Fractionalization into Merons in Quantum Dots

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We study by exact diagonalization, in the lowest Landau level approximation, the Coulomb interaction problem of $N = 4$ and $N = 6$ quantum dots in the limit of zero Zeeman coupling. We find that meron excitations constitute the lowest lying states of the quantum dots. This is based on a mapping between the excitations of the dot and states of the Haldane-Shastry spin chain.

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Introduction.—Quasiparticles with fractional charge and fractional statistics make a hallmark of the fractional quantum Hall (QH) effect [1,2]. They are usually found in spin polarized systems, but they can be found also in the systems where Zeeman energy is negligible and spin degree of freedom plays a role. In these systems, quasiparticles, which carry charge, can be identified with topological objects, special configurations of spin in space. An example of this is the skyrmion, the topological object well known from classical ferromagnetism [3], which is also a quasiparticle in the $\nu = 1$ QH system [4]. We will show in this Letter that, as the size of the $\nu = 1$ system shrinks to a quantum dot (QD), other quasiparticles of a topological nature, merons [5], enter the stage and constitute the lowest lying excitations.

Each meron is a half of a skyrmion [6]. If in the center of skyrmion excitation in a 2D plane the spin points up, it slowly tumbles down to the down configuration on its circular boundary. On the other hand, in the case of a meron, spin does not reach the down position on its boundary but it is halfway between the up and down positions; i.e., it is in the plane and winds for 2π along its boundary; see Fig. 1.

It is hard to prove the existence of a quasiparticle in a finite, small system. In order to prove the existence of merons in QDs, we used a mapping between seemingly uncorrelated physical systems: a 2D QD with zero Zeeman coupling and a 1D Haldane-Shastry (HS) spin chain [7]. It is not utterly surprising that it exists and maps the physics of spinon excitations of the HS chain to the one of merons in QD. It is well known that the edge of QH systems can be mapped to one-dimensional models (charge excitations to the Calogero-Sutherland model [8]), and this may exist even in small systems such as QDs where the distinction between the bulk and the edge is blurred.

Model.—We model a quantum dot in the regime of high magnetic fields in the lowest Landau level (LLL) approximation . Accordingly, the Hamiltonian takes the following form:

$$H = H_{sp} + H_{int}, \tag{1}$$

where H_{sp} denotes the single particle part, without the Zeeman term in our case, $H_{sp} = \hbar[\omega N + (\omega - \frac{1}{2}\omega_c)L]$, where $\omega_c = eB/m^*$, the cyclotron frequency $\omega = \sqrt{\omega_o^2 + (\omega_c^2/4)}$, where ω_o is the frequency of the harmonic confining potential, L and N are the total orbital angular momentum and the number of particles of the dot, respectively, and H_{int} denotes the interaction part

$$H_{int} = \frac{1}{2}\sum a^\dagger_{m_1\sigma}a^\dagger_{m_2\sigma'}a_{m_3\sigma'}a_{m_4\sigma}\langle m_1|\langle m_2|V_{12}|m_4\rangle|m_3\rangle, \tag{2}$$

where $a^\dagger_{m\sigma}$ and $a_{m\sigma}$ create and annihilate electrons with the spin projection σ in the single particle state of the LLL with angular momentum $m \geq 0$,

$$\langle \mathbf{r}|m\rangle = \frac{1}{\sqrt{2\pi 2^m m!}} r^m \exp\{-im\phi\}\exp\left\{-\frac{r^2}{4}\right\}, \tag{3}$$

where $\hbar = 1$ and $2m^*\omega = 1$. V_{12} is the Coulomb interaction operator $V_{12} = (e^2/4\pi\epsilon)(1/|\vec{r}_1 - \vec{r}_2|)$ between two particles. As H_{sp} is trivially diagonalized and accounted for, we will refer in the following to the energies of H_{int} as those of H.

