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2012 Chinese Phys. B 21 070302

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Chin. Phys. B Vol. 21, No. 7 (2012) 070302

Dynamical time versus system time in quantum mechanics*

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(Received 12 September 2011; revised manuscript received 21 March 2012)

Properties of an operator representing the dynamical time in the extended parameterization invariant formulation of quantum mechanics are studied. It is shown that this time operator is given by a positive operator measure analogously to the quantities that are known to represent various measurable time operators. The relation between the dynamical time of the extended formulation and the best known example of the system time operator, i.e., for the free one-dimensional particle, is obtained.

Keywords: time operators, extended state space, positive operator valued measure

PACS: 03.65.Ca, 03.65.Ta, 03.65.Xp **DOI:** 10.1088/1674-1056/21/7/070302

1. Introduction

Fundamental dynamical theories of classical and quantum nonrelativistic physics treat the time as an evolution parameter, i.e., as an independent variable conceptually and mathematically different from dependent dynamical variables or observables such as the coordinate, the momentum, and the functions thereof. However, the relativistic covariance,[1−4] on one hand, and the direct measurements of the time of occurrence and the duration time,[7−13] on the other, require that the time is treated in the same way as the standard measurable quantities.

Corresponding to the two main motivations, namely, the relativistic covariance and the measurements of time, there are two fundamentally different approaches to the representation of time as a physical measurable quantity. Our paper explores properties of the time operator in the reparameterization invariant formulation of Hamiltonian dynamics and is directed towards an explanation of the relations between different time observables as they might be and have been introduced following the two motivations.

The approach motivated by the relativistic covariance is based on the so-called parametric form of the Hamiltonian dynamics,[1,2,14] with two analogous formulations for classical[14] and quantum[1] systems. The time variable/observable is represented and treated as an additional degree of freedom. Original degrees of freedom and the time form the so-called extended phase space. However, the parametric invariance represents the gauge symmetry, which is treated as a constraint on the extended phase space. The constraint involves the Hamiltonian of the extended systems. In the quantum case, the physical interpretation of the constraint is specially problematic, leading to what has been called the problem of time in quantum gravity.[4]

In the second approach, a specific solvable system, such as free particle or harmonic oscillator, is considered, and the time variable/observable proportional to the evolution parameter[10,11] or the time of specific events occurring in the system[7−9] is expressed in terms of the basic dynamical variables/observables, say the canonical coordinates and momenta. In order to let such a time variable/observable be proportional to the evolution parameter, the condition of covariance is imposed on the Poisson bracket $\{H,T\}=1$ or the commutator $[\hat{H},\hat{T}]=\mathrm{i}$ of the time and the Hamiltonian. It is well known that neither the classical nor the quantum form of the covariance conditions can be satisfied with physically acceptable Hamiltonians. In the classical case, the relation with a generic H can be satisfied only locally; and in the quantum case, the spectrum of $\hat{H}$ would have to be the whole of $\mathbf{R}$. Similarly, expressions for the time of occurrence of different

*Project supported by the Ministry of Science and Education of the Republic of Serbia (Grant Nos. 171017, 171028, and 171006).
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dynamical events, like the arrival time, have been expressed in terms of canonical variables/operators via expressions that are singular in the classical case or results with non-self adjoined operators in the quantum case. A general consensus in the quantum case is that when the time is expressed in terms of canonical operators, the resulting expression is an object that does not give a projector measure (PM) on $\mathbf{R}$, while a self-adjoined operator does. Instead, the mathematical object representing the time operator is defined via nonorthogonal resolution of unity by positive operators, i.e., by a positive operator valued measure (POVM).

The purpose of this paper is to explore relations between the two approaches in the context of quantum mechanics. We shall start with the parametric Hamiltonian formulation and study the mathematical properties and the physical interpretation of the naturally defined dynamical time. We shall see that the dynamical time, introduced quite generally in the parametric formulation formalism, is given by a nonorthogonal resolution of unity. This is the typical property of operators representing various time observables in terms of the canonical operators. It will then be shown for the example of a free particle that an operator formulated using the dynamical time of the parametric formulation reproduces the probability distributions given by the well-known POVM obtained by representing the time in terms of the canonical variables.

