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Classical behaviour of various variables in an open Bose–Hubbard system*

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Quantum dispersions of various sets of dynamical variables of an open Bose–Hubbard system in a classical limit are studied. To this end, an open system is described in terms of stochastic evolution of its quantum pure states. It is shown that the class of variables that display classical behaviour crucially depends on the type of noise. This is relevant in the mean-field approximation of open Bose–Hubbard dynamics.

Keywords: quantum trajectories, decoherence, classical limit

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1. Introduction

The dynamics of a Bose–Einstein condensate (BEC)[1,2] in optical lattices has been extensively studied recently,[3−9] mainly because the system is an experimentally realizable representation of model systems from many fields like condensed matter physics and nonlinear dynamics. In particular, it enables the experimental investigation of quantum-to-classical transition. In this respect, the Bose–Hubbard (BH) model,[10] which strongly correlated with many particle bosonic systems, has played a prominent role. The primary theoretical approach to the problem of the quantum-to-classical transition in the BH model is the mean-field approximation of many particle dynamics in the limit of a large particle number.[11]

BEC in realistic experimental conditions must be considered as an open quantum system exposed to several types of perturbations that can be treated as environmental noise,[12−14] and the noise might influence in a crucial way the mean-field approximation. Furthermore, the influence of environment is expected to play a fundamental role in the full explanation of the quantum-to-classical transition.[15] Evolution of an open system is commonly described by the dynamics of mixed states represented by the density matrix $\rho(t)$ and the corresponding master equation.[16] Appearance of the classical behaviour of an observable A is usually analysed using the ratio $\Delta_\rho A/\,\mathrm{Tr}[\,\rho A]$ where $\Delta_\rho A$ is the dispersion of A in the state ρ. In general, if this ratio is small for all times the expectation $\langle A\rangle$ can be considered as a classical variable. However, the density matrix $\rho(t)$ represents a classical ensemble of pure quantum states $\{|\psi_i\rangle\langle\psi_i|\}$, and the influence of the environment is crucial in the choice of the most appropriate ensemble. $\Delta_\rho A$ combines the purely classical dispersion over the classical ensemble and the dispersion in each of the pure quantum states $|\psi_i\rangle$. In other words, $\Delta_\rho A$ could be quite large because the dispersion of quantum expectations $\langle\psi_i|A|\psi_i\rangle$ over the ensemble $\{|\psi_i\rangle\langle\psi_i|\}$ is large even if the dispersion in each single quantum pure state is small. This complicates the analyses of the appearance of classical behaviour in an open quantum system, in particular, in open BEC.

However, there is an alternative description of the evolution of an open quantum system in terms of pure states only,[16−18] often used in quantum optics.[19] The price that has to be paid in order to have the open system evolution described in terms of pure states is that the corresponding dynamical equation contains stochastic terms, and if the norm of the stochastic pure state is to be preserved the stochastic evolution equation must be nonlinear. Thus, the evolution of the open system is described by a nonlinear stochastic Schrödinger equation, and subsequent average over the ensemble of stochastic orbits in the pure state space of the system. In this description the roles of

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purely quantum and classical dispersions are clearly separated.[16,20−23] As pointed out, this is crucial in the study of classical behaviour of an open quantum system.

Our goal is to use the stochastic pure state description of the evolution as provided by quantum state diffusion (QSD) theory,[17] to study the mean-field approximation and the appearance of classical behaviour of different variables in the open Bose–Hubbard model with realistic types of noise. In order to study a classical limit of a quantum system one needs to select a set of physical observables, according to some classical model, which depend on a classicality parameter β such that in the limit of small β the expectations of the chosen set of observables behave like variables of the classical model. Parameter β should have a natural physical interpretation within the physical model corresponding to the chosen variables. For example, suppose that the natural dynamical variables for the classical model are the coordinate and momentum of an oscillator, that is, operators q and p satisfying $[q,p] = \mathrm{i}\,\hbar$. Rescaling q and p by a parameter β as $q \to \beta q$ and $p \to \beta p$ and taking the units such that $\hbar = 1$ enable one to study the classical limit as the condition $\beta \to 0$. This implies large ratios of dynamical variations in $\langle q\rangle$ and $\langle p\rangle$ to the phase space cell of fixed size $\hbar = 1$. Physically, the same effect as that of the limit $\beta \to 0$ is in the oscillator model achieved by taking the mass of the oscillator to be very large and its frequency to be very small, corresponding to a macroscopic oscillator. On the other hand, if the observables of the classical model satisfy $SU(2)$ commutation relations $[J_x, J_y] = J_z$ then rescaling $J_{x,y,z}$ by a small positive parameter denoted β^2 corresponds to the classical limit of large angular momentum $J^2 = J_z^2 + J_x^2 + J_y^2$. As we shall see, different environments suggest variables that belong to different classical models as the candidates for classical behaviour. The classical limits in different cases will be modeled by the same formal small parameter β that will be used to rescale the creation operator and the annihilation operator of the two modes $a_i \to \beta a_i$ and $a_i^\dagger \to \beta a_i^\dagger$ $(i = 1, 2)$ that appear in the formulation of the BH model.