HS spin chain.—The HS Hamiltonian is

$$H_{HS} = J\left(\frac{2\pi}{N}\right)^2 \sum_{\alpha<\beta}^{N} \frac{\vec{S}_\alpha \cdot \vec{S}_\beta}{|z_\alpha - z_\beta|^2}, \tag{4}$$

where sites $\alpha, \beta = 1, \ldots, N$ are positioned on a unit circle

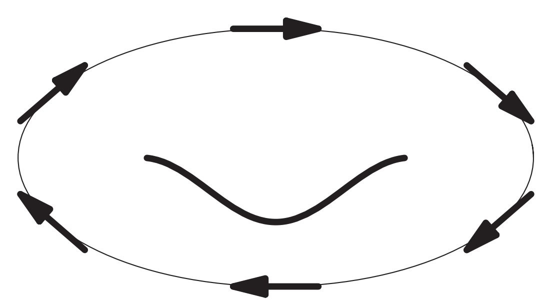

FIG. 1. A meron quasihole.

so that each site coordinate z_α fulfills $z_\alpha^N - 1 = 0$, and $\vec{S}_\alpha$'s are spin $1/2$ operators. Any state of the chain can be represented as a function of complex numbers satisfying $z_\alpha^N = 1$ and representing sites for which $S_{\alpha z} = +1/2$. In this way, the ground state wave function, a spin singlet, is a function of $N/2$ complex numbers $\Psi(z_1, \ldots, z_{N/2}) = \prod_{j<k}^{N/2}(z_j - z_k)^2 \prod_{j=1}^{N/2} z_j$. Spinons, elementary excitations, are quasiparticles of spin $1/2$. The hallmark of the HS spin chain is the existence of "supermultiplets," degenerate energy eigenstates of the same spinon number but different spin. The structure of eigenstates is built on so-called "fully (spin-)polarized spinon gas" (FPSG) states [7] with a definite spinon number and a maximum spin equal to half of the spinon number. Their corresponding, degenerate states can be found by acting with an operator of the Yangian algebra, inherent to the model for which the FPSG states are the highest weight states. In the case of two spinons, we have two degenerate states, a triplet and a singlet, and the latter wave function is $\Psi'(z_1, \ldots, z_{N/2}) = \prod_{j<k}^{N/2}(z_j - z_k)^2[1 - \prod_{j=1}^{N/2} z_j^2]$.

Motivation behind numerical calculations.—We will present our numerical work later on, but here we will give just a synopsis of what can be established on the basis of the numerical work and how this can be used to prove the existence of merons.

In the case of polarized electrons, a QD at $\nu = 1$ is in a stable state, the so-called maximum density droplet (MDD) state, where each angular momentum orbital until its maximum value $N - 1$ is filled. Therefore, its total angular momentum is $L = N(N-1)/2$. In this system, the existence of a vortex (dressed hole) quasiparticle excitation is firmly established [9–11]. As we slowly increase the magnetic field (from the MDD value), a particle-hole vortex excitation is formed, where the vortex occupies an inner orbital of the MDD state and at the same time pushes charge outside and creates a particle on the edge. This process is followed by a gradual increase of L. If the QD is small enough, this description may persist to the point when the vortex quasiparticle is created at the center, which can be associated with the total increase of $\Delta L = N$ of the angular momentum as implied by the shift register counting of Laughlin [12] or, therefore, an increase of one flux quantum in the magnetic flux through the system. Therefore, the lowest lying excitations of the dot as the magnetic field is increased can be described through vortex excitations, which makes the vortex an eigenstate of the polarized system.

We will show that a similar process happens with unpolarized electrons, starting from the MDD configuration [13], where the quasiparticle that slowly sweeps the inner orbitals, pushes charge outside, and finally is created at the center is a meron. Therefore, even in this case we will recover the usual quasiparticle-hole phenomenology of a QH system and therefore prove the existence of merons.

The meron is half of a skyrmion, and to the skyrmion an increase of one flux quantum is associated because skyrmions are analogs of hole (vortex) excitations in the limit of weak Zeeman coupling [4]. Therefore, if we create a meron at the center, we expect a half flux quantum increase in the flux through the system and an associated increase of $\Delta L = N/2$ in the total angular momentum. That should be also the period for which we expect the appearance of merons of a higher winding number in very small dots; see the $N = 4$ example below.

