2i)^n \varepsilon_{\mu_1\mu_2\ldots\mu_{2n}}$. We also use the fact that all the traces of γ_5 with less then $2n$ γ_μ's are zero. The final result for the index is most compactly written in term of forms. We have thus derived the Atiyah-Singer index theorem

$$\mathrm{ind}(\not{D}\,) = \int_{\mathcal{M}} \frac{1}{n!}\left(\frac{i}{2\pi}\right)^n \mathrm{Tr}\; F^n = \int_{\mathcal{M}} \chi(F) \;. \tag{23.57}$$

In the last step we have introduced the Chern character $\chi(F)$ as the formal expression $\chi(F) = \mathrm{Tr}\,\left(\exp \frac{i}{2\pi}F\right)$. This index theorem is valid if the manifold is flat. Its generalization to manifolds with curvature is known as the Atiyah-Patodi-Singer index theorem. For us it is just important to mention that in the case of the sphere in arbitrary dimensions we again get back the flat space result.

Let us now look at the index theorem when our manifold is the $d = 4$ dimensional sphere S_4. We may write $S_4 = S_4^+ \cup S_4^-$, with $\partial S_4^+ = -\partial S_4^- = S_3$. It is not possible to define A on all of S_4, however, we can do this on each of the two patches. On S_4^+ we have A_+, and on S_4^- the gauge field is A_-. Since they correspond to the same physical field we have

$$A_+ = G(A_- + d)G^{-1} \ , \tag{23.58}$$

for some G in the gauge group. We may now write the index theorem as

$$\begin{aligned} \mathrm{ind}(\not{D}\,) &= -\frac{1}{8\pi^2}\int_{S_4} \mathrm{Tr}\ F^2 = \\ &= -\frac{1}{8\pi^2}\int_{S_4^+} d\omega_3^0(A_+) - \frac{1}{8\pi^2}\int_{S_4^-} d\omega_3^0(A_-) = \\ &= -\frac{1}{8\pi^2}\int_{S_3}\left(\omega_3^0(A_+) - \omega_3^0(A_-)\right) \ . \end{aligned} \tag{23.59}$$

Using (23.50) this becomes

$$\mathrm{ind}(\not{D}\,) = -\frac{1}{24\pi^2}\int_{S_3} \mathrm{Tr}\ V^3 = Q \ . \tag{23.60}$$

The expression on the right hand side is the Pontryagin index — the winding number for the map from S_3 to the gauge group. From the above we see that it can take only values in $\mathbb{Z}$. More importantly, we see that $\mathrm{ind}(\not{D}\,) \neq 0$ if and only if the gauge group has a non trivial third homotopy group Π_3.

Let us look at the forms that are related through the descent equations. We shall focus on the $n = 2$ descent chain.

$$\mathrm{Tr}\ F^2 = d\omega_3^0 \tag{23.61}$$
$$\Delta\omega_3^0 = -d\omega_2^1 \tag{23.62}$$
$$\Delta\omega_2^1 = -d\omega_1^2 \tag{23.63}$$
$$\Delta\omega_1^0 = -d\omega_0^3 \ . \tag{23.64}$$

We have met most of these objects in previous lectures. The descent chain above starts with the Pontryagin density. The integral of this is a possible addition to the action of a $d = 4$ dimensional gauge theory. Because of the first descent equation we see that this is a topological term, *i.e.* it does not influence the equations of motion. The next form is the Chern-Simons density. Out of it we can construct the action

$$I_{\mathrm{cs}} = \frac{k}{4\pi}\int_{\mathcal{M}} \omega_3^0 \ , \tag{23.65}$$

where $\partial\mathcal{M} = 0$, and k is a coupling constant. From (23.50) we have

$$I_{\mathrm{cs}}^U = I_{\mathrm{cs}} - 2\pi k Q \ . \tag{23.66}$$

For gauge groups such with $\Pi_3 \neq 0$ we have $Q \in \mathbb{Z}$, and hence $I_{\rm cs}$ is not gauge invariant. However, the phase $e^{iI_{\rm cs}}$ is gauge invariant if we choose $k \in \mathbb{Z}$. This quantization of the coupling constant is a purely topological effect. If $\Pi_3 = 0$ then $Q = 0$ and we have no restrictions on the coupling constant k. The Chern-Simons action is an alternate action for $d = 3$ dimensional gauge theory. Its equation of motion is $F = 0$. Localy A can be gauged away. However, the solutions are in general not trivial as they are sensitive to global (topological) effects.

As before we will now write the Pontryagin index in yet another way. To do this we take $S_3 = S_3^+ \cup S_3^-$, with $\partial S_3^+ = -\partial S_3^- = S_2$. To make contact with the next section we introduce the notation $S_2 = \Sigma$, $S_3^+ = B$, $-S_3^- = B'$. Note that both B and B' represent $d = 3$ dimensional extensions of the $d = 2$ dimensional manifold Σ. We now have

$$2\pi Q = \frac{1}{12\pi}\int_B \mathrm{Tr}\ V^3 - \frac{1}{12\pi}\int_{B'} \mathrm{Tr}\ V^3\ . \tag{23.67}$$

It follows that the phase $e^{ik\Gamma}$, where $\Gamma = \frac{1}{12\pi}\int_B \mathrm{Tr}\ V^3$ does not depend on B but only on the Σ. As before, $k \in \mathbb{Z}$ if $\Pi_3 \neq 0$, and k may take arbitrary values if $\Pi_3 = 0$. In the next section this will enable us to treat $k\Gamma$ as an addition to the action of a theory in $d = 2$ dimensions — the Wess-Zumino-Witten model.

Let us now look at the second descent equation given above. Using (23.50) and integrating over B we find

$$\Delta \frac{k}{4\pi}\int_B \omega_3^0 = 2\pi\,\alpha\ , \tag{23.68}$$

where $2\pi\,\alpha = k\Gamma + \frac{k}{4\pi}\int_\Sigma \mathrm{Tr}\ (VA)$. We immediately see that $\Delta\alpha = 0$. Note that α is not trivial since it is given as a coboundary of an expression that is non-local on Σ. Localy we may write $\alpha \propto \int_\Sigma \omega_2^1$, and so α represents the anomaly of a $d = 2$ dimensional gauge theory. An equivalent solution of the cocycle equation differs from the one above by the action of Δ on a local functional of A. It is customary to choose this local functional to be $\frac{1}{2\lambda^2}\int_\Sigma \mathrm{Tr}\ (A^*A)$. We have $\Delta\int_\Sigma \mathrm{Tr}\ (A^*A) = \int_\Sigma \mathrm{Tr}\ (V^*V - 2A^*V)$, and so the anomaly may be written as

$$2\pi\alpha[A\,;G] = I_{\rm wzw}[G] + \frac{k}{4\pi}\int_\Sigma \mathrm{Tr}\ \left((V - \frac{4\pi}{k\lambda^2}{}^*V)A\right) \tag{23.69}$$

$$I_{\rm wzw}[G] = -\frac{1}{2\lambda^2}\int_\Sigma \mathrm{Tr}\ (V^*V) + k\Gamma\ . \tag{23.70}$$

We are free to choose λ^2, so we set $\lambda{=}\frac{4\pi}{k}$. Given a gauge group we see that the anomaly is uniquely determined up to k. In a similar way we may get the anomaly in $d = 4$ from the $n = 3$ descent chain. In general, the anomaly in $d = 2n$ dimensions follows from the $n + 1$ descent chain.

23.4 The Wess-Zumino-Witten Model

The Wess-Zumino-Witten model, over a given group, has the action

$$I_{\text{wzw}} = I_0 + k\Gamma \ . \tag{23.71}$$

The first term is just a σ-model action

$$I_0 = \frac{1}{2\lambda^2}\int_\Sigma dx \ \text{Tr}\ (\partial_\alpha G^{-1}\partial^\alpha G) = -\frac{1}{2\lambda^2}\int_\Sigma dx \ \text{Tr}\ (V_\alpha V^\alpha) \ , \tag{23.72}$$

where the fields G take their values in the group, and live on a closed $d = 2$ dimensional manifold Σ. The interaction term is given by the Wess-Zumino functional

$$\Gamma = \frac{1}{12\pi}\int_B \text{Tr}\ V^3 \ . \tag{23.73}$$

Our original manifold Σ represents the boundary of the $d = 3$ dimensional manifold B. G's are now extended to the whole of B.

Let us first look at the symmetries of the above expression. The σ-model action is invariant under *inversion* $G \to G^{-1}$, *naive parity* $x^0 \to x^0$, $x^1 \to -x^1$ as well under global *chiral transformations* given by $G \to XGY^{-1}$. It is easy to see that Γ changes sign under inversion, as well as under naive parity, while chiral transformations leave it unchanged. The WZW action is therefore invariant under chiral transformations, as well as under *parity* given by the joint transformations

$$G \to G^{-1} \tag{23.74}$$
$$x^0 \to x^0 \tag{23.75}$$
$$x^1 \to -x^1 \ . \tag{23.76}$$

In lightcone coordinates $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$ we have $x^\pm \to x^\mp$. Let us now define the right chiral current as $J_+ = G^{-1}\partial_+ G$. Under parity we need to have $J_+ \to J_-$, and this determines the expression for the left chiral current to be $J_- = G\partial_- G^{-1}$.

Now let us look at the equations of motion of the WZW model. It is easy to show that

$$\delta I_0 = \frac{1}{\lambda^2}\int_\Sigma dx \ \text{Tr}\ (\delta G G^{-1}\partial_\alpha V^\alpha) \ . \tag{23.77}$$

An equaly simple exercise gives us the variation of Γ. Using $dV = -V^2$ and Stokes' theorem we find

$$\delta\Gamma = \frac{1}{4\pi}\int_\Sigma \text{Tr}\ (\delta G G^{-1} dV) \ . \tag{23.78}$$

Here we see explicitly that Γ only contributes to the dynamics on the boundary of B. In lightcone coordinates we have

$$\partial_+ V_- + \partial_- V_+ + \frac{k\lambda^2}{4\pi}(\partial_+ V_- - \partial_- V_+) = 0 \ . \tag{23.79}$$

From now on we choose $\lambda^2 = \frac{4\pi}{k}$ (the infra red fixed point of the WZW model). The equation of motion is now simply $\partial_+ V_- = 0$. In terms of the current we have

$$\begin{aligned} \partial_+ J_- &= 0 && (23.80) \\ \partial_- J_+ &= 0\ . && (23.81) \end{aligned}$$

These two equations are not independent. One follows from the other either by parity, or by using the identity $G\partial_- J_+ G^{-1} = \partial_+ J_-$. The equation of motion for the WZW model turns out to be extremely simple, much simpler than the σ-model equation $\partial_\alpha V^\alpha = 0$. The general solution of the WZW equation of motion is

$$G(x^+, x^-) = G_-(x^-) G_+(x^+)\ . \tag{23.82}$$

The fact that a model is integrable is usualy taken to imply that there exists a field redefinition that maps it to a free field theory. Witten has shown for that this is indeed true for the $SU(N)$ and $SO(N)$ groups, and that the WZW model represents a free Fermion theory written in terms of Bose fields. That such a bosonization is even in principle possible is a consequence of the fact that in $d = 2$ dimensions spin is not quantized — we can continuosly deform a Fermion into a Boson. Goddard, Nahm and Olive have given a classification of all groups for which the WZW model is equivalent to a free Fermi theory.

Let us now decompose the field according to

$$G = LR^{-1}\ . \tag{23.83}$$

The chiral symmetry of our theory $G \to XGY^{-1}$ is now represented by $L \to XL$ as well as $R \to YR$. The above decomposition has enabled us to separate the actions of the right and left chiral symmetries. Under parity we have $LR^{-1} \to RL^{-1}$. The simplest way to get this is to have $L \to R$, and $R \to L$. In this way L determines R and hence the whole field G. Our decomposition has as a byproduct the local symmetry $L \to LU^{-1}$, $R \to RU^{-1}$. If we introduce $A_+ = L^{-1}\partial_+ L$, and $A_- = R^{-1}\partial_- R$, then under parity we have $A_\pm \to A_\mp$, hence A is a vector. Under the above local symmetry we have

$$A_\pm \to U(A_\pm + \partial_\pm)U^{-1}\ , \tag{23.84}$$

so that A represents the gauge potential for this symmetry. Note that A is not pure gauge since in general $L \neq R$.