Also, behaviour of the selected variables might depend on the type of the initial state. It is natural to use as the initial states those that minimize the dispersions of the basic variables of the classical model, i.e. the appropriate (generalized) coherent states determined by the dynamical group of system.[24−26] However, the set of coherent states is seldom invariant on the full unitary dynamics, and furthermore the noise might favour other asymptotic states. Similar arguments and techniques of analyses have been used recently to suggest an efficient way of simulating quantum evolution of a restricted set of observables singled out by weak measurement.[27]

In the next section we recapitulate the BH model with typical noise and provide a summary of the QSD theory of an open system. In Section 3 we study the QSD description of the open BH system in the classical limit with various environments. We see that the class of observables that behaves classically, in what can be considered as the classical limit, crucially depends on the type of noise. The numerical results are given in Section 4. In Section 5 we present a short summary of our analyses.

2. Open Bose–Hubbard system

A two-mode BH model is given by the following Hamiltonian with $\hbar = 1$:

$$H = \epsilon_1 a_1^\dagger a_1 + \epsilon_2 a_2^\dagger a_2 + \alpha(a_1^\dagger a_2 + a_2^\dagger a_1) + c(a_1^{\dagger 2} a_1^2 + a_2^{\dagger 2} a_2^2), \tag{1}$$

where a_i and $a_i^\dagger$ $(i = 1, 2)$ are both bosonic operators. The physical model with Hamiltonian (1) could be realized by confining a BEC into a double well trap obtained by e.g. superimposing an optical lattice on an optical dipole trap[3,4] or by other means.[28] Hamiltonian (1) corresponds to the case where only one mode in each of the two traps is significantly populated. All other modes are to be considered as a heat bath. α is the tunnelling parameter and c is the coupling constant of the nonlinear local interaction between the atoms. ϵ_j $(j = 1, 2)$ are the site energies in each trap.

There are two types of noise and dissipation that might be important in experiments with trapped BEC.[12] The most important source of noise is considered to be elastic scattering on the atoms of the background gas. This causes only phase noise and heats the condensate but leaves the total number of atoms N conserved. The second type of noise and dissipation is due to inelastic scattering and induces growth and depletion of the BEC. In modelling the two types of environmental influences it is assumed that the

Markov approximation is applicable.[12,29] Then, the corresponding master equation is in the Lindblad–Kossakowski–Gorini–Sudarshan form[30,31,16] with the four independent Lindblad operators corresponding to the two types of noise and two degrees of freedom: $L_{\rm p}^{1,2} = \gamma_{\rm p} n_{1,2} \equiv \gamma_{\rm p} a^\dagger_{1,2} a_{1,2}$, $L_{\rm a} = \gamma_{\rm a} a_{1,2}$:[19,29]

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}t} =& -\mathrm{i}[H,\rho] - \frac{\gamma_{\rm p}}{2}\sum_{j=1,2}([n_j^2,\rho]_+ - 2n_j\rho n_j) \\ &+ \sum_{j=1,2}\frac{\gamma_{\rm a}}{2}\left([a_j^\dagger,\rho]_+ - 2a_j\rho a_j^\dagger\right).\end{aligned} \tag{2}$$

The term proportional to $\gamma_{\rm p}$ describes the phase noise and the term proportional to $\gamma_{\rm a}$ models inelastic scattering. Sometimes[29] the values of local site dumping rate $\gamma_{\rm a}$ are assumed to be different in different wells, which can produce interesting effects like stochastic coherence, but we always consider that the values of $\gamma_{\rm a}$ in different wells are equal. In general, in the current experimental conditions the phase noise is much more effective than the amplitude noise.[29,14]

QSD theory[17] provides an alternative description of an open quantum system with continuous evolution in a Markov environment. The evolution equation is a nonlinear stochastic Schrödinger equation in the following form:

$$\begin{aligned}|\mathrm{d}\psi\rangle =& -\mathrm{i}\,H|\psi\rangle\mathrm{d}t + \frac{1}{2}\sum_i(2\langle L_i^\dagger\rangle L_i \\ &- L_i^\dagger L_i - \langle L_i^\dagger\rangle\langle L_i\rangle)|\psi\rangle\mathrm{d}t \\ &+ \sum_i(L_i - \langle L_i\rangle)|\psi\rangle\mathrm{d}W_i(t),\end{aligned} \tag{3}$$

where H and L_i are the same Hamiltonian and Lindblad operators as those in Lindblad master equation (2), and $\mathrm{d}W_i$ represents a differential increment of complex Winer process, i.e.

$$\mathrm{d}W_i\overline{\mathrm{d}W}_k = \delta_{ik}\mathrm{d}t,\ \ \mathrm{d}W_i\mathrm{d}W_k = 0,\ \ E[\mathrm{d}W_i] = 0, \tag{4}$$

where $E[\ ]$ denotes the expectation over realizations of the stochastic process (3), $\overline{\mathrm{d}W}_k$ is the complex conjugate of $\mathrm{d}W_k$, and $\delta_{i,j}$ is the Kronecker delta. Equation (4) is an Ito–Langevin equation of diffusion in the Hilbert space of pure states. The drift term, proportional to $\mathrm{d}t$, describes unitary evolution generated by Hamiltonian H and the dissipation given by L_i. Fluctuation is described by the term proportional to $\mathrm{d}W_i$. The basic relation between the solutions of Eqs. (2) and (3) is given by