Our findings support that the MDD state, with respect to its spin content, can be viewed as a condensate of N merons spin $\frac{1}{2}$, where each two merons pair to one hole (a vortex) [14]. We will call these quasiparticle (not quasihole) kind of merons condensate merons. We find, while establishing a mapping between the dot and the HS chain, that the ferromagnetically ordered MDD state corresponds to the $2S = N$, maximum number of spinons, HS N site spin chain state. This state, except for the spin degeneracy, is a unique state of the chain. When a quasiparticle, more precisely a meron quasihole, enters the dot, the number of condensate merons is reduced, by creation of a quasiparticle-hole pair, to $N - 2$. We find that, as an effect, the successive quasihole orbital states as it enters a dot can be associated to half of (see [15]) the $N - 2$ spinon sector of the HS N site spin chain. Therefore, merons map to spinons and we are establishing merons as eigenstates of the dot problem [16].

The $N = 4$ dot.—In Fig. 2, we show spin-spin correlations along fixed radii, of lowest lying, at fixed L, spin-singlet states. On the right-hand side of Fig. 2 are shown

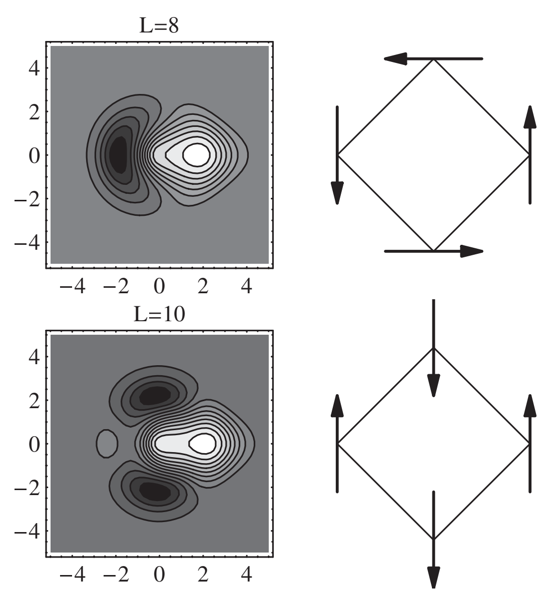

FIG. 2. Spin-spin correlations along fixed radii of the lowest lying, spin-singlet states of the $N = 4$ dot. The reference points are on the positive side of the x axis. On the right-hand side are two possible classical meron configurations where only the characteristic spin windings along the meron edge are shown.

two possible types of (classical) configurations, windings of merons at their edges projected to four sites. The occurrence of black regions corresponds to our expectations of where to find antiferromagnetic correlations according to the classical configurations at some radii, which turn out to be near or at the maximum (radial) density radius. Similar results already appeared in Ref. [17] but not with the emphasis on the $\frac{N}{2}$ periodicity (the L of MDD is at $L = \frac{N(N-1)}{2} = 6$) and the meron interpretation we are pointing out here. Indeed, after a detailed investigation we found also resonating valence bond states RVB^+ and RVB^- four-site spin states [17] to describe these states, $L = 10$ and $L = 8$, respectively, but in addition we identified these states with the ground and the two spinon lowest lying singlet state of the HS four-site spin chain, respectively. Moreover, at $L = 8$ we identified, with respect to the lowest lying, a nearly degenerate $S = 1$ state with the triplet, z^2 state of the HS, and this is in accordance with the mapping to the two degenerate states, a singlet (RVB^-) and a triplet (z^2) of the $N - 2 = 2$ spinon sector of the HS chain. Also, at $L = 7$ we found the triplet z^3 two spinon state of the chain as expected [7].