We next look at the WZW action under the above decomposition. A direct calculation gives us the Polyakov-Wiegmann identity

$$I_{\text{wzw}}[LR^{-1}] = I_{\text{wzw}}[L] + I_{\text{wzw}}[R^{-1}] + \frac{k}{2\pi}\int_\Sigma dx\ \text{Tr}\ (A_+ A_-)\ . \tag{23.85}$$

It is important to note that the term that mixes L and R is just a $d = 2$ integral.

Now let us consider G, L, and R as independent fields. We look at the expression

$$I[G,A] = I_{\rm wzw}[LGR^{-1}] - I[LR^{-1}] \ . \tag{23.86}$$

Using the Polyakov-Wiegmann identity we find

$$I[G,A] = I_{\rm wzw}[G] + \frac{k}{2\pi}\int_\Sigma {\rm Tr}\ (J_+A_- + J_-A_+ - A_+A_- + A_+GA_-G^{-1}) \ . \tag{23.87}$$

We have $I[G,0] = I_{\rm wzw}[G]$. Further, $I[G,A]$ is invariant under the gauge transformation $G \to UGU^{-1}$, $L \to LU^{-1}$ and $R \to RU^{-1}$, or equivalently, in terms of G and A, under

$$G \ \to \ UGU^{-1} \tag{23.88}$$
$$A_\pm \ \to \ U(A_\pm + \partial_\pm)U^{-1} \ . \tag{23.89}$$

Therefore, the action $I[G,A]$ represents the gauged WZW model.

Lecture 24

Vacuum Polarization

24.1 Schwinger's Solution

Let us look at the dynamics of a Dirac Fermion in $d = 4$ dimensions moving in a external electro-magnetic field. The action is

$$I = \int dx\, \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi \ . \tag{24.1}$$

We want to evaluate $W[A] = \int dx\, w(x)$ defined by

$$e^{iW} = \frac{\int [d\psi d\bar{\psi}]\, e^{iI}}{\int [d\psi d\bar{\psi}]\, e^{iI_0}} = \frac{\det(i\partial\!\!\!/ - eA\!\!\!/ - m)}{\det(i\partial\!\!\!/ - m)} \ . \tag{24.2}$$

Note that I_0 is just the free Fermion action. The charge conjugation matrix C satisfies $C\gamma_\mu C^{-1} = -\gamma_\mu^{\mathrm{T}}$, so that we have

$$\begin{aligned} \det(i\partial\!\!\!/ - m) &= \det\left(C(i\partial\!\!\!/ - m)C^{-1}\right) = \\ &= \det\left(-(i\partial\!\!\!/ + m)^{\mathrm{T}}\right) = \det(i\partial\!\!\!/ + m) \ . \end{aligned} \tag{24.3}$$

Using this we find

$$e^{2iW} = \det\left(\frac{(i\partial\!\!\!/ - eA\!\!\!/)^2 - m^2}{(i\partial\!\!\!/)^2 - m^2}\right) \ . \tag{24.4}$$

A very simple calculation gives us

$$\begin{aligned} (i\partial\!\!\!/ - eA\!\!\!/)^2 &= \gamma_\mu\gamma_\nu(i\partial^\mu - eA^\mu)(i\partial^\nu - eA^\nu) = \\ &= \gamma_\mu\gamma_\nu(-\partial^\mu\partial^\nu - ie\partial^\mu A^\nu - ieA^\nu\partial^\mu - ieA^\mu\partial^\nu + e^2A^\mu A^\nu) = \\ &= (i\partial_\mu - eA_\mu)^2 - \frac{1}{2}\, e\sigma_{\mu\nu}F^{\mu\nu} \ . \end{aligned} \tag{24.5}$$

We have used the fact that

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} \tag{24.6}$$

$$\frac{i}{2}\,[\gamma_\mu, \gamma_\nu] = \sigma_{\mu\nu} \ . \tag{24.7}$$

Therefore, we have

$$2iW = \text{Tr}\ \ln\left(\frac{(P_\mu - eA_\mu)^2 - \frac{e}{2}\,\sigma_{\mu\nu}F^{\mu\nu} - m^2}{P^2 - m^2}\right)\ , \tag{24.8}$$

where we have introduced the familiar operator notation $P_\mu = i\partial_\mu$. Note that $[P_\mu, x_\nu] = ig_{\mu\nu}$. We will evaluate the trace in the coordinate representation. We find

$$w(x) = \frac{1}{2i}\ \text{tr}\ \langle x|\ln\left(\frac{(P_\mu - eA_\mu)^2 - \frac{e}{2}\,\sigma_{\mu\nu}F^{\mu\nu} - m^2}{P^2 - m^2}\right)|x\rangle\ . \tag{24.9}$$

The remaining trace ‘ tr ’ is over spinor indices. To simplify this, we write the log in terms of the proper time integral[1] formula

$$\ln\left(\frac{a + i\varepsilon}{b + i\varepsilon}\right) = \int_0^{+\infty}\frac{ds}{s}\left(e^{is(b+i\varepsilon)} - e^{is(a+i\varepsilon)}\right)\ . \tag{24.10}$$

The $\varepsilon \to 0+$ limit is understood. From now on we will drop the ε's. They always come in correctly via the usual $m \to m - i\varepsilon$ prescription. We now have

$$w(x) = \frac{i}{2}\int_0^{+\infty}\frac{ds}{s}\,e^{-ism^2}\left(M_e(x) - M_0(x)\right)\ , \tag{24.11}$$

Where

$$M_e(x) = \text{tr}\ \langle x|\exp\left(is\left((P_\mu - eA_\mu)^2 - \frac{e}{2}\,\sigma_{\mu\nu}F^{\mu\nu}\right)\right)|x\rangle\ , \tag{24.12}$$

and $M_0(x)$ is the same matrix element with $e = 0$. This is easy to evaluate for constant electro-magnetic fields. $F_{\mu\nu}$ then commutes with $(P_\mu - eA_\mu)$ and we have

$$M_e(x) = \text{tr}\ \left(e^{-is\,\frac{e}{2}\,\sigma_{\mu\nu}F^{\mu\nu}}\right)\langle x|e^{is(P_\mu - eA_\mu)^2}|x\rangle\ . \tag{24.13}$$

For a constant electric field along the x-axis we have

$$F^{01} = -F^{10} = -E\ , \tag{24.14}$$

while all other components of the field strength vanish. In terms of the gauge field we take

$$\begin{aligned} A_1 &= E\,x_0 \\ A_0 &= A_2 = A_3 = 0\ . \end{aligned} \tag{24.15}$$

For the above trace we find

$$\text{tr}\ (e^{-is\,\frac{e}{2}\,\sigma_{\mu\nu}F^{\mu\nu}}) = \text{tr}\ (e^{+ise\,\sigma_{01}E}) = 4\cosh(seE)\ , \tag{24.16}$$

---

[1]The formula for the log follows by integration from the simpler proper time integral representation of the propagator, namely $\int_0^{+\infty} ds\, e^{is(a+i\varepsilon)} = \frac{i}{a+i\varepsilon}$.

where we have used the trace identities

$$\text{tr }(\sigma_{01}^{2n+1}) = 0 \tag{24.17}$$

$$\text{tr }(\sigma_{01}^{2n}) = 4\,(-)^n \ . \tag{24.18}$$

Now we are ready to evaluate the remaining matrix element $\langle x|e^{is(P_\mu - eA_\mu)^2}|x\rangle$. To do this let us note that for the field given in (24.15) we have

$$\begin{aligned}(P_\mu - eA_\mu)^2 &= P_0^2 - (P_1 - eEx_0)^2 - P_2^2 - P_3^2 = \\ &= e^{i\frac{P_0P_1}{eE}}\left(P_0^2 - e^2E^2x_0^2 - P_2^2 - P_3^2\right)e^{-i\frac{P_0P_1}{eE}} \ .\end{aligned} \tag{24.19}$$

The last step follows from the translation identity $e^{i\ell P_0}x_0e^{-i\ell P_0} = x_0 - \ell$. Our matrix element is thus equal to

$$\begin{aligned}&\langle x|e^{i\frac{P_0P_1}{eE}}e^{is(P_0^2-e^2E^2x_0^2-P_2^2-P_3^2)}e^{-i\frac{P_0P_1}{eE}}|x\rangle = \\ &= \int\frac{dp}{(2\pi)^4}\frac{dp'}{(2\pi)^4}\langle x|e^{i\frac{P_0P_1}{eE}}|p\rangle\langle p'|e^{-i\frac{P_0P_1}{eE}}|x\rangle\,\cdot \\ &\quad\cdot\langle p|e^{is(P_0^2-e^2E^2x_0^2-P_2^2-P_3^2)}|p'\rangle \ .\end{aligned} \tag{24.20}$$

The intermediate matrix elements are easily calculated. We have

$$\langle x|e^{i\frac{P_0P_1}{eE}}|p\rangle = e^{-ip_0\left(x_0+\frac{p_1}{eE}\right)+i\vec{p}\cdot\vec{x}} \tag{24.21}$$

$$\langle p'|e^{-i\frac{P_0P_1}{eE}}|x\rangle = e^{ip'_0\left(x_0+\frac{p'_1}{eE}\right)-i\vec{p}'\cdot\vec{x}} \tag{24.22}$$

$$\begin{aligned}\langle p|e^{is(P_0^2-e^2E^2x_0^2-P_2^2-P_3^2)}|p'\rangle &= (2\pi)^3\,\delta\,(\vec{p}-\vec{p}')\,e^{-is(p_2^2+p_3^2)}\,\cdot \\ &\quad\cdot\langle p_0|e^{is(P_0^2-e^2E^2x_0^2)}|p'_0\rangle \ .\end{aligned} \tag{24.23}$$

Three of the integrals in (24.20) are now easily done

$$\int\frac{dp_2dp_3}{(2\pi)^2}\,e^{-is(p_2^2+p_3^2)} = \frac{1}{4\pi is} \tag{24.24}$$

$$\int\frac{dp_1}{2\pi}\,e^{-i(p_0-p'_0)\frac{p_1}{eE}} = eE\,\delta\,(p_0-p'_0) \ . \tag{24.25}$$

Finally we find

$$\begin{aligned}&\langle x|e^{is(P_\mu-eA_\mu)^2}|x\rangle = \\ &= \frac{eE}{8\pi^2is}\int\frac{dp_0}{2\pi}\langle p_0|e^{is(P_0^2-e^2E^2x_0^2)}|p_0\rangle = \\ &= \frac{eE}{8\pi^2is}\,\text{Tr}\,\left(e^{is(P_0^2-e^2E^2x_0^2)}\right) \ .\end{aligned} \tag{24.26}$$

The remaining trace is almost that of a harmonic oscillator. If we define

$$P_0 = \frac{1}{\sqrt{2}}P \tag{24.27}$$

$$x_0 = -\sqrt{2}\,Q \tag{24.28}$$

$$\omega_0 = 2ieE \ , \tag{24.29}$$

then Q and P satisfy the correct commutation relation, *i.e.* $[Q,P]=i$, while the trace in (24.26) becomes

$$
\begin{aligned}
\mathrm{Tr}\,\left(e^{is(P_0^2-e^2E^2x_0^2)}\right) &= \mathrm{Tr}\,\left(e^{is(\frac{1}{2}P^2+\omega_0^2Q^2)}\right) = \\
&= \sum_{n=0}^{\infty} e^{is(n+\frac{1}{2})\omega_0} = \frac{e^{i\frac{s}{2}\omega_0}}{1-e^{is\omega_0}} = \\
&= \frac{i}{2\sin(\frac{s}{2}\omega_0)} = \frac{1}{2\sinh(seE)} \ . \qquad (24.30)
\end{aligned}
$$

Therefore, we have found that

$$
\langle x|e^{is(P_\mu-eA_\mu)^2}|x\rangle = \frac{eE}{16\pi^2 is}\,\frac{1}{\sinh(seE)} \ . \qquad (24.31)
$$