$$\rho(t) = E[|\psi(t)\rangle\langle|\psi(t)|]. \tag{5}$$

There are other forms of stochastic Schrödinger equation for pure state evolution that satisfies the unravelling property (5), but the QSD equation (3) is the only one with the same symmetries as the Lindblad master equation. As pointed out in the introduction, the QSD equation enables one to separate purely quantum dispersions $\Delta_{\rm q}A$ from those due to averaging over the classical ensemble in Eq. (5). QSD theory and other stochastic unravellings of an open quantum system dynamics have often been used to study the quantum-to-classical transition.[32−40] The quantum, classical and total dispersions are calculated using stochastic pure states as follows:[16,22,23]

$$\begin{aligned}\Delta_{\rm q}^2A =& E[\langle\psi(t)|A^2|\psi(t)\rangle \\ &- \langle\psi(t)|A|\psi(t)\rangle^2] = E[\Delta_\psi^2A],\end{aligned} \tag{6}$$

$$\begin{aligned}\Delta_{\rm c}^2A =& E[\langle\psi(t)|A|\psi(t)\rangle^2] \\ &- (E[\langle\psi(t)|A|\psi(t)\rangle])^2,\end{aligned} \tag{7}$$

$$\Delta^2A = \Delta_{\rm q}^2A + \Delta_{\rm c}^2A. \tag{8}$$

Notice that $\Delta_{\rm q}A$ and $\Delta_{\rm c}A$ cannot be defined using only $\rho(t)$, but can be experimentally determined.[16] $\Delta_{\rm q}^2A$ represents the average variance in pure states that appear in the unravelling of ρ. Thus, it is a measure of average intrinsic quantum variance. On the other hand, $\Delta_{\rm c}^2A$ is the variance of the c-number $\langle\psi|A|\psi\rangle$ and represents statistical fluctuation of this classical quantity. $\Delta_{\rm q}A(t)$ and $\Delta_{\rm c}A(t)$ can be computed separately using Eq. (3), while master equation (2) enables one to calculate only the total dispersion $\Delta^2A = \mathrm{Tr}[\rho(t)A^2] - (\mathrm{Tr}[\rho(t)A])^2$. It is the behaviour of $\Delta_\psi A(t)$ along typical sample paths $|\psi(t)\rangle$ that is important for the classical appearance of the variable $\langle\psi|A|\psi\rangle$.

Besides the fundamental insight provided by the pure state description of the open system evolution, equation (3) is a very powerful computational tool.[17,41] The master equation (2) is an equation of d^2 variable, where d is the effective dimension of the Hilbert space of the system. On the other hand, the QSD equation involves only d variable. Furthermore, due to the localization properties of the open system dynamics, the number of effective dimensions d' that are needed to solve Eq. (3) numerically can be explicitly and significantly reduced using a moving bases technique.[41] Thus, the effective dimension d' is much smaller than d: $d' \ll d \ll d^2$.

In the next section we use the QSD equation to study the pure state mean-field approximation and the quantum pure state dispersion $\Delta_\psi A$ of the open BH model with the Lindblad operator as in Eq. (2).

3. Pure state dynamics of an open BH system

Our objective is to obtain relevant conclusions about the classical limit from a comparison of the evolution between expectation $\langle A\rangle$ and dispersion $\Delta_\psi^2 = \langle A^2\rangle - \langle A\rangle^2$ in the pure state $|\psi\rangle$ that describes the stochastic evolution of a single BH system for different types of noises and different observables. We are especially interested in the dynamics of the open BH that can be considered as a small variant of some classical mechanical model under the influence of the appropriate noise.

Our main conclusions are obtained from numerical solutions of the QSD equation and are presented in the second part of this section. First we consider the stochastic evolution equations for expectations of the basic set of dynamical observables. The equation of motion for the expectation of an arbitrary operator A in the state that satisfies Eq. (4) is given by

$$\begin{aligned}\mathrm{d}\langle A\rangle &= \mathrm{i}\langle[H,A]\rangle\mathrm{d}t - \frac{1}{2}\langle L^\dagger[L,A]+[A,L^\dagger]L\rangle\mathrm{d}t\\ &\quad+\sigma(A^\dagger,L)\mathrm{d}W+\sigma(L,A)\overline{\mathrm{d}W}, \qquad (9)\end{aligned}$$

where $\sigma(A,B)=\langle A^\dagger B\rangle - \overline{\langle A\rangle}\langle B\rangle$ and we abbreviate $\sum_i L_i \equiv L$.

The Hamiltonian and the Lindblad operators of the open BH system are defined in terms of bosonic operators a_i and $a_i^\dagger$. So we first consider the evolution of the expectation (10) for the operators a_i and $a_i^\dagger$ and introduce formally the classical limit by rescaling the basic operators with common positive parameter β so that the relevant commutator is proportional to β^2. Thus, formally when β is small the system is close to its classical limit.