The $N = 6$ dot.—Also in the case of the $N = 6$ dot, we will recover meron quasiparticle-hole states as the lowest lying ones for angular momenta higher than, but still near, the MDD state. First we can notice in Fig. 3 at $\Delta L = N/2 = 3$, $L = 18$ a meron quasihole created at the center of the dot. Before reaching that state, as the magnetic field is gradually increased from the MDD state, the system passes through the states at $L = 16$ and $L = 17$ analyzed in Fig. 4. As announced, we expect that the $N - 2 = 4$ spinon sector of the HS chain is associated with the states at $L = 16$, 17, and 18. Indeed, just above $L = \frac{N(N-1)}{2} = 15$ at $L = 16$ and $L = 17$, we easily identified these HS states as in Fig. 4, where the corresponding states are noted. Plotted are ratios of the calculated real parts of S^+S^- correlations as functions of radius of the dot states subtracted by the same ratios for the HS $N = 6$ chain for the expected HS states. We take ratios because of the different electron densities at different dot radii. In general, we define the quantities displayed in Fig. 4 as

$$f_{ab} = \frac{\mathrm{Re}[S^+(r,0)S^-(r,a)]}{\mathrm{Re}[S^+(r,0)S^-(r,b)]} - \frac{\mathrm{Re}[S^+(0)S^-(a)]|_{\mathrm{HS}}}{\mathrm{Re}[S^+(0)S^-(b)]|_{\mathrm{HS}}}, \quad (5)$$

where a and b take values 1, 2, and 3 denoting the angles, with respect to the positive x axis, represented on the hexagon in the upper right corner in Fig. 3. Other HS states with respect to those shown do not show the nice confluences of all three lines f_{12}, f_{13}, and f_{23} at their simultaneous value zero at a single radius; they are completely off and uncorrelated. Similarly, we can define quantities with the imaginary parts of S^+S^- correlations. Some of them are identically zero, due to the same real correlation property of HS and QD states, and the rest are compatible to and very suggestive of the trend around the special radius that exists in real parts (Fig. 4). That again is the behavior we see only for these special HS states. We find that the HS states' conditions, including S_zS_z correlations, are satisfied at the radius (Fig. 4) slightly beyond the maximum density radius.

For $L = 17$, we found the $S = 2$, z^4 closely lying, first excited state that belongs to the expected [7] HS two state degenerate supermultiplet (Figs. 4 and 5). We emphasize that, although the results of the $N = 4$ case may be justified or expected because of early (for small L) Wigner crystal formation [18], we checked that for $L = 15$–18, in the $N = 6$ case there is no underlying Wigner crystal configuration [19]. The mapping does not require underlying crystal structure and may persist in dots with N higher than 6.

In Fig. 5, we see clearly, beside the $L = 17$ multiplet, the existence of an additional multiplet at $L = 18$, with three states, that we expect [7] if we identify the $S = 0$ state as the lowest lying $S = 0$ state in the 4-spinon sector

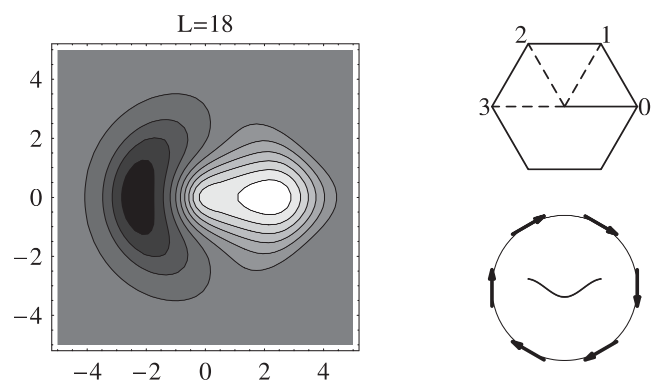

FIG. 3. The map of the spin-spin correlations of the lowest lying, spin-singlet state at $L = 18$ of the $N = 6$ dot. In the lower right corner is the classical meron edge configuration for comparison.

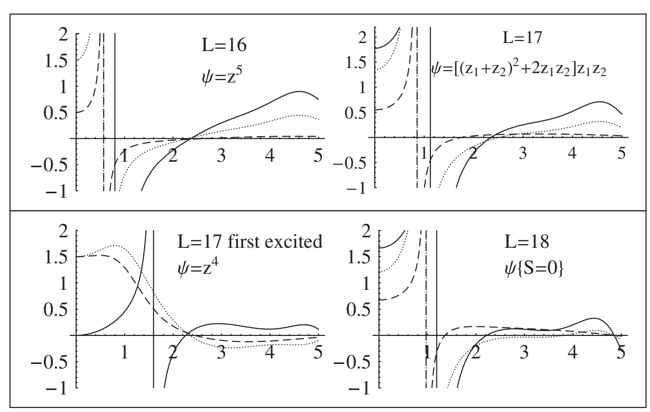

FIG. 4. Quantities f_{12} (solid line), f_{13} (dotted line), and f_{23} (dashed line)—for the definitions see the text—as functions of radius. The quantities should go to zero, simultaneously if the corresponding HS state (in the figure denoted for each angular momentum; for $L = 18$, see the text) describes the dot state correlations along a fixed radius. (Asymptotes are also shown.)