Finally,

$$
M_e(x) = \frac{eE}{4\pi^2 si}\coth(seE) \ . \qquad (24.32)
$$

Note that M_e does not depend on x. This is to be expected — the particles are in a homogenous external field. We have now collected all the pieces of our calculation. Our final result

$$
w(x) = \frac{1}{8\pi^2}\int_0^{+\infty}\frac{ds}{s^2}\,e^{-ism^2}\left(eE\coth(seE)-\frac{1}{s}\right) \ . \qquad (24.33)
$$

We can take our exact calculation one step further. To do this let us calculate the imaginary part of $w(x)$. This is equal to

$$
\begin{aligned}
\mathrm{Im}\,w(x) &= -\frac{1}{8\pi^2}\int_0^{+\infty}\frac{ds}{s^2}\,\sin(sm^2)\left(eE\coth(seE)-\frac{1}{s}\right) = \\
&= -\frac{1}{8\pi^2}\int_{-\infty}^{0}\frac{ds}{s^2}\,\sin(sm^2)\left(eE\coth(seE)-\frac{1}{s}\right) \ . \qquad (24.34)
\end{aligned}
$$

The last step follows from taking $s\to -s$. We now have

$$
\begin{aligned}
\mathrm{Im}\,w(x) &= -\frac{1}{16\pi^2}\int_{-\infty}^{+\infty}\frac{ds}{s^2}\,\sin(sm^2)\left(eE\coth(seE)-\frac{1}{s}\right) = \\
&= \frac{1}{16\pi^2}\,\mathrm{Im}\int_{-\infty}^{+\infty}\frac{ds}{s^2}\,e^{-ism^2}\left(eE\coth(seE)-\frac{1}{s}\right) = \\
&= \frac{1}{16\pi^2}\,\mathrm{Im}\oint_C\frac{dz}{z^2}\,e^{-izm^2}\left(eE\coth(zeE)-\frac{1}{z}\right) \ . \qquad (24.35)
\end{aligned}
$$

The complex integral above is taken over the contour C shown in Figure 24.1. The poles of the integrand are located on the negative imaginary axis. It is easy to see that all the residues are equal to 1. Therefore,

$$
\mathrm{Im}\,w(x) = \frac{e^2E^2}{8\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^2}\,e^{-\frac{n\pi m^2}{eE}} \ . \qquad (24.36)
$$

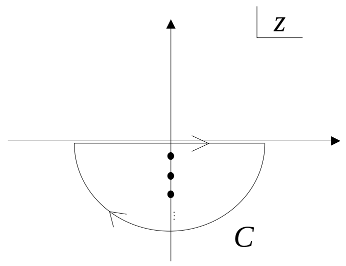


Figure 24.1: Contour and poles for calculating $\operatorname{Im} w(x)$.

We will discuss the character of this solution in the following lecture. At this moment let us just note that the fact that $\operatorname{Im} W \neq 0$ indicates that we are dealing with an unstable theory. Schwinger's solution looks like a sum of instantons, and that is precisely what it is. The instantons correspond to tunneling from the false vacuum $E \neq 0$ to the true vacuum $E = 0$. Said another way, the vacuum in the region where $E \neq 0$, *i.e.* inside the capacitator, is polarized. Virtual electron-positron pairs pick up energy from the electric field, separate and fly of to infinity. In doing this they lower the charge on the 'capacitator plates', thereby decreasing the electric field. After enough time has passed the capacitor is discharged and the system is in the true vacuum $E = 0$.

24.2 Perturbative Solution

Having found Schwinger's exact solution (24.33) let us take a step back and look at what perturbation theory would give us. We will need this result in the following lecture when we spend some time looking at the relation between perturbation theory and exact results in quantum field theory. We could do the perturbative calculation in the usual way, however, it is much simpler to just Taylor expand Schwinger's solution in powers of the coupling e. To do this note that we have

$$\coth t - \frac{1}{t} = \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!}\, t^{2n-1} \ , \tag{24.37}$$

where the B_n's are the Bernoulli numbers generated by

$$\frac{te^t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \ . \tag{24.38}$$

From this formula we see that $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, etc. It also follows that $B_{2n+1} = 0$ for $n = 1, 2, 3, \ldots$. For large n there is also the asymptotic formula

$$B_{2n} \sim (-)^{n+1} \frac{2(2n)!}{(2\pi)^{2n}} \left(1 + o(2^{-2n})\right) \ . \tag{24.39}$$

Using (24.37) we find

$$w(x) = \frac{e^2E^2}{8\pi^2} \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} \mathcal{I}_{2n-3}(\frac{m^2}{eE}) \ , \tag{24.40}$$

where $\mathcal{I}_n(\alpha) = \int_0^{+\infty} dx\, x^n\, e^{-i\alpha x}$. These integrals is easily calculated. For example, remembering the $i\varepsilon$ prescriotion, we have $\mathcal{I}_0 = \frac{1}{i\alpha}$. Also, $\mathcal{I}_{n+1} = i\frac{\partial}{\partial\alpha}\mathcal{I}_n$. Using this we get

$$\mathcal{I}_n = \left(\frac{1}{i\alpha}\right)^{n+1} n! \ , \tag{24.41}$$

for $n \in \mathbb{N}$, as well as $\mathcal{I}_{-1} = -\ln\alpha$, $\mathcal{I}_{-2} = i(\alpha\ln\alpha - \alpha)$, etc. The general property is that the $\mathcal{I}$'s with even indices are imaginary, while those with odd indices are real. Substituting the above values our perturbative result (to all orders in e) is equal to

$$\begin{aligned} w(x) = &-\frac{e^2E^2}{24\pi^2} \ln\left(\frac{m^2}{eE}\right) + \\ &+ \frac{1}{8\pi^2} m^4 \sum_{n=2}^{\infty} (-)^{n+1} \frac{B_{2n}}{(2n)!} (2n-3)! \left(\frac{2eE}{m^2}\right)^{2n} \ . \end{aligned} \tag{24.42}$$

Note that $w(x)$ is real to all orders in perturbation theory. This is to be expected — perturbation theory can't "see" tunneling. Using the asymptotic relation for Bernoulli numbers we find

$$w_{2n} \sim \frac{m^4}{4\pi^2} (2n-3)! \left(\frac{eE}{\pi m^2}\right)^{2n} \ . \tag{24.43}$$

EXERCISES

24.1 Calculate $\operatorname{Im} w(x)$ for the case of Dirac Fermions in $d = 4$ dimensions in a constant magnetic field.

24.2 Re-do the calculation of $\operatorname{Im} w(x)$ in a constant electric field, for the case of Dirac Fermions in $d = 2$ dimensions.

24.3 Re-do the calculation of $\operatorname{Im} w(x)$ in a constant electric field, for the case of scalar particles in d dimensions. Take the exact formula for $w(x)$ and expand it in powers of e to obtain the perturbative solution to all orders.

Lecture 25

Perturbative vs. Exact

25.1 Borel Summation

In quantum field theory we are interested in calculating quantites of the form

$$G(g) = \int [d\phi]\, M\, e^{-I} \ . \tag{25.1}$$

This is in general quite difficult to do. Perturbation theory offers a generic prescription for dealing with these expressions. We first write the action as a sum of a free term and an interaction, *i.e.* $I[\phi] = I_0[\phi] + g\, U[\phi]$, and then expand the integrand in a power series in the coupling constant g so that

$$G(g) = \int [d\phi] \sum_{n=0}^{\infty} \frac{(-)^n}{n!}\, M\, g^n\, U^n\, e^{-I_0} \ . \tag{25.2}$$

The next step is to switch the order of the integration and summation. This switch gives us a *different* quantity

$$G_{\text{pert}}(g) = \sum_{n=0}^{\infty} \frac{(-)^n}{n!}\, g^n \int [d\phi]\, M\, U^n\, e^{-I_0} \ . \tag{25.3}$$

Each of the terms in this sum can be calculated. In practice, however, for large n the job gets progresively more time consuming. We usually calculate only the first few terms in the above perturbative sum. For theories with a small coupling constant the sum of even the first few terms gives good agreement with experiment. Because of this success we tend to forget that $G(g)$ and $G_{\text{pert}}(g)$ are not the same thing.

In this section we shall focus on the question of how big an error we make by using $G_{\text{pert}}(g)$ instead of $G(g)$. Typically, (after renormalization) we get something like

$$G_{\text{pert}}(g) = \sum_{n=0}^{\infty} (-)^n\, n!\, g^n \ . \tag{25.4}$$

For large n we can use Stirling's asymptotic formula $\ln n! \sim n \ln n - n$. The n^{th} term is then $G_n \sim (-\frac{ng}{e})^n$. As we can see, G_{pert} is a divergent series! Dyson has shown that this is true in general: In the complex g plane $G(g)$ has a branch cut along negative real axis, so that $G_{\text{pert}}(g)$ has zero radius of convergence. Therefore, the answer to our question seems to be that we make an infinite error in using $G_{\text{pert}}(g)$ instead of $G(g)$.

The many successes of perturbation theory tell us that something must be wrong with the above conclusion. The series for $G_{\text{pert}}(g)$ is indeed divergent, however, it represents an asymptotic series. The absolute values of the terms in the series at first decrease with n up to $n \approx 1/g$. For still larger values of n the absolute values start increasing without limit. The asymptotic series of a given function has the nice property that it gives a good approximation of the starting function if one truncates the series at the term whose absolute value is smallest. For example, in electrodynamics the coupling constant is $\alpha \approx 1/137$, so we may sum Feynman diagrams up to $n = 137$ vertices, and get better and better aproximations. In this case, perturbation theory is a great success, since even sums to $n = 6$ give results to 10 decimal places. On the other hand, in the case of QCD we have $g \approx 1$, and so perturbation theory is no good. The second answer to our question would seem to be that perturbation theory gives a very small error (if g is small). In fact the absolute value of the error is less than the last term in our truncated series.

Let us remeber that the reason $G(g)$ and $G_{\text{pert}}(g)$ differ is that we interchanged the order of integration and summation. Borel summation attempts to undo this by yet again interchanging a sum and an integral. In $G_{\text{pert}}(g)$ we insert the identity written as $1 = \frac{1}{n!}\int_0^\infty dt\, t^n\, e^{-t}$, and interchange the order of the summation and integration. This gives us a third related quantity, namely

$$G_{\text{borel}}(g) = \int_0^\infty dt\, e^{-t}\, F(tg) \tag{25.5}$$

$$F(\alpha) = \sum_{n=0}^{\infty} \frac{G_n}{n!}\, \alpha^n \; . \tag{25.6}$$

Because of the extra $1/n!$ factor the Borel function F may often converge even though G_{pert} does not. In the case of the simple example given above we have $F(\alpha) = \sum_{n=0}^{\infty}(-)^n \alpha^n = \frac{1}{1+\alpha}$, and so

$$G_{\text{borel}}(g) = \int_0^\infty dt\, \frac{e^{-t}}{1+tg} \; . \tag{25.7}$$

In fact, we find $G_{\text{borel}}(g) = G(g)$, and we have extracted the *exact* result from the perturbative one. In general, if we can sum $F(\alpha)$, and if the associated Borel integral exists, then the theory is said to be Borel summable. In that case we have $G_{\text{borel}}(g) = G(g)$. We are now in the position to give the third, and final, answer to the question of the validity of perturbation theory: If a theory is Borel summable then we can extract exact results from perturbative ones.

25.2 Theories that are not Borel Summable

The prototype of a series that can't be Borel summed is

$$G_{\text{pert}}(g) = \sum_{n=0}^{\infty} n!\, g^n \ . \tag{25.8}$$

This is like the previous example, but without the oscillating factor $(-)^n$. Said differently, this series follows from the previous one by substituting $g \to -g$. We again sum the Borel function (for $|g| < 1$) and find $F(\alpha) = \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$. The Borel integral is now

$$G_{\text{borel}}(g) = \int_0^{\infty} dt\, \frac{e^{-t}}{1 - tg} \ . \tag{25.9}$$

This is now a singular integral, because there is a pole on the path of integration at $t = 1/g$. Borel's prescription does not work. Singularities of this kind are caused by instantons. To see this let us look at the partition function

$$Z(g) = \int [d\phi]\, e^{-I[\phi]} \ , \tag{25.10}$$

for a theory whose action is bounded from bellow. Without loss of generality we may set $I \geq 0$. It now follows that $1 = \int_0^{\infty} dt\, \delta\,(t - I[\phi])$. Inserting this into the expression for the partition function, and performing the obligatory change of order of integrations, we get

$$Z_{\text{borel}}(g) = \int_0^{\infty} dt\, e^{-t}\, F(tg) \tag{25.11}$$

$$F(tg) = \int [d\phi]\, \delta\,[t - I(\phi)] = \sum_{\phi^*} \frac{1}{|\frac{\delta I}{\delta \phi}|}\bigg|_{\phi=\phi^*(t)} \ , \tag{25.12}$$

where $t = I[\phi^*]$. Let ϕ_{inst} be a classical solution of I with $I < \infty$, *i.e.* an instanton. We then have that F is singular at $t = I[\phi_{\text{inst}}] = I_{\text{inst}}$.