Expectation values of the basic variables satisfy stochastic differential equation

$$\begin{aligned}\mathrm{d}\langle a_1\rangle &= \mathrm{i}\,\beta^2(-\epsilon_1\langle a_1\rangle+\alpha\langle a_2\rangle)\mathrm{d}t\\ &\quad-2\beta^4 c\langle a_1^\dagger a_1^2\rangle\mathrm{d}t\\ &\quad+(\beta^2\gamma_{\mathrm{p}}+\gamma_{\mathrm{a}})\beta^2\langle a_1\rangle\mathrm{d}t+\text{fluctuations},\\ \mathrm{d}\langle a_2\rangle &= \mathrm{i}\,\beta^2(-\epsilon_1\langle a_2\rangle+\alpha\langle a_1\rangle)\mathrm{d}t\\ &\quad-2\beta^4 c\langle a_2^\dagger a_2^2\rangle\mathrm{d}t\\ &\quad+(\beta^2\gamma_{\mathrm{p}}+\gamma_{\mathrm{a}})\beta^2\langle a_2\rangle\mathrm{d}t+\text{fluctuations},\\ \mathrm{d}\langle a_1^\dagger\rangle &= \mathrm{i}\,\beta^2(\epsilon_1\langle a_1^\dagger\rangle-\alpha\langle a_2^\dagger\rangle)\mathrm{d}t\\ &\quad+2\beta^4 c\langle a_1^{\dagger 2} a_1\rangle\mathrm{d}t\\ &\quad+(\beta^2\gamma_{\mathrm{p}}+\gamma_{\mathrm{a}})\beta^2\langle a_1^\dagger\rangle\mathrm{d}t+\text{fluctuations},\\ \mathrm{d}\langle a_2^\dagger\rangle &= \mathrm{i}\,\beta^2(\epsilon_2\langle a_2^\dagger\rangle-\alpha\langle a_1^\dagger\rangle)\mathrm{d}t\\ &\quad+2\beta^4 c\langle a_2^{\dagger 2} a_2\rangle\mathrm{d}t\\ &\quad+(\beta^2\gamma_{\mathrm{p}}+\gamma_{\mathrm{a}})\beta^2\langle a_2^\dagger\rangle\mathrm{d}t+\text{fluctuations}. \qquad (10)\end{aligned}$$

The fluctuation terms are in the following form: In $\mathrm{d}\langle a_1\rangle$ fluctuations are

$$\begin{aligned}&= \sigma(a_1^\dagger,L_{\mathrm{p}})\mathrm{d}W_1+\sigma(L_{\mathrm{p}},a_1)\overline{\mathrm{d}W}_1\\ &\quad+\sigma(a_1^\dagger,L_{\mathrm{a}})\mathrm{d}W_2+\sigma(L_{\mathrm{a}},a_1)\overline{\mathrm{d}W}_2\\ &= \beta^2\gamma_{\mathrm{p}}[\langle a_1a_1^\dagger a_1\rangle+\langle a_1a_2^\dagger a_2\rangle\\ &\quad-\langle a_1\rangle\langle a_1^\dagger a_1+a_2^\dagger a_2\rangle]\mathrm{d}W_1\\ &\quad+\beta^2\gamma_{\mathrm{p}}[\langle a_1^\dagger a_1a_1\rangle+\langle a_2^\dagger a_2a_1\rangle\\ &\quad-\langle a_1\rangle\langle a_1^\dagger a_1+a_2^\dagger a_2\rangle]\overline{\mathrm{d}W}_1\\ &\quad+\beta\gamma_{\mathrm{a}}[\langle a_1a_1\rangle+\langle a_1a_2\rangle-\langle a_1\rangle\langle a_1+a_2\rangle]\mathrm{d}W_2\\ &\quad+\beta\gamma_{\mathrm{a}}[\langle a_1^\dagger a_1\rangle+\langle a_1a_2^\dagger\rangle-\langle a_1\rangle\overline{\langle a_1+a_2\rangle}]\overline{\mathrm{d}W}_2, \qquad (11)\end{aligned}$$

and similarly for other variables.

A couple of observations about Eqs. (10) and (11) are obvious. First, the non-unitary drifts due to both the phase and the amplitude noises have the same form. Thus, in the purely quantum regime, i.e. when $\beta=1$, the drift, in terms of the basic variables $a_1, a_2, a_1^\dagger, a_2^\dagger$, due to the amplitude noise is smaller than due to the phase noise since $\gamma_{\mathrm{a}}<\gamma_{\mathrm{p}}$ for BEC in current experimental conditions. On the other hand, the drift due to the phase noise is multiplied by β^4 and due to the amplitude noise by β^2, so that as the system becomes more classical, i.e. for small β, the amplitude noise becomes more important than the phase noise. A similar conclusion is true about the fluctuation terms. Secondly, both the unitary part of the drift due to the on-site atomic interactions (proportional to $c\beta^4$), and the fluctuation terms involve higher order moments of the dynamical variables. If the expectations of the products are approximated by products of expectations, like in the mean-field approximation, the fluctuation terms completely disappear.

The classical limit of the isolated BH system (1) is commonly studied using the mean-field approximation for the dynamics of a special set of variables suggested by the symmetries of the system.[26,11] The Hamiltonian can be written entirely in terms of the angular momentum operators J_x, J_y, J_z, defined, respectively, as

$$J_x = \frac{1}{2}(a_1^\dagger a_2+a_2^\dagger a_1),$$

$$
\begin{aligned}
J_y &= \frac{i}{2}(a_1^\dagger a_2 - a_2^\dagger a_1),\\
J_z &= \frac{1}{2}(a_2^\dagger a_2 - a_1^\dagger a_1)),
\end{aligned} \tag{12}
$$

which satisfy $SU(2)$ commutation relations. The total particle number $N = n_1 + n_2$ is related to $SU(2)$ Casimir operator: $J^2 = N(N/2+1)/2$ and is conserved. The $SU(2)$ symmetry also suggests a special set of initial states, that is, the $SU(2)$ coherent states. The Hamiltonian in terms of $J_{x,y,z}$ with explicit classicality parameter β reads

$$
H = -2\beta^2\alpha J_x + 2\beta^2(\epsilon_2 - \epsilon_1)J_z + c\beta^4 J_z^2. \tag{13}
$$

Thus, $SU(2)$ is the dynamical group of the BH model. The classical limit obtained for small β corresponds here to large values of the angular momentum and the total particle number.