The paper is organized as follows. In the next section, we briefly recapitulate the parametric formulation for quantum systems. In Section 3, we first present the main properties of the dynamical time operator. We then introduce an operator based on the dynamical time and compare it with the well known POVM representation of the time for the free particle. Section 4 presents a brief discussion of the possible probabilistic interpretation of vectors from the extended space and from the dynamical space. A summary is given in Section 5. In the appendix, we provide an alternative mathematical framework, which avoids the mathematical difficulties of the Hilbert space formulation of the reparametrization invariance constraint.

2. Extended space and space of dynamical vectors

The parametric formulation of Hamiltonian dynamics based on the extended phase space enables us to treat the time formally on an almost equal footing as the other canonical coordinates. However, since the dynamics of the time must be trivial, i.e., the time must be proportional to the evolution parameter, the time appears as an unphysical degree of freedom, and the extended system is constrained on the subset of dynamical vectors by the corresponding constraint. Furthermore, the quantum mechanics of the constrained extended system introduces potential mathematical subtleties.

We shall very briefly sketch the classical Hamiltonian formulation of the extended system with the parametric invariance, and recapitulate the analogous formulation in quantum mechanics with more details.

In the parametric formulation of the Hamiltonian mechanics, the time is treated as the coordinate of an additional degree of freedom, and the evolution is parameterized by a new evolution parameter. In order to let the dynamical equations of the original and the extended systems describe the same evolution, the Hamiltonian of the extended system is defined as a sum of the original Hamiltonian H_{s} and the variable J_{t} conjugated to the time coordinate T, i.e., $H_{\mathrm{ex}} = H_{\mathrm{s}} - J_{\mathrm{s}}$. Here H_{ex} is a function on the extended phase space, for the example of one dimensional particle, it is $M_{\mathrm{ex}} = \mathbf{R}^2 \times \mathbf{R}^2$. The physical meaning of the time variable is formalized as a constraint on the extended phase space. In the classical case, the constraint is given by

$$H_{\mathrm{ex}}(q, p, J_{\mathrm{t}}) = H_{\mathrm{s}}(q, p) - J_{\mathrm{t}} = 0, \tag{1}$$

where $J_{\mathrm{t}} = h_{\mathrm{s}}(q, p)$, and $h_{\mathrm{s}}(q, p)$ is the value of H_{s} at (q, p).

The quantization of the extended system H_{ex} is often discussed as a simple example of a constrained quantization, i.e., reparametrization invariant, system.[1] In our recapitulation, we shall follow one of the possible quantization procedures, in which the extended phase space is canonically quantized as if there is no constraint. The constraints are then included in the form of conditions imposed on the space of dynamical vectors.

The classical approach suggests an analogous treatment of the extended quantum system with time as an additional physical observable. The Hilbert space $\mathscr{H}_{\mathrm{ex}}$ of the extended system is considered as the tensor product of the Hilbert space $\mathscr{H}_{\mathrm{s}}$ corresponding to the spatial degrees of freedom and the Hilbert space $\mathscr{H}_{\mathrm{t}}$ that corresponds to the time treated as an additional degree of freedom. The extended system will be defined by the commutation relations between the time and its conjugate, the extended Hamiltonian and the constraint.

The extended Hilbert space is defined as the direct product $\mathscr{H}_{\rm ex} = \mathscr{H}_{\rm s} \otimes \mathscr{H}_{\rm t}$. The structure of $\mathscr{H}_{\rm t}$ is dictated by the desired algebraic properties of the time variable represented by an operator $I \otimes \hat{T}$ acting on $\mathscr{H}_{\rm t}$. Analogy to the classical extended phase space, the Hilbert space $\mathscr{H}_{\rm t}$ should carry an irreducible representation of the same Lie algebra

$$[\hat{T}, \hat{J}_{\rm t}] = -{\rm i}. \tag{2}$$

Thus, $\hat{T}$ and $\hat{J}_{\rm t}$ are represented by multiplication and differentiation operators acting on functions from the corresponding domains in $\mathscr{H}_{\rm t}$. Of course, $\hat{T}$ and its conjugate $\hat{J}_{\rm t}$ commute with the operators acting in $\mathscr{H}_{\rm s}$. In particular, $\hat{T}$ commutes with Hamiltonian operator $\hat{H}_{\rm s}$.