As a further example, let us look at a simple model whose action is not bounded from bellow. The partition function given by

$$Z(g) = \int_{-\infty}^{+\infty} dx\, e^{-x^2 + gx^3} \ , \tag{25.13}$$

is obviously ill defined, but let us proceed naively and calculate it perturbatively. We have

$$Z_{\text{pert}}(g) = \sum_{n=0}^{\infty} \frac{g^{2n}}{(2n)!} \int_{-\infty}^{+\infty} dx\, x^{6n}\, e^{-x^2} = \sum_{n=0}^{\infty} \frac{g^{2n}}{(2n)!}\, \Gamma(3n + 1/2) \ . \tag{25.14}$$

The Borel function is thus

$$\sum_{n=0}^{\infty} \frac{1}{(2n)!\, n!}\, \Gamma(3n + 1/2)\, \alpha^n \ . \tag{25.15}$$

Stirling's formula gives $\ln \frac{\Gamma(3n+1/2)}{(2n)!\,n!} \sim n \ln \frac{27}{4}$, so that we find

$$Z_{\text{borel}}(g) = \int_0^\infty dt\, e^{-t}\, F(tg^2) = \int_0^\infty dt\, \frac{e^{-t}}{1 - 27/4\, tg^2} \;. \tag{25.16}$$

Although each term in the perturbative expansion makes sense, the Borel integral is divergent — there is a pole at $t = \frac{4}{27g^2}$. This should not be surprising as $Z(g)$ was divergent to begin with. What is nice is that, even in this case, the pole in the Borel integral is due to the instanton, since $I_{\text{inst}} = \frac{4}{27g^2}$.

It is possible to modify the Borel technique by deforming the contour of integration in some way. The important thing to note is that there are many inequivalent ways to do this. For example, for the deformed contour of Figure 25.1 the series given in (25.8) gives

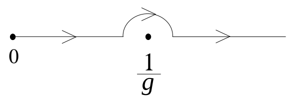


Figure 25.1: One deformation of the Borel contour

$$\text{Im}\, G_{\text{borel}}(g) = \frac{\pi}{g}\, e^{-\frac{1}{g}} \;. \tag{25.17}$$

On the other hand, bypassing the pole from bellow, as in Figure 25.2, we get

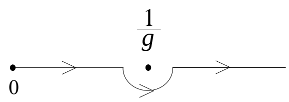


Figure 25.2: Another deformation of the Borel contour

$$\text{Im}\, G_{\text{borel}}(g) = -\,\frac{\pi}{g}\, e^{-\frac{1}{g}} \;. \tag{25.18}$$

Taking the principal value of $G_{\text{borel}}(g)$ we average between the two previous cases, and we get $\text{Im}\, G_{\text{borel}}(g) = 0$. In all these cases the real part of Borel sum remains the same, namely

$$\text{Re}\, G_{\text{borel}}(g) = \text{P} \int_0^{+\infty} dt\, \frac{e^{-t}}{1 - gt} \;. \tag{25.19}$$

25.3 Getting Around Instantons

In the previous lecture we looked at charged particles in a fixed constant electric field E. For Dirac Fermions in $d = 4$ we found

$$W = \frac{1}{8\pi^2}\, \Omega \int_0^{+\infty} \frac{ds}{s^2}\, e^{-ism^2} \left(eE \coth(seE) - \frac{1}{s} \right) \;. \tag{25.20}$$

W is the generating functional of connected graphs, and Ω is the volume of space-time in which the electric field is equal to E. The above expression has a small imaginary part given by

$$\mathrm{Im}\, W = \frac{e^2 E^2}{8\pi^3}\, \Omega \sum_{n=1}^{\infty} \frac{1}{n^2}\, e^{-\frac{n\pi m^2}{eE}} \ . \tag{25.21}$$

The fact that $\mathrm{Im}\, W \neq 0$ is a signal of the quantum instability of the $E \neq 0$ state. To see this let us look at the vacuum inside this capacitator. The vacuum in a quantum field theory is a seething mass of virtual particles. At some point, a virtual e^+e^- pair can get energy from the electric field and be promoted into a pair of real particles. The particles then separate to infinity and thereby decrease the charges on the "condenser plates", *i.e.* decrease the external electric field. A pair of particles of mass m interact over a typical distance $\frac{1}{m}$. In doing this they soak up from the electric field the ammount of energy $\frac{eE}{m}$. In order to make a pair of physical particles, this must greater than $2m$ (the rest mass of the two). Therefore, the probability that a single virtual pair is promoted into a pair of real particles must be given in terms of the ratio $\frac{m^2}{eE}$. This probability must be negligible for small E, and must tend to 1 for $E \to \infty$. From equation (25.21) we see that it is, in fact, equal to $e^{-\frac{\pi m^2}{eE}}$. Subdominant terms like $e^{-\frac{n\pi m^2}{eE}}$ correspond to the simultaneous extraction of n e^+e^- pairs from the vacuum.

Vacuum polarization is a tunneling, or instanton effect. We have seen that perturbative calculations miss these kind of effects. In the above case we found

$$\begin{aligned}\frac{W}{\Omega} = &-\frac{e^2E^2}{24\pi^2} \ln\left(\frac{m^2}{eE}\right) + \\ &+ \frac{1}{8\pi^2}\, m^4 \sum_{n=2}^{\infty} (-)^{n+1} \frac{B_{2n}}{(2n)!}\, (2n-3)! \left(\frac{2eE}{m^2}\right)^{2n} \ .\end{aligned} \tag{25.22}$$

Not surprisingly, perturbation only sees the real part of W. Using the asymptotic relation for Bernoulli numbers, to leading order the above perturbative result may be written as

$$\begin{aligned}\frac{W}{\Omega} \sim &-\frac{e^2E^2}{24\pi^2} \ln\left(\frac{m^2}{eE}\right) + \\ &+ \sum_{n=2}^{\infty} \frac{m^4}{4\pi^2}\, (2n-3)! \left(\frac{eE}{\pi m^2}\right)^{2n} \ .\end{aligned} \tag{25.23}$$

Note that this is of the same form as equation (25.8). For the same reason the above sum is also not Borel summable. This is also to be expected — the culprits are again instantons.

Before we proceed further, let us note that vacuum polarization exists for all types of particles in all dimensions. For example, for charged scalar particles in d

dimensions we find

$$\mathrm{Im}\,\frac{W}{\Omega} = -\pi \left(\frac{eE}{4\pi i}\right)^{d/2} \sum_{n=1}^{\infty} (-)^n \frac{1}{n^{d/2}} \exp\left(-\frac{m^2\pi}{eE} n\right) . \tag{25.24}$$

If we introduce the notation $X = \exp\left(-\frac{m^2\pi}{eE}\right)$ then we have

$$\mathrm{Im}\,\frac{W}{\Omega} = \begin{cases} \frac{\pi X}{1+X} & \text{for d=0} \\ \frac{eE}{4\pi} \ln(1+X) & \text{for d=2} \end{cases} \tag{25.25}$$

On the other hand, perturbation theory gives us

$$\frac{W_{\mathrm{pert}}}{\Omega} \sim -2 \left(\frac{m^2}{4\pi}\right)^{d/2} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (-)^k \, k^{-2n} \, (2n - d/2 - 1)! \left(\frac{eE}{\pi m^2}\right)^{2n} . \tag{25.26}$$

As before, we have used an asymptotic formula for Bernulli numbers in order to get the above expression. It is easy to see that the subleading terms that have been dropped sum into a small analytic expression that will not interest us. All the instanton effects are in the above sum. Borel summation gives us

$$\frac{W_{\mathrm{borel}}}{\Omega} = \int_0^{\infty} dt \, e^{-t} \, F \tag{25.27}$$

$$F = 2 \left(\frac{m^2}{4\pi}\right)^{d/2} \sum_{k=1}^{\infty} (-)^k \frac{t^{-d/2-1} \, t_k^2}{(t - t_k)(t + t_k)} , \tag{25.28}$$

with the poles on the Borel contour given by $t_k = \frac{m^2\pi}{eE} k$. As we have already anticipated, (25.26) is not Borel summable. However, if we re-instate the $i\varepsilon$ term in the usual way, *i.e.* $m^2 \to m^2 - i\varepsilon$ this moves all the poles off the contour of integration. The Borel integral is now easily done, and one gets

$$\mathrm{Im}\,\frac{W_{\mathrm{borel}}}{\Omega} = -\pi \left(\frac{eE}{4\pi i}\right)^{d/2} \sum_{n=1}^{\infty} (-)^n \frac{1}{n^{d/2}} \exp\left(-\frac{m^2\pi}{eE} n\right) . \tag{25.29}$$

This is in fact our exact result! It is important to note that the same modification of the Borel proceedure works for all dimensions and all spins. To conclude, we note that it is possible to modify Borel summation by using the standard $i\varepsilon$ prescription. This modification allows us to extract non-perturbative information (tunelling) from perturbative results. In this way the modified Borel proceedure highlights the relation between tunelling and instantons in yet another way.

EXERCISES

25.1 Start from the perturbative result for Dirac Fermions given in (25.22). Show that it differs from the sum of leading terms (25.23) by a small analytic function. Using the modified Borel prescription sum (25.23) and obtain the exact result (25.21).

Lecture 26

Quantizing Gauge Theories

26.1 Faddeev-Popov Quantization

In this lecture we finally get around to the task of quantizing gauge theories. We shall assume that there are no gauge anomalies. Therefore, we have $I[A^G] = I[A]$ and $[dA]^G = [dA]$. Because of this symmetry we are performing an infinite overcounting of physical configurations. Therefore, the naive deffinition of Z

$$Z = \int [dA] e^{iI[A]} \ , \tag{26.1}$$

is horribly divergent. We will get rid of this problem by choosing an appropriate gauge fixing condition $\chi[A] = 0$ that picks out one field on each gauge orbit. To do this, we first consider the expression

$$\triangle_{\text{fp}}^{-1}[A] = \int [dG] \, \delta[\chi[A^G]] \ , \tag{26.2}$$

where $[dG]$ is the Haar measure, and so $[dG] = [dGH]$. From this definition it follows that $\triangle_{\text{fp}}[A]$ is gauge invariant. Namely,

$$\begin{aligned} \triangle_{\text{fp}}^{-1}[A^H] &= \int [dG] \, \delta[\chi[(A^H)^G]] = \\ &= \int [dGH] \, \delta \left[\chi[A^{GH}]\right] = \int [dG] \, \delta \left[\chi[A^G]\right] = \triangle_{\text{fp}}^{-1}[A] \ . \end{aligned} \tag{26.3}$$

Following Faddeev and Popov let us insert the identity

$$1 = \triangle_{\text{fp}}[A] \int [dG] \, \delta \left[\chi[A^G]\right] \tag{26.4}$$

into our naive expression for Z. We now have

$$Z = \int [dA][dG] \ \triangle_{\text{fp}} \, [A] \, \delta \left[\chi[A^G]\right] e^{iI[A]} =$$

$$= \int [dA]^G [dG] \; \triangle_{\rm fp}\, [A^G]\, \delta\, [\chi[A^G]] e^{iI[A^G]} =$$
$$= \int [dG] \int [dA] \; \triangle_{\rm fp}\, [A]\, \delta\, [\chi[A]] e^{iI[A]} \; . \qquad (26.5)$$

Note that $\int [dG]$ has factored out. This is an infinite constant term. It is precisely the source of our overcounting. Therefore, the correct definition of the partition function is

$$Z = \int [dA] \; \triangle_{\rm fp}\, [A]\, \delta\, [\chi[A]] e^{iI[A]} \; . \qquad (26.6)$$