Angular momentum variables $\langle J_{x,y,z}\rangle$ play a crucial role in the mean-field approximation for the isolated BH model.[11] However, notice that the Hamiltonian is a nonlinear expression of the group generator, so that the set of $SU(2)$ coherent states is not invariant during the evolution of the isolated BH system. Nevertheless, asymptotically as $\beta \to 0$ the dispersions of all basic variables $\langle J_{x,y,z}\rangle$ during the unitary evolution are negligible compared with the averages, so that the mean-field approximation is valid. However, different noise can influence the mean-field dynamics in different ways. Notice that although the Linblad operators $L_{\mathrm{p}}^{1,2}$ and $L_{\mathrm{a}}^{1,2}$ cannot be written in terms of angular variables, the Linblad operators for the phase noise $L_{\mathrm{p}}^{1,2} = \gamma_{\mathrm{p}} a_{1,2}^\dagger a_{1,2}$ commute with J_z and N. This suggests that the J_z variable has a special status in the BH system with the phase noise. The phase noise can significantly speed up evolution away from the set of the $SU(2)$ coherent states towards the eigenstate of J_z, which crucially affects the ratio of the dispersions to the averages of the basic variables.

We also analyse the evolution of dynamical variables that appear when the BH model is considered as a pair of coupled nonlinear oscillators. The basic dynamical variables are given by $q_j = (a_j^\dagger + a_j)/\sqrt{2}$, and $p_j = \mathrm{i}(a_j^\dagger - a_j)/\sqrt{2}$, $(j = 1, 2)$. The Hamiltonian and the Lindblad operators with the explicit scaling parameter β, respectively, read

$$
\begin{aligned}
H &= \beta^2\epsilon_1(p_1^2+q_1^2)/2 + c\beta^4(p_1^2+q_1^2)^2/4\\
&\quad + \beta^2\epsilon_1(p_1^2+q_1^2)/2 + c\beta^4(p_1^2+q_1^2)^2/4\\
&\quad + \beta^2\varDelta(p_1p_2 + q_1q_2),
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
L_{\mathrm{p}}^{1,2} &= \beta^2\gamma_{\mathrm{p}}(p_{1,2}^2 + q_{1,2}^2),\\
L_{\mathrm{a}}^{1,2} &= \beta\gamma_{\mathrm{p}}[(q_{1,2} + \mathrm{i}\,p_{1,2})/\sqrt{2}.
\end{aligned} \tag{15}
$$

The classical limit, obtained for small β, corresponds to large values of basic variables $\langle q_i\rangle$ and $\langle p_i\rangle$ as compared with the unit cell in the phase space, or with the large masses and low frequencies of the two oscillators. From various well-known studies of decoherence in model systems it is expected that the dominant amplitude noise acting independently on the two oscillators' degrees of freedom will favour the oscillator coherent states as the asymptotic states.

The dynamical equations of expression (9) can be written for the averages of the dynamical variables $q_{1,2}, p_{1,2}$ or $J_{x,y,z}$, but, as is illustrated in Eqs. (10) and (11), involve moments of higher orders. Also, in the case of $J_{x,y,z}$ the equations cannot be written solely in terms of $\langle J_{x,y,z}\rangle$. We do not write down these equations since they have not been used in our numerical computations, which are based on the solutions of basic QSD equation (3).

4. Numerical results

Evolution of the quantum dispersion, i.e. dispersion in pure states, of various observables is studied using the numerical solution of the QSD equation (3). We are interested in the connection between different environments on one hand and the behaviour along stochastic orbits of the pure state dispersions $\varDelta_\psi A(t)$ for various observables on the other. We study the dependence of this relation on (i) the value of the classicality parameter β, (ii) on the nonlinearity parameter c, and (iii) on the type of the initial state.

Small values of the parameter β slow down evolution, that is, rescale the time variable, besides their primary role which is to distinguish different terms in the evolution equation multiplied by different powers of β. However, since there exists no non-autonomous term, i.e. an explicitly time-dependent term in the Hamiltonian, there is no need to rescale the time.

Our conclusions are presented in Figs. 1–4, which illustrate typical results of numerical computation.

Dynamics of the averages $\langle A\rangle$ and the corresponding dispersions $\varDelta_\psi A$ for the isolated BH model are illustrated in Fig. 1 for different values of the classicality parameter β. The purpose of these figures is to illustrate the conclusion that the ratio $\varDelta_\psi A/\langle A\rangle$ increases during evolution irrespective of the values of classicality β (or nonlinearity c). Of course, smaller β implies slower evolution, but $\Delta_\psi A$ increases for any nonzero β. Similarly, other nonzero values of c comparable to $\epsilon_{1,2}$ and α ($c=\alpha=\epsilon_{1,2}=1$) or much smaller ($c=0.01\epsilon_{1,2}$) always imply growth of the amplitude of quantum dispersion with time. This is true for observables of either of the two kinds of classical models, that is for q_1, q_2, p_1, p_2 or $J_{x,y,z}$. This conclusion is illustrated in Fig. 1 using as the initial states the direct product of coherent states for the two oscillators q_1, p_1 and q_2, p_2 but the behaviour of the dispersions is qualitatively the same for a generic initial state. However, the class of $SU(2)$ coherent states displays a different behaviour (not illustrated). With these initial states, the ratio $\Delta_\psi J_{x,y,z}/\langle J_{x,y,z}\rangle$ is proportional to β so that $\langle J_{x,y,z}\rangle$ asymptotically as $\beta\to 0$ displays classical behaviour. This is the basic assumption underlying the mean-field approximation and the derivation of the Gross–Pitaevsky equation.[11]