Operators $\hat{T}$ and $\hat{J}_{\rm t}$ have continuous spectra on $\mathscr{H}_{\rm ex}$. Nevertheless, we shall often use the terminology eigenvalues and eigenvectors for those operators, these should be understood in the generalized sense. Let $\int |E\rangle\langle E|{\rm d}E$ denote the spectral resolution of unity associated with the Hamiltonian operator $\hat{H}_{\rm s}$. The integral is understood as a sum or as an integral over the spectra of $\hat{H}_{\rm s}$ depending on the types of the spectra. Similarly $\int_{\rm R} |J_{\rm t}\rangle\langle J_{\rm t}|{\rm d}J_{\rm t}$ and $\int_{\rm R} |t\rangle\langle t|{\rm d}t$ denote the continuous orthogonal resolutions of unity corresponding to operators $J_{\rm t}$ and T on $\mathscr{H}_{\rm t}$.

Consider a system with the Hamiltonian $\hat{H}_{\rm s} = \hat{H}_{\rm s}(\hat{q},\hat{p})$. In order to let the original system on $\mathscr{H}_{\rm s}$ and the extended one on $\mathscr{H}_{\rm ex}$ describe the same quantum evolution, the Hamiltonian of the extended system is defined as

$$\hat{H}_{\rm ex} = \hat{H} - \hat{J}_{\rm t}. \tag{3}$$

The Hamiltonian $\hat{H}_{\rm ex}$ and the operator $\hat{T}$ satisfy

$$\Delta H_{\rm ex}\Delta T \geq \langle {\rm i}[\hat{H}_{\rm ex}, \hat{T}]/2\rangle = 1/2. \tag{4}$$

Of course, the physical interpretation of Eq. (4) is not that of the time–energy uncertainty relation, since the system's energy is represented by the Hamiltonian $\hat{H}_{\rm s}$, which commutes with $\hat{T}$.

The classical constrain $H_{\rm ex} = 0$ is introduced into quantum mechanics by using a constraint that must be satisfied by the vectors from $\mathscr{H}_{\rm ex}$ as follows. It is declared that not all vectors from $\mathscr{H}_{\rm ex}$ should be considered as representing the dynamically possible vectors but only those that satisfy the following condition analogous to the classical equation of the constraint:

$$\hat{H}_{\rm ex}|\psi\rangle = (\hat{H}_{\rm s} - \hat{J}_{\rm t})|\psi\rangle = 0, \tag{5}$$

which is, for the self-adjoined operators, equivalent to

$$\langle\psi|(\hat{H}_{\rm s} - \hat{J}_{\rm t})^2|\psi\rangle = 0. \tag{6}$$

Since $\hat{H}_{\rm s}$ and $\hat{J}_{\rm t}$ are linear, the set of dynamical vectors is a linear subspace of $\mathscr{H}_{\rm ex}$. We denote the space of dynamical vectors, i.e., those that satisfy Eq. (5), by $\mathscr{H}_{\rm dyn} \subset \mathscr{H}_{\rm ex}$. However, the vectors satisfying the constraint might not have finite norms, which is the case if zero is an eigenvalue in the continuous spectrum of $H_{\rm ex}$. Mathematically, the rigorous formulation of the constraint (5) is provided by the geometric Hamiltonian formulation of quantum dynamics,[15–19] as is shown in Appendix A.