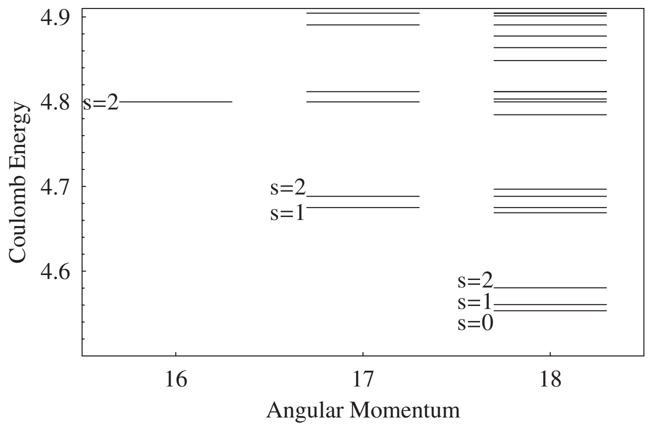

FIG. 5. The spectrum of the $N = 6$ dot as a function of angular momentum. The lowest energy, closely lying levels correspond to HS supermultiplets [7].

$$\Psi_{S=0} = \prod_{i<j}^{M}(z_i - z_j)^2 \prod_{i}^{M} z_i + 15\left\{\prod_{i}^{M} z_i + \prod_{i}^{M} z_i^3 + \prod_{i}^{M} z_i^5\right\}, \tag{6}$$

where $M = 3$. Indeed, we were able to do that; the analysis is given in Fig. 3, where we see a clear correspondence between the classical meron edge configuration (2π winding of the spin vector in the plane) and the spin-spin correlation map, and in Fig. 4 but less easily than the lower momentum states. The reason is that we are already approaching a transition region to the Wigner regime [20,21] where the Wigner structure with a pentagon and an electron in the middle competes with the hexagonal structure [22]. Therefore, the transition region may well consist of liquid states of merons similarly to what happens in the polarized case with vortices [11].

Final remarks.—The physics of exchange [4,6] that favors smooth variations of spin in space favors meron solutions instead of skyrmion ones in small systems, in the limit of zero Zeeman coupling. Merons distort spin at a slower rate than skyrmions, although they acquire also energy due to winding over the space of the system. This second contribution is suppressed in small systems [23] where merons, as we demonstrated here, constitute the lowest lying eigenstates.

The meron physics should be detectable in lateral quantum dots, in which interaction effects are strong, in their not yet explored regime beyond the MDD state [24]. The same conclusions are valid for rapidly rotating Fermi atoms, where there is no Zeeman effect to disguise the fractionalization into merons, and, therefore, the RVB and spinon physics implied by the mapping to the HS chain may reveal more easily.

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[13] Here we use the notion of the MDD state not as of a maximum possible density state, but as the one in which in the orbital space, each orbital until the value $N - 1$ is filled by exactly one electron.
[14] We noticed the presence of vortex excitations in the low lying spectrum.
[15] As may be expected, only one chirality (angular momentum direction) branch maps to the states of QD in a magnetic field.
[16] It is not surprising that the mapping we find is to the HS chain. The chain is a paradigm of quantum antiferromagnet fractionalization and its excitations, spinons, have semionic statistics. Merons as halves of skyrmions, which at $\nu = 1$ carry fermionic statistics, are expected to have semionic statistics. In principle, it is hard to talk about statistics in small systems, but if we have signatures of statistics in small chains [7], to which mapping is possible, they should carry information of the expected quasiparticle statistics in QDs.
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