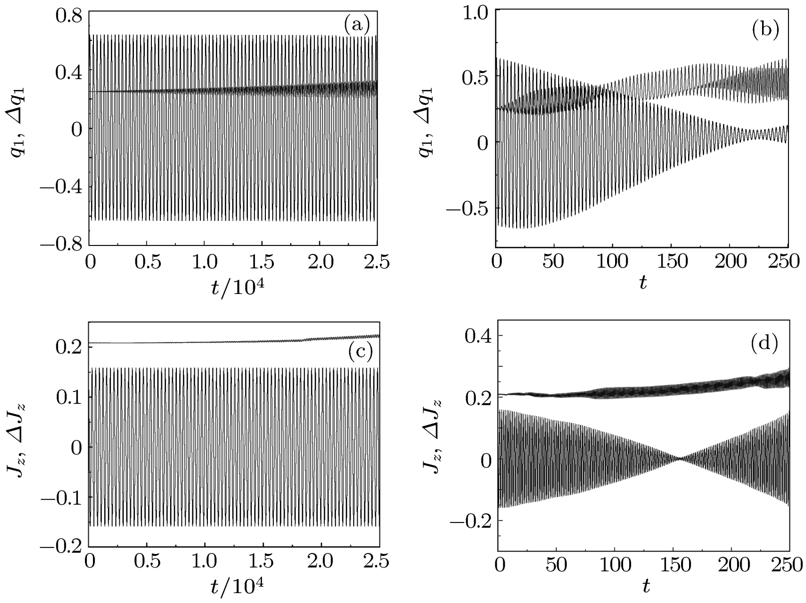

Fig. 1. Evolution of the averages $\langle q_1\rangle$ ((a) and (b), black line) and $\langle J_z\rangle$ ((c) and (d), black line) and pure state dispersions $\Delta_\psi q_1$ (Figs. 1(a) and 1(b), grey line) and ΔJ_z ((c) and (d) grey line) for the classicality parameter $\beta=0.01$ ((a) and (c)) and $\beta=1$ ((b) and (d)). Other parameters are $\epsilon_1=\epsilon_2=1$, $\alpha=1$, $c=0.01$, and t is dimensionless time $t\equiv t\epsilon_1$.

Thus, we conclude that in an isolated BH system small values of β and/or small values of c are not sufficient to imply classical behaviour for all initial states of any of the interesting observables with potentially classical interpretation.

The influence of the phase noise $L_{\rm p}^{1,2}$ is illustrated in Fig. 2. The asymptotic states with such a noise are the number states. The initial states in Figs. 2 are products of oscillators coherent states in Figs. 2(a), 2(c), 2(d) and the number states in Fig. 2(b). Dispersions of q_1 (and of other variables of the two-oscillator model) become and stay larger than the corresponding averages. On the other hand, we see that the dispersion of the variable J_z (and similarly of n_1, n_2) becomes small irrespective of the type of initial state. Thus, $\langle J_z\rangle$ behaves as a classical variable, but the other two components $J_{x,y}$ have large dispersions. Qualitatively the same behaviour is observed for all sufficiently large $\gamma_{\rm p}$ and sufficiently small classicality parameter β. Then the conclusion is the same even when nonlinearity is not small, say $c=\epsilon_{1,2}$. However, in the purely quantum case $\beta=1$, classical behaviour of the $\Delta_\psi J_z$ induced by phase noise does not occur. There is no special behaviour of the angular momentum dispersion for the $SU(2)$ coherent initial states (not illustrated), i.e. only ΔJ_z quickly becomes very small while the other two are large, which is contrary to the case of the isolated BH model.

Now, we consider the effects of the amplitude noise $L_{\rm a}=\gamma_{\rm p}(a_1+a_2)$ illustrated in Fig. 3. Here, we choose to illustrate the dynamics from the generic initial state using the number of initial states. Thus, initially $\Delta_\psi J_z=0=\langle q_j\rangle=\langle p_j\rangle$ and $\Delta_\psi q_1\gg 1$, which cannot be seen clearly in the figure. The parameter $\gamma_{\rm a}$ is chosen to be much smaller than $\gamma_{\rm p}$ in the previous figure. We see that the variables $\langle q_1\rangle, \langle p_1\rangle$

(and $\langle q_2\rangle, \langle p_2\rangle$) can be considered to be classical, since the dispersions of all these variables quickly become small and do not increase. On the other hand, the dispersions of $\Delta_\psi J_{z,x,y}$ and the corresponding averages $\langle J_{z,x,y}\rangle$ simultaneously converge to zero, due to dissipation, but the dispersions are always larger than the averages $[\Delta_\psi J_{z,x,y}/\langle J_{z,x,y}\rangle] \gg 1$. These qualitative conclusions are obtained with other values of $\gamma_{\rm p}$ and c. However, like in the case of phase noise, when β is not small, $\beta \approx 1$ no classical behaviour of dispersions of any of the observables q_i, p_i or $J_{x,y,z}$ is observed. The dynamics from $SU(2)$ coherent initial states is illustrated in Fig. 4.