The constraint (5) demands that operators $\hat{H}_{\rm s}\otimes 1$ and $1 \otimes \hat{J}_{\rm t}$ are equal when restricted on the subspace of the dynamical vectors. Vector $|E_0\rangle\otimes|J_{\rm t}\rangle$ represents a dynamical vector if and only if $J_{\rm t} = E_0$, i.e., if the eigenvector $|J_{\rm t}\rangle$ of $\hat{J}_{\rm t}$ has the eigenvalue of $J_{\rm t}$ numerically equaling to the eigenvalue E_0 of the Hamiltonian. We denote such vectors by $|E\rangle \otimes |E\rangle \equiv |E, E\rangle$. A general vector from the extended space is given by

$$|\psi\rangle = \iint {\rm d}E\,{\rm d}J_{\rm t} f(E, J_{\rm t})|E\rangle \otimes |J_{\rm t}\rangle, \tag{7}$$

and the dynamical vectors are of the form

$$|\psi\rangle_{\rm dyn} = \int {\rm d}E f(E)|E\rangle \otimes |E\rangle. \tag{8}$$

The projector on the subspace of dynamical vectors $\mathscr{H}_{\rm dyn}$ is written as

$$\hat{P}_{\rm dyn} = \int {\rm d}E|E, E\rangle\langle E, E|\frac{1}{\delta(0)}, \tag{9}$$

where $1/\delta(0)$ cancels the norm of eigenvector $|J_T = E\rangle$. Obviously, the subspace of dynamical vectors is invariant under the Schrödinger evolution generated by $H_{\rm ex}$.

Notice that the second component of the elementary dynamical vector $|E\rangle\otimes|E\rangle$ is a generalized eigenvector of operator $\hat{J}_{\rm t}$. This operator has only a continuous spectrum, and its eigenvectors are not normalizable and do not belong to the Hilbert space $\mathscr{H}_{\rm t}$, but are properly speaking elements of the corresponding rigged Hilbert space. In the same way, the eigenspace $\mathscr{H}_{\rm dyn}$ of the extended hamiltonian $H_{\rm ex}$ must be understood in the sense of generalized vectors. The expression of $\hat{P}_{\rm dyn}$ given by Eq. (9) represents a projection operator only formally and in the generalized sense. Despite this fact, we shall continue to address $P_{\rm dyn}$ as the projection operator bearing in mind the mathematical subtleties involved in its definition.

The physical meaning of $\hat{P}_{\rm dyn}$ is seen in the fact that it associates with an arbitrary vector of the form $|\psi_0\rangle|t_0\rangle$ with its image $\hat{P}_{\rm dyn}|\psi_0\rangle|t_0\rangle \in \mathscr{H}_{\rm dyn}$, and the latter can be identified with a solution of the Schrödinger equation in $\mathscr{H}_{\rm s}$, i.e., the system's orbit that goes through the state $|\psi_0\rangle \in \mathscr{H}_{\rm s}$ at time t_0. In other words, the Cauchy problem for the Schrödinger evolution equation in $\mathscr{H}_{\rm s}$ is replaced by two algebraic equations for the vectors of the extended $\mathscr{H}_{\rm ex}$. The first one is the constraint (5) satisfied by each dynamical vector, i.e., each Schrödinger orbit on $\mathscr{H}_{\rm s}$. A particular orbit, which goes through $\psi_0\rangle_{\rm s} \in \mathscr{H}_{\rm s}$ at $t = t_0$, is described by the particular dynamical vector $|\psi\rangle_{\rm dyn} \in \mathscr{H}_{\rm dyn}$, which satisfies, besides constraint (5), the second condition

$$1 \otimes |t_0\rangle\langle t_0|\psi\rangle_{\rm dyn} = |\psi_0\rangle_{\rm s} \otimes |t_0\rangle. \tag{10}$$

The probabilistic interpretation of general vectors from $\mathscr{H}_{\rm ex}$ and $\mathscr{H}_{\rm dyn}$ is discussed in Section 4.

3. Dynamical time and system time

The parameter time of the Schrödinger equation is in the extended space related to operator $1\otimes\hat{T}$. The projection of this operator on the space of dynamical vectors $\mathscr{H}_{\rm dyn}$ gives an operator, which we shall call the dynamical time $\hat{T}_{\rm dyn}$

$$\hat{T}_{\rm dyn} = \hat{P}_{\rm dyn}(1 \otimes \hat{T})\hat{P}_{\rm dyn}. \tag{11}$$