To evaluate $\triangle_{\rm fp}$ note that

$$\delta\, [\chi(A^G)] = \frac{1}{\det\left(\frac{\delta\chi^G}{\delta G}\right)}\, \delta[G-1] \; . \qquad (26.7)$$

This is just the generalization of the well known delta function identity $\delta\,(f(x)) = \frac{1}{f'}\,\delta\,(x - x^*)$ where $f(x^*) = 0$. As a consequence of this we get

$$\triangle_{\rm fp}[A] = \det\left(\left.\frac{\delta\chi^G}{\delta G}\right|_{G=1}\right) = \det\left(\left.\frac{\delta\chi^\omega}{\delta\omega}\right|_{\omega=0}\right) \; . \qquad (26.8)$$

In the last step we have written the gauge elements as $G = e^{i\omega}$. $\triangle_{\rm fp}[A]$ is called the Faddeev-Popov determinant. In fact, it is just the Jacobian for the transformation from the gauge in which we have only physical (transverse) fields to the general gauge given by $\chi[A] = 0$. In general, the Faddeev-Popov determinant is quite easy to calculate. For example, for electrodynamics in the Lorentz gauge we have $\chi = \partial_\mu A^\mu$. The gauge transformation is $\delta\, A^\mu = \partial^\mu \omega$, so

$$\chi^\omega = \partial_\mu (A^\mu + \partial^\mu \omega) = \partial^2 \omega \; . \qquad (26.9)$$

In the second step we have used the gauge fixing condition $\chi = 0$. Therefore, in this case the Faddeev-Popov determinant is simply

$$\triangle_{\rm fp} = \det(\partial^2) \; . \qquad (26.10)$$

This is a quite trivial example. The Faddeev-Popov determinant does not depend on A, and may be dropped, as it will not influence normalized Green's functions. In general, however, the Faddeev-Popov determinant depends on A. By introducing a pair of auxilliary anticommuting fields c and $\bar{c}$ we may write it as

$$\triangle_{\rm fp} = \det\left(\frac{\delta\chi^\omega}{\delta\omega}\right) = \int [dc d\bar{c}] \; \exp\left(\int dx\, \bar{c}\, \frac{\delta\chi^\omega}{\delta\omega}\, c\right) \; . \qquad (26.11)$$

The auxilliary fields c, and $\bar{c}$ are called ghosts. They are anti-comuting bosonic fields. This is not a violation of the spin-statistics law that states that bosons commute and fermions anti-commute. That law is only applicable for physical fields. The ghosts are auxilliary fields. They exist inside Feynman diagrams, but not as external lines[1]. By introducing one further auxilliary field — the Lagrange multiplier field b we can write the gauge fixing delta functional as

$$\delta[\chi] = \int [db]\, e^{i \int dx\, b\chi} \ . \tag{26.12}$$

The partition function now becomes

$$Z = \int [dA][dcd\bar{c}][db] \ \exp\left(iI - i \int dx\, \bar{c}\, \frac{\delta\chi^\omega}{\delta\omega}\, c + i \int dx\, b\chi \right) \ . \tag{26.13}$$

This is the most convenient form of the partition function to work with. Before we finish this section let us make contact with a related form for Z. Consider a class of gauge fixings given by $\chi[A] = \lambda(x)^2$. In the new gauge, the partition function is

$$Z_\lambda = \int [dA] \triangle_{\rm fp} [A]\, \delta\, [\chi - \lambda] e^{iI} \ . \tag{26.14}$$

The Faddeev-Popov determinant is the same as before, *i.e.* it does not depend on λ. By taking the average over all λ's with Gaussian weights $\exp\left(-\frac{i}{2a}\int dx\, \lambda^2\right)$ we find

$$\begin{aligned} Z_a = \int [dA][d\lambda] \ \triangle_{\rm fp} [A] \ \exp\left(-\frac{i}{2a} \int dx\, \chi^2 \right) \, \delta\, [\chi - \lambda] e^{iI} = \\ = \int [dA] \ \triangle_{\rm fp} [A] \ \exp i \left(I - \frac{1}{2a} \int dx\, \chi^2 \right) \ . \end{aligned} \tag{26.15}$$

For historic reasons the choice of weight factor a is also called a choice of gauge. For electrodynamics in the Lorentz gauge we find

$$Z_a = \int [dA]\, e^{iI_{\rm eff}} \ . \tag{26.16}$$

We have dropped the constant Faddeev-Popov determinant. The effective Lagrangian is

$$\begin{aligned} \mathcal{L}_{eff} &= -\frac{1}{4}\, F_{\mu\nu} F^{\mu\nu} - \frac{1}{2a}\, (\partial \cdot A)^2 = \\ &= \frac{1}{2}\, A^\mu \left(\eta_{\mu\nu} \partial^2 - (1 - \frac{1}{a}) \partial_\mu \partial_\nu \right) A^\nu \ . \end{aligned} \tag{26.17}$$

The choice $a = 1$ is called the Feynman gauge, while $a = 0$ is known as the Landau gauge.

[1]The reason for this is quite simple — we do not couple ghosts to external fields. In fact, their only reason for being is to allow us to write the Faddeev-Popov determinant in a convenient form.

[2]If $\chi = 0$ is a good gauge, then it is easy to show that (for any $\lambda(x)$) $\chi = \lambda$ also represents a good gauge choice.

26.2 BRST Symmetry

Let us consider a gauge theory given by the action $I[\varphi]$ and gauge fields φ^i, where $i = 1, 2, \cdots, n$. The model is invariant under the infinitesimal gauge transformations

$$\delta\varphi^i = T^i{}_\alpha \xi^\alpha \ . \tag{26.18}$$

The gauge transformations are indexed by $\alpha = 1, 2, \cdots, m$, and $m < n$. These transformations can be written as

$$\delta\varphi^i = [\xi^\alpha \Gamma_\alpha, \varphi^i] \ , \tag{26.19}$$

where $\Gamma_\alpha = T^i{}_\alpha \delta_i$ are the associated generators. We have introduced the shorthand notation $\delta_i = \frac{\delta}{\delta\varphi^i}$. Invariance of the action under (26.18) gives us the Noether identities

$$\delta_i \, I \, T^i{}_\alpha \xi^\alpha \ . \tag{26.20}$$

If we introduce $F_i = \delta_i \, I[\varphi]$, then the Noether identities can be written as

$$F_i \, T^i{}_\alpha = 0 \ . \tag{26.21}$$

If the expressions $T^i{}_1, T^i{}_2, \cdots, T^i{}_m$ are linearly independent the associated gauge theory is said to be irreducible. On the other hand, the gauge algebra is closed if

$$[\Gamma_\alpha, \Gamma_\beta] = f_{\alpha\beta}{}^\gamma \Gamma_\gamma \ . \tag{26.22}$$

If this is the case then we not only have a gauge algebra, but it also exponentiates into a gauge group. Note that, in general, the $f_{\alpha\beta}{}^\gamma$ are not constants, but depend on the fields.

Gauge fixing picks a unique representative in each class of equivalent gauge fields. We do this by imposing conditions

$$\chi_\alpha[\varphi] = 0 \ , \tag{26.23}$$

one for each independent parameter of the gauge symmetry. χ must be such that for every φ^i there is a unique ξ^α such that

$$\chi_\alpha[\varphi^i + T^i{}_\alpha \xi^\alpha] = 0 \ . \tag{26.24}$$

Taylor expanding this we find $\chi_\alpha[\varphi^i] = -\delta_i \, \chi_\alpha T^i{}_\beta \xi^\beta$. We need to be able to solve this in terms of ξ. To do this we must be able to invert the matrix $\delta_i \, \chi_\alpha T^i{}_\beta$. Therefore, a good gauge fixing satisfies

$$\det(\delta_i \chi_\alpha T^i{}_\beta) \neq 0 \ . \tag{26.25}$$

For an irreducible gauge theory with closed algebra we can use the Faddeev-Popov procedure to obtain the correct partition function

$$\begin{aligned} Z &= \int [d\varphi] \, \det(\delta_i \chi_\alpha T^i{}_\beta) \, \delta \, [\chi[\varphi]] \, e^{iI[\varphi]} = \\ &= \int [d\varphi][dc d\bar{c}][db] \, \exp\left(I_{\text{eff}}[\varphi, c, \bar{c}, b]\right) \ . \end{aligned} \tag{26.26}$$

The effective action is

$$I_{\text{eff}} = I[\varphi] + \int dx \, \left(b^\alpha \chi_\alpha - i\bar{c}^\alpha \, \delta_i \, \chi_\alpha T^i{}_\beta c^\beta \right) \ . \tag{26.27}$$

For later convenience let us list some properties of the above fields. φ, c, b are Hermitian, $\bar{c}$ is anti-Hermitian. Under a φ, b are Grassmann even, $c, \bar{c}$ are Grassmann odd. The effective action I_{eff} is also invariant under a $U(1)$ rotation of ghost fields

$$c \ \rightarrow \ e^{i\psi} c \tag{26.28}$$

$$\bar{c} \ \rightarrow \ e^{-i\psi} \bar{c} \ . \tag{26.29}$$

The corresponding conserved charge is called the ghost number. The fields have the following ghost number assignments: $g(\varphi) = g(b) = 0$, $g(c) = 1$ and $g(\bar{c}) = -1$. The effective action has $g(I_{\text{eff}}) = 0$. The above symmetry is relatively trivial. In addition, the quantum theory is invariant under an important symmetry — a remnant of gauge symmetry — called BRST invariance. The action of the BRST transformation on gauge fields follows directly from the gauge transformations (26.18) through the substitution $\xi^\alpha \rightarrow c^\alpha \theta$. Here θ is a Grassmann odd constant. Therefore $\delta\varphi^i \rightarrow \delta_{\text{brst}}\varphi^i = s\varphi^i\theta$. From this it follows that s is a Grassmann odd operator, and that

$$s\varphi^i = T^i{}_\alpha c^\alpha \ . \tag{26.30}$$

BRST transformations turn gauge fields into ghosts. As a consequence of gauge invariance we have $sI[\varphi] = 0$.

The BRST variation s corresponds to infinitesimal transformations, so it satisfies the graded Leibnitz rule. A consequence of this is the chain rule

$$s\chi_\alpha = \partial_i \chi_\alpha \, s\varphi^i \ . \tag{26.31}$$

Using this we may write the effective action as

$$I_{\text{eff}} = I[\varphi] + \int dx \, \left(b^\alpha \chi_\alpha - i\bar{c}^\alpha s\chi_\alpha \right) \ . \tag{26.32}$$

Now we see that it is useful to impose that $s\bar{c}^\alpha = -ib^\alpha$, *i.e.* anti-ghosts transform into Lagrange multipliers. With this we get

$$I_{\text{eff}} = I[\varphi] + i \int dx \, s \left(\bar{c}^\alpha \chi_\alpha \right) \ . \tag{26.33}$$

We want BRST to be an invariance of the full action I_{eff}, that is, we want $sI_{\text{eff}} = 0$. Therefore, we need to have $s^2(\bar{c}^\alpha \chi_\alpha) = 0$. We will now elevate this nilpotence to an operator statement $s^2 = 0$. This will completely determine the action of s on all the remaining fields.

Note that $s\varphi^i$ mimicks gauge invariance, while $s\bar{c}^\alpha = -ib^\alpha$ as well as its consequence (due to nilpotence) $sb^\alpha = 0$ are purely kinematic, *i.e.* don't depend on the gauge algebra. The algebra comes in from the action of s on the ghost fields.