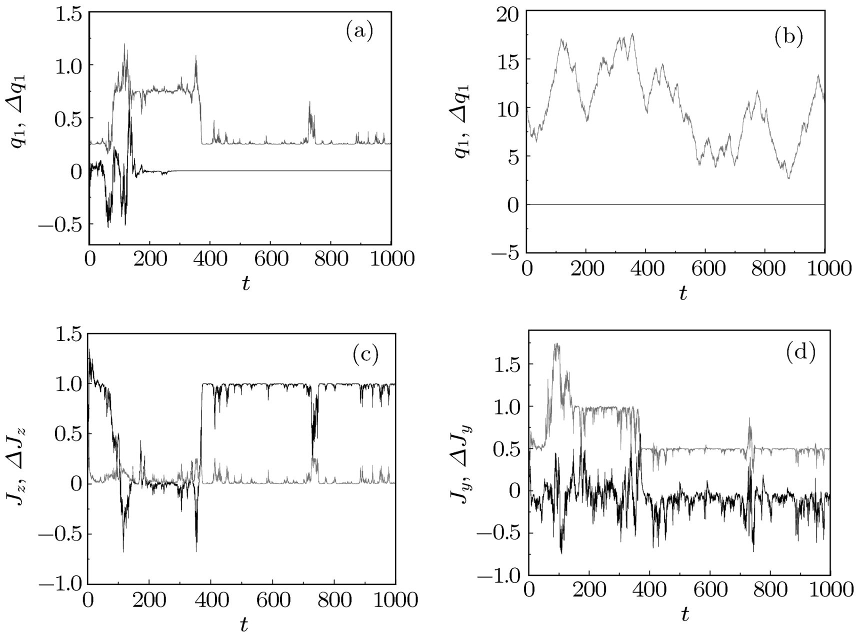

Fig. 2. Influence of phase noise with $\gamma_{\rm p} = 0.075$. Evolution of the averages $\langle q_1\rangle$ (panels (a) and (b) black line) and $\langle J_z\rangle$ ((c), black line), $\langle J_y\rangle$ ((d), black line) and pure state dispersions $\Delta_\psi q_1$ ((a) and (b) grey line) and ΔJ_z ((c), grey line) and ΔJ_y ((d), grey line). The classicality parameter $\beta = 0.01$. Initial states are oscillator coherent states ((a), (c), and (d)) and the number state (b). Other parameters are $\epsilon_1 = \epsilon_2 = 1$, $\alpha = 1$, $c = 0.01$, and t is dimensionless time $t \equiv t\epsilon_1$.

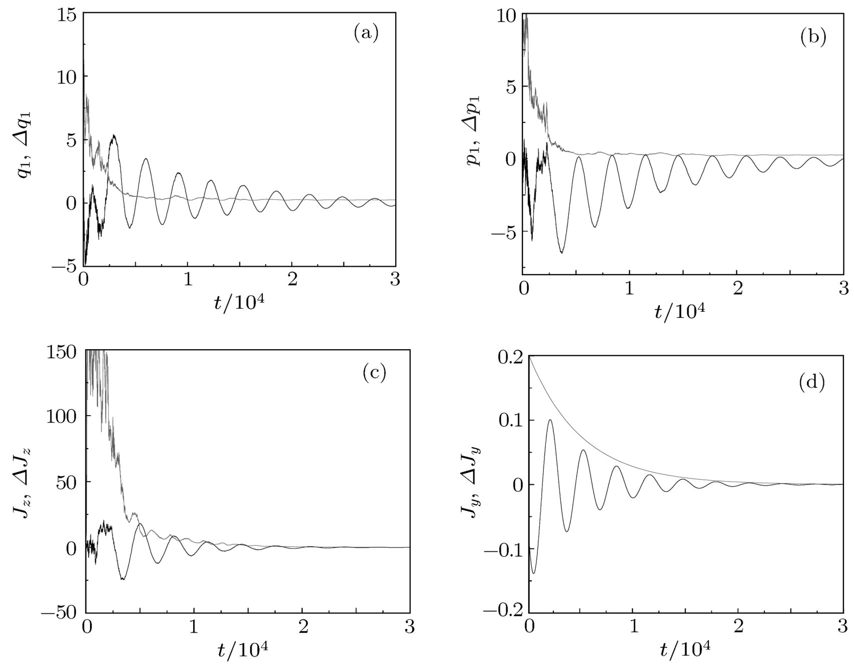

Fig. 3. Influence of the amplitude noise with $\gamma_{\rm a} = 0.001$. Evolution of the averages $\langle q_1\rangle$, $\langle p_1\rangle$ (panels (a) and (b), black line) and $\langle J_{z,y}\rangle$ ((c) and (d) black line), and pure state dispersions $\Delta_\psi q_1$, $\Delta_\psi p_1$ ((a) and (b), grey line) and $\Delta J_{x,y}$ ((c) and (d), grey line) for the classicality parameter $\beta = 0.01$. The initial state is the number state. Other parameters are $\epsilon_1 = \epsilon_2 = 1$, $\alpha = 1$, $c = 0.01$, and t is dimensionless time $t \equiv t\epsilon_1$.