Properties of the particular system enter into the definition of $\hat{T}_{\rm dyn}$ through projector $\hat{P}_{\rm dyn}$, i.e., through the constraint (5). However, the constraint is a consequence of the parametric invariance, and this is a property of the formalism not of the particular system. Therefore, rather peculiar properties of operator $\hat{T}_{\rm dyn}$ are valid for all Hamiltonian systems and are a consequence of the fact that the dynamical time $\hat{T}_{\rm dyn}$ is not a proper dynamical variable (constraint (5)), and it does not correspond to an independent degree of freedom. The properties are consistent with the interpretation of $\hat{T}_{\rm dyn}$ as an object representing the universal time external to any system. Let us formulate the main properties of $\hat{T}_{\rm dyn}$ all following from the explicit form (9) of projector $\hat{P}_{\rm dyn}$. In fact, by using Eq. (9), $\hat{T}_{\rm dyn}$ becomes

$$\begin{aligned}\hat{T}_{\rm dyn} &= \int |E,E\rangle\langle E,E|\frac{{\rm d}E}{\delta(0)}(1\otimes\hat{T}) \\ &\quad\times \int |E',E'\rangle\langle E',E'|\frac{{\rm d}E'}{\delta(0)} \\ &= \int |E,E\rangle\langle E,E|\frac{{\rm d}E}{\delta(0)}[\langle E|\hat{T}|E'\rangle/\delta(0)] \\ &= -{\rm i}\frac{\delta'(0)}{\delta(0)}\hat{P}_{\rm dyn}.\end{aligned} \tag{12}$$

The crucial properties of $\hat{T}_{\rm dyn}$ easily follow from the observation that operator $\hat{T}_{\rm dyn}$ acts trivially on the space of dynamical states $\mathscr{H}_{\rm dyn} = \hat{P}_{\rm dyn}\mathscr{H}_{\rm ex}$. It is easily seen that $\langle\hat{T}_{\rm dyn}\rangle = \delta'(0)/\delta(0)$ for any vector $|\psi\rangle$, which can be informally read as $\langle\hat{T}_{\rm dyn}\rangle = 0$. Furthermore, the projection on the dynamical vectors of the spectral resolution of operator $1\otimes\hat{T}$ generates a non-orthogonal resolution of unity. Consider the product of $\hat{P}_{\rm dyn}(1\otimes|t\rangle\langle t|)\hat{P}_{\rm dyn}$ for two different values of $t = t_1, t_2$,

$$\hat{P}_{\rm dyn}(1\otimes|t_1\rangle\langle t_1|)\hat{P}_{\rm dyn}(1\otimes|t_2\rangle\langle t_2|)\hat{P}_{\rm dyn}. \tag{13}$$

Substituting formula (9) for $P_{\rm dyn}$ gives (up to the normalization)

$$\begin{aligned}&\int {\rm d}E{\rm d}E'{\rm d}E''|E,E\rangle\langle E,E|(1+\otimes|t_1\rangle\langle t_1|) \\ &\quad\times |E',E'\rangle\langle E',E'|(1\otimes|t_2\rangle\langle t_2|)|E'',E''\rangle\langle E'',E''| \\ &= \int {\rm d}E{\rm d}E'{\rm d}E''\delta(E-E')\delta(E'-E'') \\ &\quad\times \exp[{\rm i}t_1(E-E')+{\rm i}t_2(E'-E'')]|E,E\rangle\langle E'',E''| \\ &= \int {\rm d}E|E,E\rangle\langle E,E| \neq 0.\end{aligned} \tag{14}$$

Thus, it can be concluded that operator $\hat{T}_{\rm dyn}$ is not given by a PM, but in a rather trivial sense by a POVM. However, let us stress that the trivial action of operator $\hat{T}_{\rm dyn}$ matches the trivial character of the dynamical time considered as a physical observable.

We have seen that quantity $\hat{T}_{\rm dyn}$ representing the external time has rather trivial properties. However, the framework of the space of dynamical vectors offers the possibility to introduce operators of the form

$$\hat{P}_{\rm dyn}(|\psi\rangle\langle\psi|\otimes\hat{T})\hat{P}_{\rm dyn} \tag{15}$$

with suitable $|\psi\rangle \in \mathscr{H}_{\rm s}$, which have nontrivial properties and a rather interesting interpretation. In particular, statistics generated by the well-known system time operators can be reproduced with operators of the form (15). Before we explore this possibility, let us very briefly recapitulate the well-known example[10,11] of the system time operator for the case of a free quantum particle.