Before we do this let us derive two consequences of our our gauge algebra. Closure gives

$$T^i{}_\alpha\, \delta_i\, T^j{}_\beta - T^i{}_\beta\, \delta_i\, T^j{}_\alpha = f_{\alpha\beta}{}^\gamma T^j{}_\gamma \ , \tag{26.34}$$

while the Jacobi identity $[[\Gamma_\alpha, \Gamma_\beta], \Gamma_\gamma] + \text{cyclic} = 0$ gives[3]

$$f_{\alpha\beta}{}^\delta f_{\delta\gamma}{}^\varepsilon + \text{cyclic in } \alpha, \beta, \gamma = 0 \ . \tag{26.35}$$

We continue with our derivation of BRST transformations. We impose $s^2\varphi^i = 0$. This gives $0 = s(T^i{}_\alpha c^\alpha) = \delta_j\, T^i{}_\alpha\, s\varphi^j c^\alpha + T^i{}_\alpha s c^\alpha$. Therefore,

$$\begin{aligned} T^i{}_\alpha s c^\alpha &= -\delta_j\, T^i_\alpha T^j{}_\beta c^\beta c^\alpha = \\ &= -\frac{1}{2}\left(\delta_j\, T^i{}_\alpha T^j{}_\beta - \delta_j\, T^i{}_\beta T^j{}_\alpha\right) c^\beta c^\alpha == -\frac{1}{2} f_{\beta\alpha}{}^\gamma T^i{}_\gamma c^\beta c^\alpha \ . \end{aligned} \tag{26.36}$$

The last step follows from closure. Finaly we find

$$s c^\gamma = -\frac{1}{2} f_{\alpha\beta}{}^\gamma c^\alpha c^\beta \ . \tag{26.37}$$

We have completely determined the action of s on all fields. Let us check if nilpotency realy works. To do that we calculate

$$\begin{aligned} s^2 c^\alpha &= -\frac{1}{2} f_{\alpha\beta}{}^\gamma \left(s c^\alpha c^\beta - c^\alpha s c^\beta\right) = \\ &= -f_{\alpha\beta}{}^\gamma s c^\alpha c^\beta = \frac{1}{2} f_{\alpha\beta}{}^\gamma f_{\delta\varepsilon}{}^\alpha c^\delta c^\varepsilon c^\beta = 0 \ . \end{aligned} \tag{26.38}$$

The last step follows from the Jacobi identity, so s is indeed nilpotent.

To recapitulate — we have determined the BRST transformations to be

$$s\varphi^i = T^i{}_\alpha c^\alpha \tag{26.39}$$

$$s c^\alpha = -\frac{1}{2} f_{\beta\gamma}{}^\alpha c^\beta c^\gamma \tag{26.40}$$

$$s\bar{c}^\alpha = -i b^\alpha \tag{26.41}$$

$$s b^\alpha = 0 \ . \tag{26.42}$$

These transformations are an invariance of I_{eff}. The BRST variation s is nilpotent and satisfies the graded Leibnitz rule. The fields are shown in Figure general-fields The above transformations are generated by the BRST charge

$$Q = \int dx\, \left(c^\alpha \Gamma_\alpha - \frac{1}{2} f_{\alpha\beta}{}^\gamma c^\alpha c^\beta \frac{\delta}{\delta c^\gamma} - i b^\alpha \frac{\delta}{\delta \bar{c}^\alpha}\right) \ . \tag{26.43}$$

Therefore, we have $sA = [Q, A\}$ for all fields A. The nilpotence of the BRST variation s implies the nilpotence of the BRST charge, *i.e.* $Q^2 = 0$. Q is the conserved charge corresponding to the Noether current J_{brst}. In the operator formalism, after quantization, Q is given in terms of operators. In general we find that the quantized BRST operator is no longer nilpotent. In fact, the BRST operator will remain nilpotent if and only if our theory has no gauge anomalies.

[3] Here we have used the linear independence of the T's.

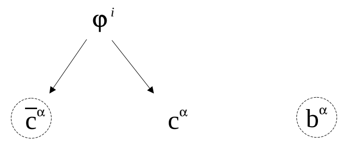


Figure 26.1: Fields for an irreducible theory with closed gauge algebra.

26.3 Examples

As a first example let us consider electrodynamics. Here

$$I = -\frac{1}{4}\int dx\, F_{\mu\nu}F^{\mu\nu} \ , \tag{26.44}$$

and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The gauge symmetry is simply

$$\delta A_\mu = \partial_\mu \xi \ , \tag{26.45}$$

so that $i = \mu$, $\alpha = 1$, and $T^i{}_\alpha = \partial_\mu$. There is only one T, so it is irreducible. There is also only one generator Γ so the gauge algebra is trivialy closed. The fields are shown Figure 26.2 Quantum electrodynamics is invariant under the following

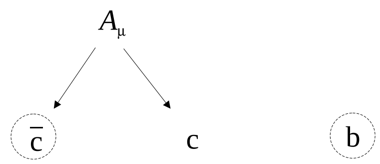


Figure 26.2: Fields for electrodynamics

BRST transformations

$$\begin{aligned} sA_\mu &= \partial_\mu c && (26.46)\\ sc &= 0 && (26.47)\\ s\bar{c} &= -ib && (26.48)\\ sb &= 0 \ . && (26.49)\end{aligned}$$

The full action may be written as $I_{\text{eff}} = I[A] + s\Psi$, where the Grassmann odd functional Ψ is called the gauge Fermion. It determines the gauge fixing that one uses. The usual choice of gauge Fermion is

$$\Psi = i\int dx\, \bar{c}\chi[A] \ , \tag{26.50}$$

where $\chi[A] = 0$ is the gauge fixing. In the Lorentz gauge we have $\chi = \partial^\mu A_\mu$, and then

$$I_{\text{eff}} = I[A] + \int dx \left(b\chi - i\bar{c}\frac{\delta\chi}{\delta A_\mu}\partial_\mu c \right) \ . \tag{26.51}$$

This is precisely what we get from Faddeev-Popov quantization. Another choice of gauge Fermion is

$$\Psi = -i \int dx\, \bar{c} \left(\frac{1}{2}\, ab - \chi[A] \right) \ , \tag{26.52}$$

where a is a constant. The BRST variation of this gauge Fermion gives

$$s\Psi = -\int dx\, b \left(\frac{1}{2}\, a\, b - \chi \right) - i \int dx\, \bar{c} s \chi \ . \tag{26.53}$$

We next integrate out the b field to get $a\, b = \chi$. Then, the full action becomes

$$I_{\text{eff}} = I[A] + \frac{1}{2a^2} \int dx\, \chi^2 + \text{Faddeev} - \text{Popov piece} \ . \tag{26.54}$$

As before, for $\chi = \partial \cdot A$, $a = 1$ is the Feynman gauge, $a = 0$ the Landau gauge. Let us see what has happened to the BRST symmetry. $sA_\mu = \partial_\mu c$ stays the same, as does $sc = 0$. The BRST transformation of the anti-ghost field now goes over into $s\bar{c} = -i\frac{1}{a}\chi$, while sb is no longer defined, since we have integrated over it. Checking nilpotence we now find

$$s^2 \bar{c} = \frac{1}{a} \frac{\delta I_{\text{eff}}}{\delta \bar{c}} \ . \tag{26.55}$$

This is zero only on-shell, *i.e.* on the equations of motion. This is the price we pay for using equations for the b field.

For a more interesting example we now consider QCD. The gauge symmetry is now $\delta A^a_\mu = D^{ab}_\mu \xi^b$, so that $i = \{\mu, a\}$, $\alpha = b$, $T^i{}_\alpha = D_\mu{}^{ab}$ and $f_{\alpha\beta}{}^\gamma = \varepsilon^{abc}$. The BRST transformations are now

$$\begin{aligned} sA^a_\mu &= D_\mu{}^{ab} c^b && (26.56)\\ sc^a &= -\frac{1}{2}\, \varepsilon^{abc} c^b c^c && (26.57)\\ s\bar{c}^a &= -ib^a && (26.58)\\ sb^a &= 0\ . && (26.59) \end{aligned}$$

For example, for $\Psi = i \int dx\, \bar{c}^a \partial^\mu A^a_\mu$ we get

$$I_{\text{eff}} = I[A] + \int dx\, \left(b^a \partial^\mu A^a_\mu - i \bar{c}^a \partial^\mu D^{ab}_\mu c^b \right) \ . \tag{26.60}$$

Note that now we have ghost-gauge coupling since $D^{ab}_\mu = \delta^{ab}\partial_\mu + g\varepsilon^{acb} A^c_\mu$. The ghost-gauge coupling is shown in Figure 26.3 In this case ghost fields can't be just dropped, or we would loose unitarity.

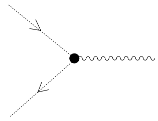


Figure 26.3: Ghost-gauge coupling for QCD in the Lorentz gauge.

26.4 The $U(1)$ Antisymmetric Tensor Model

The Faddeev-Popov method generalizes to off-shell reducible gauge theories. To see an example of this we will consider the following model

$$I = -\frac{1}{8}\int dx\ (\varepsilon^{\mu\nu\rho\sigma} B_{\mu\nu} F_{\rho\sigma} - A^\mu A_\mu)\ . \tag{26.61}$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The antisymmetric tensor $B^{\mu\nu}$ is the gauge field, while A^μ is an auxilliary field. This action is invariant under

$$\delta_0 B^{\mu\nu} = \partial^\mu \xi^\nu - \partial^\nu \xi^\mu\ . \tag{26.62}$$

Let us check this

$$\begin{aligned} \delta_0 I &= -\frac{1}{8}\int dx\, \varepsilon^{\mu\nu\rho\sigma}(\partial_\mu \xi_\nu - \partial_\nu \xi_\mu) F_{\rho\sigma} = \\ &= \frac{1}{4}\int dx\, \varepsilon^{\mu\nu\rho\sigma} \xi_\nu \partial_\mu F_{\rho\sigma} = \frac{1}{2}\int dx\, \varepsilon^{\mu\nu\rho\sigma} \xi_\nu \partial_\mu \partial_\rho A_\sigma = 0\ . \end{aligned} \tag{26.63}$$

The indices are $i = [\mu,\nu]$ (antisymmetrized), $\alpha_1 = \rho$, $T^i{}_{\alpha_1} = T^{\mu\nu}{}_\rho = \partial^\mu \delta^\nu_\rho - \partial^\nu \delta^\mu_\rho$. These T's are not linearly independent, since $T^{\mu\nu}{}_\rho T^\rho = 0$ where $T^\rho = \partial^\rho$. In the general notation we would write $T^i{}_{\alpha_1} T^{\alpha_1}{}_{\alpha_2} = 0$. Now, however, $\alpha_2 = 1$. Since there is only one T^ρ the theory is not further reducible. The reducibility doesn't depend on the equations of motion, *i.e.* it is off-shell.

For BRST we proceed as with irreducible theories

$$\begin{aligned} s_0 B^{\mu\nu} &= \partial^\mu c^\nu - \partial^\nu c^\mu && (26.64) \\ s_0 c^\mu &= 0 && (26.65) \\ s_0 \bar{c}^\mu &= -i b'^\mu && (26.66) \\ s_0 b'^\mu &= 0\ . && (26.67) \end{aligned}$$

The notation b' will be usefull later. Gauge fixing this according to $\chi_\mu = \partial^\nu B_{\mu\nu} = 0$ corresponds to using the gauge Fermion

$$\Psi = i\int dx\, \bar{c}^\mu \chi_\mu = i\int dx\, \bar{c}^\mu \partial^\nu B_{\mu\nu}\ . \tag{26.68}$$

Thus, the effective action becomes

$$I_{\text{eff}} = I + s\Psi = I + \int dx\ \left(b'^\mu \partial^\nu B_{\mu\nu} - i\bar{c}^\mu(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)c^\nu\right)\ . \tag{26.69}$$

Note, however, that the Faddeev-Popov term is now itself gauge invariant! This should not surprise us since we have seen that the theory is reducible. We have two invariances

$$\delta_1 c^\mu = \partial^\mu \xi \tag{26.70}$$
$$\delta_1 \bar{c}^\mu = 0 \ , \tag{26.71}$$

and

$$\delta_1^E c^\mu = 0 \tag{26.72}$$
$$\delta_1^E \bar{c}^\mu = \partial^\mu \xi^E \ . \tag{26.73}$$

For the first invariance we go to BRST in the standard way $\xi \to c\theta$. The new ghost field c is Grassmann even and has ghost number $g(c) = 2$. We gauge fix this with $\chi = \partial_\mu c_\mu$. Therefore, $g(\chi) = 1$ and $g(b) = -1$. Similarly,for the second invariance we have $\xi^E \to c_E\theta$. From this we find $g(c_E) = 0$. We gauge fix with $\chi_E = \partial_\mu \bar{c}^\mu$. It follows that $g(b_E) = 1$. Using the fact that s acting on anti-ghosts gives -1 times the appropriate Lagrange multiplier, we find $g(\bar{c}) = g(b) - 1 = -2$, and $g(\bar{c}_E) = g(b_E) - 1 = 0$. The fields needed are depicted in Figure 26.4 The

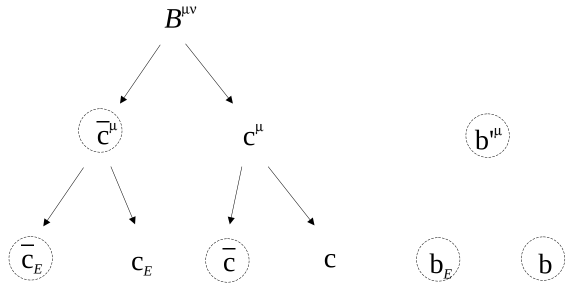


Figure 26.4: Fields for $U(1)$ gauge theory of the antisymmetric tensor field.