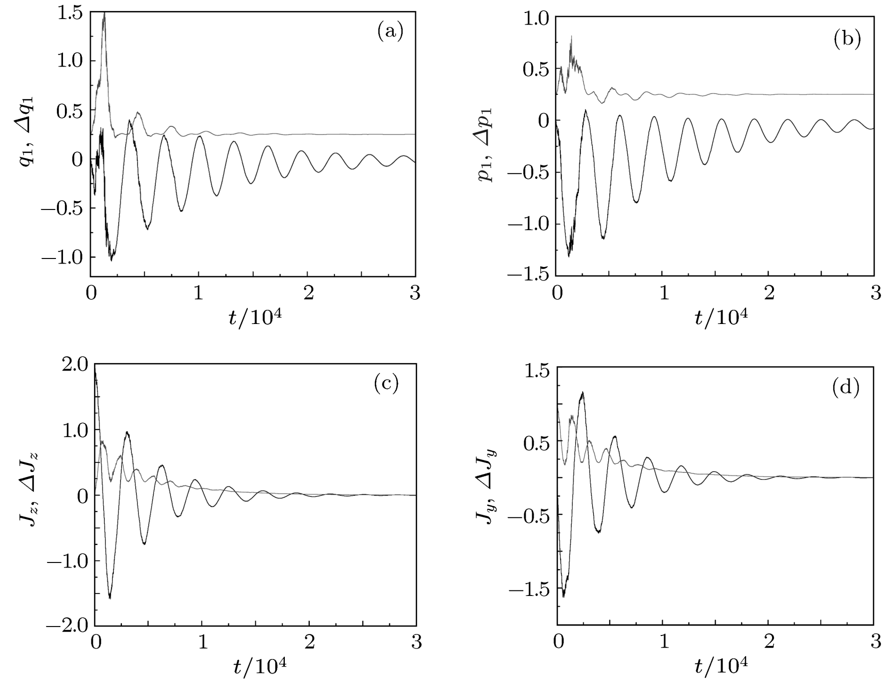

Fig. 4. The same as in Fig. 3 but the initial state is an $SU(2)$ coherent state.

Dispersions $\Delta_\psi J_y$ and $\Delta_\psi J_z$ are always smaller than the corresponding averages, and together converge to zero. Dispersions of the variables $\langle q_i\rangle$, $\langle p_i\rangle$ behave in the same way as with the number initial states, i.e. quickly converge to a small value characteristic of the oscillators coherent states. We can conclude that with the amplitude noise and any initial condition, for small β, the oscillator variables $\langle q_i\rangle$ and $\langle p_i\rangle$ behave approximately classically, while the variable $\Delta_\psi J_{y,z}$ appears classical only if the initial state is one of the $SU(2)$ coherent initial states. For the described behaviour from any of the initial states to occur the classicality parameter must be sufficiently small, i.e. for $\beta = 1$ no variable displays classical behaviour.

5. Summary

We studied an open Bose–Hubbard system with two types of noise that are relevant in the current experiments. Our goal is to examine which physical variables under what conditions can be considered to be classical in the sense that the dispersions remain negligible compared with the averages during evolution. This question is crucial for the mean-field approximation of open system dynamics.

The open system can be described by its density matrix or by an ensemble of pure states. In the description by the density matrix the classical and the quantum contributions to the averages and the dispersions are inseparable, but in the stochastic pure state description the two contributions can be clearly distinguished and can be computed separately. We use the description in terms of stochastic pure states as provided by quantum state diffusion theory, and the pure state quantum averages and dispersions of various operators are calculated.

In our analyses three parameters β, c, and γ play important roles. The classicality parameter β corresponds to different physical quantities, depending on the classical model, but small values of β always correspond to the classical limit. $\beta = 1$, on the other hand, corresponds to the purely quantum system. The nonlinearity parameter c measures how much the Hamiltonian deviates from a linear expression of the main dynamical variables. The parameters $\gamma_{\rm p}$ and $\gamma_{\rm a}$ represent the strengths of the phase or the amplitude noise respectively.

Our main conclusions can be summarized as follows.

(I) No variable $\langle A\rangle$ of an isolated BH system can be considered to be classical for general initial states and any nonzero values of β and c. The angular variable behaves as a classical one asymptotically in the limit $\beta \to 0$ if the initial state is one of the $SU(2)$ coherent states.

(II) In the case of phase noise $L_{\rm p}^{1,2}$ the variable $\langle J_z\rangle$ can be considered as a classical variable since $\Delta_\psi J_z$ is much smaller than $\langle J_z\rangle$. q_i and p_i (and J_x, J_y)

do not display classical behaviour. This is true for all initial states and sufficiently small nonzero β.

(III) In the case of the amplitude noise $L_{\rm a}^{1,2}$, and for small β, variables $\langle q_i \rangle, \langle p_i \rangle$ behave classically with small dispersions. Angular variables in general satisfy $\Delta_\psi J_{x,y,z}/J_{x,y,z} > 1$ so that they cannot be considered to be classical for any nonzero β. In the case of $SU(2)$ the coherent initial state $\Delta_\psi J_{x,y,z}/J_{x,y,z} < 1$.

Thus, the type of noise crucially determines which of the variables of the open BH model can be considered to be classical. From the point of view of decoherence theory this is not an unexpected result, but it is important in the formulation of mean-field approximation of the open BH system.

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