BRST transformations at this level are

$$s_1 c^\mu = \partial^\mu c \tag{26.74}$$
$$s_1 \bar{c}^\mu = \partial^\mu c_E \tag{26.75}$$
$$s_1 c = 0 \tag{26.76}$$
$$s_1 c_E = 0 \tag{26.77}$$
$$s_1 \bar{c} = -ib \tag{26.78}$$
$$s_1 \bar{c}_E = -ib_E \tag{26.79}$$
$$s_1 b = 0 \tag{26.80}$$
$$s_1 b_E = 0 \ . \tag{26.81}$$

The full BRST is simply $s = s_0 + s_1$, therefore

$$sB^{\mu\nu} = \partial^\mu c^\nu - \partial^\nu c^\mu \tag{26.82}$$

$$sc^\mu = \partial^\mu c \tag{26.83}$$
$$s\bar{c}^\mu = -ib'^\mu + \partial^\mu c_E \tag{26.84}$$
$$sc = 0 \tag{26.85}$$
$$s\bar{c} = -ib \tag{26.86}$$
$$sc_E = 0 \tag{26.87}$$
$$s\bar{c}_E = -ib_E \tag{26.88}$$
$$sb'^\mu = sb = sb_E = 0 \ . \tag{26.89}$$

It is convenient to redefine $b^\mu = b'^\mu + i\partial^\mu c_E$. Now we get the usual rule: antighost goes into Lagrange multiplier, Lagrange multiplier goes to zero. The new BRST scheme is

$$sB^{\mu\nu} = \partial^\mu c^\nu - \partial^\nu c^\mu \tag{26.90}$$
$$sc^\mu = \partial^\mu c \tag{26.91}$$
$$s\bar{c}^\mu = -ib^\mu \tag{26.92}$$
$$s\bar{c} = -ib \tag{26.93}$$
$$s\bar{c}_E = -ib_E \tag{26.94}$$
$$sb^\mu = sb = sb_E = 0 \ . \tag{26.95}$$

Notice that c_E falls out of the picture. It is BRST inert, and also s on any field does not give c_E. This is in fact a general rule: Antighosts have only antighosts and not ghosts fields. The fields for this new BRST scheme are shown in Figure 26.5 Finally, the full action equals

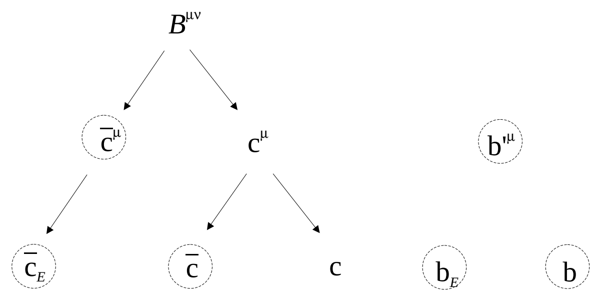


Figure 26.5: New fields for $U(1)$ gauge theory of the antisymmetric tensor field.

$$I_{\text{eff}} = I + s\Psi \ , \tag{26.96}$$

where

$$\Psi = i \int dx \ (\bar{c}^\mu \chi_\mu + \bar{c}\chi + \bar{c}_E \chi_E) \ . \tag{26.97}$$

The gauge fixings are

$$\chi_\mu = \partial^\nu B_{\mu\nu} \tag{26.98}$$

$$\chi = \partial_\mu c^\mu \quad (26.99)$$
$$\chi_E = \partial_\mu \bar{c}^\mu \ . \quad (26.100)$$

Therefore, we find

$$I_{\text{eff}} = I + \int dx \ (b^\mu \partial^\nu B_{\mu\nu} + b\partial_\mu c^\mu + b_E \partial_\mu \bar{c}^\mu) + \\ + \int dx \ \left(-i\bar{c}^\mu(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)c^\nu + i\bar{c}\partial^2 c + \bar{c}_E \partial_\mu b^\mu\right) \ . \quad (26.101)$$

We end with a brief aside on how we count degrees of freedom in the BRST scheme. To be concrete, let us do this in $d = 4$ dimensions. For the vector field we have four degrees of freedom at the zeroth level (the gauge field A_μ), and two at the first level (c and $\bar{c}$). The total number of degrees of freedom is $4 - 2 = 2$. The counting is quite simple — we just have to remember that adjecent levels contribute with opposite sign. For the antisymmetric tensor we have six degrees of freedom at the zeroth level ($B^{\mu\nu}$), eight at the first level (c^μ and $\bar{c}^\mu$), and three at the second level (c, $\bar{c}$ and $\bar{c}_E$). This gives a total of $6-8+3 = 1$ degrees of freedom. In fact this theory can be written in terms of a single scalar field. Similarly, for the symmetric tensor field (gravitation) we have ten degrees of freedom at the zeroth level ($h^{\mu\nu}$), and eight (c^μ and $\bar{c}^\mu$) at the first level. The total number of degrees of freedom is therefore $10 - 8 = 2$.

The proceedure for determining the BRST invariance of a given gauge theory is purely algebraic. It may easily be generalized to the case of n-stage reducible theories (even infinitely reducible theories like strings) as well as to theories whose algebras only close on shell. We shall not get into this here. The important thing is that the logic of this section can be turned upside down. We can construct the BRST transformations (insisting on nilpotence), and then determine the full action $I_{\text{eff}} = I + s\Psi$, by choosing an appropriate gauge Fermion. In recent years this method — called BRST quantization — has been applied to models for which the standard Faddeev-Popov proceedure does not work.

EXERCISES

26.1 Derive Feynman rules for QCD in the Lorentz gauge.

Lecture 27

Background Field Method

27.1 One Loop Counterterms

The basic path integral formula in quantum field theory

$$e^{-\frac{1}{\hbar}W[J]} = \int [d\phi]\, e^{-\frac{1}{\hbar}(I[\phi]-J_a\phi_a)} \tag{27.1}$$

may be written in terms of the effective action. We had

$$\Gamma[\varphi] = W[J] + J_a\varphi_a \tag{27.2}$$

$$\frac{\partial\Gamma}{\partial\varphi_a} = J_a\ , \tag{27.3}$$

so that we find

$$e^{-\frac{1}{\hbar}\Gamma[\varphi]} = \int [d\eta]\, e^{-\frac{1}{\hbar}\left(I[\varphi+\eta]-\frac{\partial\Gamma}{\partial\varphi_a}\eta_a\right)}\ . \tag{27.4}$$

The main drawback of this result is that the effective action appears on both sides of the equation. Things simplify if we evaluate the path integral on the right to one loop. To get the one loop result we need to expand $I[\varphi+\eta]-\frac{\partial\Gamma}{\partial\varphi_a}\eta_a$ up to quadratic terms in η, and perform the remaining Gaussian path integral. In fact, $\Gamma = I + o(\hbar)$, so the one loop result follows from expanding

$$I[\varphi+\eta] - \frac{\partial I}{\partial\varphi_a}\eta_a \tag{27.5}$$

up to quadratic terms. Therefore, we find

$$I[\varphi+\eta] - \frac{\partial I}{\partial\varphi_a}\eta_a = I[\varphi] + \int dx\, \mathcal{L}^{(2)}(\varphi|\eta) + \ldots \tag{27.6}$$

For a general Lagrangian of the form $\mathcal{L} = \mathcal{L}(\phi_i, \partial_\mu\phi_i)$, after partial integrations, we may put $\mathcal{L}^{(2)}(\varphi|\eta)$ into the form

$$\mathcal{L}^{(2)}(\varphi|\eta) = \frac{1}{2}\,\partial_\mu\eta_i W^{\mu\nu}_{ij}(\varphi)\partial_\nu\eta_j - \eta_i N^{\mu}_{ij}(\varphi)\partial_\mu\eta_j + \frac{1}{2}\,\eta_i M_{ij}(\varphi)\eta_j\ , \tag{27.7}$$

where

$$W_{ij}^{\mu\nu} = W_{ji}^{\mu\nu} = W_{ij}^{\nu\mu} \tag{27.8}$$
$$N_{ij}^{\mu} = -N_{ji}^{\mu} \tag{27.9}$$
$$M_{ij} = M_{ji} \ . \tag{27.10}$$

Up to one loop we thus have

$$e^{-\frac{1}{\hbar}(\Gamma[\varphi]-I[\varphi])} = \int [d\eta]\, e^{-\frac{1}{\hbar}\int dx\, \mathcal{L}^{(2)}(\varphi|\eta)} \ . \tag{27.11}$$

We now set $\hbar = 1$ for the remainder of this lecture.

Our aim is to evaluate the ultraviolet divergent part of the above Gaussian integral. In this lecture we will consider theories for which

$$W_{ij}^{\mu\nu} = \delta^{\mu\nu}\,\delta_{ij} \ . \tag{27.12}$$

The general case will be studied in the following lecture. Using the above form of $W_{ij}^{\mu\nu}$, the above path integral now becomes

$$\int [d\eta]\, e^{-\int dx\, \mathcal{L}^{(2)}(\varphi|\eta)} = \exp\left(-\frac{1}{2}\ \mathrm{Tr}\ \ln(-\partial^2 - 2N_\mu\partial_\mu + M)\right) \ . \tag{27.13}$$

We will tackle this trace in the next section. At this moment we focus on the case when $N_{ij}^{\mu} = 0$. We then have

$$\begin{aligned}\mathrm{Tr}\ \ln(-\partial^2 + M) &= \mathrm{Tr}\ \ln\left(1 - \frac{1}{\partial^2}M\right) = \\ &= -\,\mathrm{Tr}\ \left(\frac{1}{\partial^2}M + \frac{1}{2}\frac{1}{\partial^2}M\frac{1}{\partial^2}M + \dots\right) \ .\end{aligned} \tag{27.14}$$

In the first step we dropped an infinite constant term $\ln(-\partial^2)$. In the second step we Taylor expanded the log. Note that $\frac{1}{\partial^2}$ is just the massless propagator. $\mathrm{Tr}\ \ln(-\partial^2+M)$ is shown as a sum of diagrams in Figure 27.1. In $d = 4$ dimensions

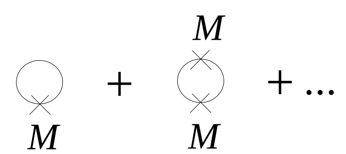


Figure 27.1: $\mathrm{Tr}\ \ln(-\partial^2 + M)$ written as a sum of Feynman diagrams.

the first diagram in this sum diverges as Λ^2, where Λ is a large momentum cut off. The second diagram diverges as $\ln\Lambda$. All the other diagrams are convergent. Log divergent diagrams play a special role in quantum field theory. We will look into this when we study the renormalization group equations. For this reason we will

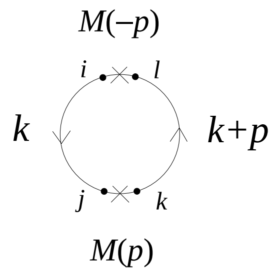


Figure 27.2: Log divergent diagram.

now restrict ourselves to evaluating the log divergent part of the one loop effective action. As we have seen, log divergence comes from the diagram in Figure 27.2 Writing this out we find

$$
\begin{aligned}
&\int \frac{dk}{(2\pi)^4} \int \frac{dp}{(2\pi)^4} \frac{\delta_{ij}}{k^2} M_{jk}(p) \frac{\delta_{kl}}{(k+p)^2} M_{li}(-p) = \\
&\quad = \int \frac{dp}{(2\pi)^4} \ \mathrm{tr}\ \left(M(p)M(-p)\right) \int \frac{dk}{(2\pi)^4} \frac{1}{k^2(k+p)^2} = \\
&\quad = \int dx \ \mathrm{tr}\ \left(M(x)^2\right) \int \frac{dk}{(2\pi)^4} \frac{1}{k^2(k+p)^2} \ . \qquad (27.15)
\end{aligned}
$$

Note that ‘ Tr ’ traced over matrix indices as well as space time, while ‘ tr ’ is just a trace over the matrix indices. We simplify the remaining k integration by using the Feynman parametrization formula

$$
\begin{aligned}
\mathcal{I} &= \int \frac{dk}{(2\pi)^4} \frac{1}{k^2(k+p)^2} = \\
&= \int_0^1 d\alpha_1 d\alpha_2 \delta(\alpha_1+\alpha_2-1) \int \frac{dk}{(2\pi)^4} \frac{1}{\left(\alpha_1 k^2 + \alpha_2 (k+p)^2\right)^2} \ . \qquad (27.16)
\end{aligned}
$$

All that is left is to use the basic dimensional regularization formula

$$
\int \frac{d\ell}{(2\pi)^d} \frac{1}{(\ell^2 + 2\ell\cdot p + M^2)^A} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(A-d/2)}{\Gamma(A)} \frac{1}{(M^2-p^2)^{A-d/2}} \ . \qquad (27.17)
$$

In $d = 4 - \varepsilon$ dimensions we get

$$
\mathcal{I} = \int_0^1 d\alpha_1 d\alpha_2 \, \delta(\alpha_1+\alpha_2-1) \left(\frac{2}{\varepsilon} \frac{1}{(4\pi)^2} + \ldots \right) = \frac{1}{8\pi^2} \frac{1}{4-d} + \ldots \qquad (27.18)
$$

Dots indicate terms that are finite in the $d \to 4$ limit. To one loop we thus have

$$
\Gamma[\varphi] = S[\varphi] + \frac{1}{8\pi^2} \frac{1}{4-d} \int dx \ \mathrm{tr}\ \left(M(\varphi)^2\right) + \ldots \qquad (27.19)
$$

The additional factor of two is a symmetry factor. Along with the contribution of Figure 27.2 which we evaluated, we have to consider the contribution of the crossed

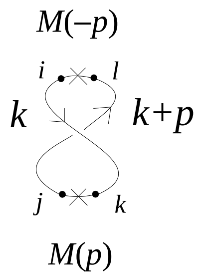


Figure 27.3: Another log divergent diagram.

diagram given in Figure 27.3. Both of these diagrams give the same contribution. In order for Γ to be finite we need to introduce a counterterm to the starting Lagrangian equal to

$$\Delta\mathcal{L} = -\frac{1}{8\pi^2}\frac{1}{4-d}\ \mathrm{tr}\ M^2\ . \tag{27.20}$$

This is the general one loop counterterm (due to a $\ln\Lambda$ divergence) for a theory with $N^{\mu}_{ij} = 0$. We could determine the Λ^2 counterterm in a similar manner.

As an example, let us look at the $O(N)$ vector model with ϕ^4 interaction

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\vec{\phi})^2 + \frac{1}{2}m^2\vec{\phi}^2 + \frac{\lambda}{4!}(\vec{\phi}^2)^2\ . \tag{27.21}$$

We now have

$$\begin{aligned}
\mathcal{L}(\varphi+\eta) - \frac{\partial\mathcal{L}}{\partial\varphi_i}\eta_i &= \\
&= \frac{1}{2}(\partial_\mu\vec{\varphi})^2 + \frac{1}{2}m^2\vec{\varphi}^2 + \frac{1}{2}(\partial_\mu\vec{\eta})^2 + \frac{1}{2}m^2\vec{\eta}^2 + \\
&+ \frac{\lambda}{4!}\left(\varphi^4 + 2\vec{\eta}^2\vec{\varphi}^2 + (\vec{\varphi}\cdot\vec{\eta})^2\right) + o(\eta^3) = \\
&= \mathcal{L}(\varphi) + \frac{1}{2}\partial_\mu\eta_i(\delta_{ij}g^{\mu\nu})\partial_\nu\eta_j + \frac{1}{2}m^2\eta_i\delta_{ij}\eta_j + \\
&+ \frac{\lambda}{4!}2\vec{\varphi}^2\eta_i\delta_{ij}\eta_j + \frac{\lambda}{4!}\eta_i\varphi_i\varphi_j\eta_j\ .
\end{aligned} \tag{27.22}$$

Therefore,

$$W^{\mu\nu}_{ij} = \delta_{ij}g^{\mu\nu} \tag{27.23}$$

$$N^{\mu}_{ij} = 0 \tag{27.24}$$

$$M_{ij} = +\left(m^2 + \frac{\lambda}{3!}\vec{\varphi}^2\right)\delta_{ij} - \frac{1}{2}\frac{\lambda}{3!}\varphi_i\varphi_j\ . \tag{27.25}$$

Using this we immediately find

$$\begin{aligned}
\text{tr } M^2 &= \\
&= \left(m^2 + \frac{\lambda}{3!}\,\vec{\varphi}^2\right)^2 N + \frac{\lambda}{3!}\vec{\varphi}^2\left(m^2 + \frac{\lambda}{3!}\,\vec{\varphi}^2\right) + \left(\frac{1}{2}\,\frac{\lambda}{3!}\right)^2 = \\
&= m^4 N + m^2\,\frac{\lambda}{3!}\,(2N+1)\vec{\varphi}^2 + \left(\frac{\lambda}{3!}\right)^2\left(N + \frac{5}{4}\right)\left(\vec{\varphi}^2\right)^2 \ . \qquad (27.26)
\end{aligned}$$

The Lagrangian counterterm is

$$\begin{aligned}
\Delta\mathcal{L} = &- \frac{1}{8\pi^2(4-d)}\cdot \\
&\cdot\left(m^4 N + m^2\,\frac{\lambda}{3!}\,(2N+1)\vec{\varphi}^2 + \left(\frac{\lambda}{3!}\right)^2\left(N + \frac{5}{4}\right)\left(\vec{\varphi}^2\right)^2\right) \ . \qquad (27.27)
\end{aligned}$$

As we can see, the counterterm can be absorbed into changes of mass and coupling constant. We have

$$\Delta m^2 = -\frac{1}{8\pi^2(4-d)}\,2m^2\,\frac{\lambda}{3!}\,(2N+1) \qquad (27.28)$$

$$\Delta\lambda = -\frac{1}{8\pi^2(4-d)}\left(\frac{\lambda}{3!}\right)^2 4!\left(N + \frac{5}{4}\right) \ . \qquad (27.29)$$

27.2 An Auxilliary Gauge Symmetry

We are ready to go back to the more general case when $N^{\mu}_{ij} \neq 0$. The quadratic Lagrangian can be written as

$$\begin{aligned}
\mathcal{L}^{(2)}(\eta) &= \frac{1}{2}\,\partial_\mu\eta^T\partial_\mu\eta - \eta^T N_\mu\partial_\mu\eta + \frac{1}{2}\,\eta^T M\eta = \\
&= \frac{1}{2}\,(D_\mu\eta)^T(D_\mu\eta) + \frac{1}{2}\,\eta^T X\eta \ , \qquad (27.30)
\end{aligned}$$

where we have introduced obvious matrix notation and supressed explicitly writting the background field φ. In addition we have introduced

$$D_\mu = \partial_\mu + N_\mu \qquad (27.31)$$

$$X = M + N_\mu N_\mu + \partial_\mu N_\mu \ . \qquad (27.32)$$

In this form it is easy to check that $\mathcal{L}^{(2)}(\eta)$ is invariant under a gauge symmetry. This is true even when $\mathcal{L}$ is not gauge invariant. To show this, let us consider the local transformations

$$\eta \rightarrow e^{-\omega}\eta \qquad (27.33)$$

$$\eta^T \rightarrow \eta^T e^{\omega} \ , \qquad (27.34)$$

where $\omega^T = -\omega$. Imposing that $D_\mu\eta \rightarrow e^{-\omega}D_\mu\eta$ we determine the gauge transformation for N_μ. For infinitesimal transformations we have

$$\delta N_\mu = \partial_\mu\omega + [N_\mu, \omega] = D_\mu\omega \ . \qquad (27.35)$$

To make the second term gauge invariant we impose that X transforms according to

$$X \to e^{-\omega} X e^{\omega} \ . \tag{27.36}$$

In terms of infinitesimal transformations this is simply

$$\delta X = [X, \omega] \ . \tag{27.37}$$

From this and δN_μ we can determine δM. This completely specifies our gauge symmetry. Note that X transforms homogenously, just like the field strength. As we know from Yang-Mills, the field strength

$$Y_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu N_\nu - \partial_\nu N_\mu + [N_\mu, N_\nu] \tag{27.38}$$

transforms according to

$$\delta Y_{\mu\nu} = [Y_{\mu\nu}, \omega] \ . \tag{27.39}$$

In terms of finite gauge transformations we have

$$Y_{\mu\nu} \to e^{-\omega} Y_{\mu\nu} e^{\omega} \ . \tag{27.40}$$

The fact that $\mathcal{L}^{(2)}$ is invariant implies that the counterterm $\Delta\mathcal{L}$ must also be gauge invariant.[1] The counterterm must be constructed out of gauge invariant objects like $\text{tr}\ (X^2)$ and $\text{tr}\ (Y_{\mu\nu}Y_{\mu\nu})$. When $N^\mu \to 0$ we have

$$\text{tr}\ (X^2) \quad \to \quad \text{tr}\ (M^2) \tag{27.41}$$

$$\text{tr}\ (Y_{\mu\nu}Y_{\mu\nu}) \quad \to \quad 0 \ . \tag{27.42}$$

It is easy to convice oneself that there are no other invariant objects that can make up the counterterm Lagrangian. For example, $\text{tr}\ X$ is gauge invariant, however for $N^\mu \to 0$ it goes over into $\text{tr}\ M$, and we have seen that this term is not present in the log divergent counterterm. At the same time $\varepsilon_{\mu\nu\rho\sigma}\, \text{tr}\ (Y_{\mu\nu}Y_{\rho\sigma})$ is also not present. In fact it is a total divergence (Pontryagin density) and so it can be dropped from the Lagrangian. Finaly, we have

$$\Delta\mathcal{L} = a\, \text{tr}\ (X^2) + b\, \text{tr}\ (Y_{\mu\nu}Y_{\mu\nu}) \ . \tag{27.43}$$

If we let $N^\mu \to 0$ we determine a to be equal to

$$a = -\frac{1}{8\pi^2}\frac{1}{4-d} \ . \tag{27.44}$$

Similarly we could determine b. The final result for the counterterm is

$$\Delta\mathcal{L} = -\frac{1}{8\pi^2(4-d)}\left(\text{tr}\ (X^2) + \frac{1}{24}\ \text{tr}\ (Y_{\mu\nu}Y_{\mu\nu})\right) \ . \tag{27.45}$$

[1] The measure $[d\eta]$ is invariant just as in Yang-Mills theories.

EXERCISES

27.1 Show that $\mathcal{L}(\eta)$ and $\mathcal{L}^{(2)}(\varphi|\eta)$ are equivalent as far as one loop diagrams are concerned.

27.2 Calculate $W_{ij}^{\mu\nu}$, N_{ij}^{μ} and M_{ij} for ϕ^4 theory and for Yang-Mills theory.

27.3 Prove that $\mathcal{L}^{(2)}(\eta)$ is gauge invariant. Determine the gauge transfrmation of M.

27.4 Determine the b coefficient in (27.43).

27.5 Determine the general Λ^2 counterterm in $d = 4$ dimensions. Assume that $W_{ij}^{\mu\nu} = \delta_{ij}\delta^{\mu\nu}$.

27.6 Determine the general one loop counterterm in $d = 2$ dimensions. As in the lecture, assume that $W_{ij}^{\mu\nu} = \delta_{ij}\delta^{\mu\nu}$.