The free particle Hamiltonian $\hat{H} = \hat{P}^2/2m$ is doubly degenerate. The basis of the energy representation is given by two component eigenvectors

$$\hat{H}|E,\pm\rangle = E|E,\pm\rangle,$$
$$\hat{P}|E,\pm\rangle = \pm\sqrt{2mE}|E,\pm\rangle, \quad E \in (0,\infty). \quad (16)$$

Vectors

$$|t_H,\alpha\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int_0^\infty \mathrm{d}E \exp[-\mathrm{i}Et_H/\hbar]|E,\alpha\rangle, \quad (17)$$

which in the energy representation read

$$\psi_{t_H,\alpha}(E,\beta) = \langle E,\beta|t,\alpha\rangle_H$$
$$= \delta_{\alpha,\beta}\frac{1}{\sqrt{2\pi\hbar}}\exp[-\mathrm{i}Et_H/\hbar], \quad (18)$$

form a nonorthogonal resolution of unity. Indeed,

$$\langle t_H|\alpha|t'_H,\beta\rangle = \delta_{\alpha,\beta}[\delta(t_H - t'_H) + \frac{\mathrm{i}\mathscr{P}}{2\pi(t_H - t'_h)}. \quad (19)$$

Thus formula

$$\hat{F}_\mu = \sum_\alpha \int_\mu \mathrm{d}t_H |t_H,\alpha\rangle\langle\alpha|t_H|, \quad \mu \subset (-\infty,\infty) \quad (20)$$

gives a POVM on **R**. The POVM and the free particle Hamiltonian satisfy the covariance, i.e., the commutation relation, which is the basic condition set on the object that can represent the system time.

Using the POVM (20), we can formally define an operator

$$\hat{T}_H = \sum_\alpha \int_{-\infty}^\infty t_H \mathrm{d}t|t_H,\alpha\rangle\langle\alpha,t_H|. \quad (21)$$

The operator $\hat{T}_H$ has been called the event time or the screen time. We shall call it the system time. The index H in $\hat{T}_H$ and in all other formulas is to remind us that the corresponding objects are constructed for the system with a particular Hamiltonian. To the best of our knowledge, the first person to treat the free particle time in this way was Prof. Holevo in his monograph.[10]

The operator $\hat{T}_H$ is represented as differentiation in the energy bases and has a pleasing property

$$\langle\psi|\hat{T}_H|\psi\rangle(t) = t, \quad (22)$$

but is not self-adjoined. It is self-adjoined on the domain $D = \{|\psi\rangle \in \mathscr{H}_\mathrm{s}| < 0,\ \alpha|\psi\rangle = 0\}$. With this remark, we end our brief recapitulation.

To explore the possible relation with an operator of the form (15) and the system time $\hat{T}_H$, let us first consider operators of the form

$$\hat{P}_\mathrm{dyn}(\hat{B}\otimes|t_0\rangle\langle t_0|)\hat{P}_\mathrm{dyn}, \quad (23)$$

where $\hat{B}$ represents some physical observable. The expression

$$\langle\psi|\hat{P}_\mathrm{dyn}(\hat{B}\otimes|t_0\rangle\langle t_0|)\hat{P}_\mathrm{dyn}|\psi\rangle$$
$$= \langle\psi_\mathrm{s}(t_0)|\hat{B}|\psi_\mathrm{s}(t_0)\rangle \quad (24)$$

reproduces the expectation of quantity $\hat{B}$ in quantum state $|\psi_\mathrm{s}(t_0)\rangle_\mathrm{s} \in \mathscr{H}_\mathrm{s}$ in terms of dynamical vectors $P_\mathrm{dyn}|\psi\rangle \in \mathscr{H}_\mathrm{dyn} \subset \mathscr{H}_\mathrm{ex}$ and operator $\hat{B}\otimes|t_0\rangle\langle t_0|$. Notice that the relevant quantity is $\hat{B}\otimes|t_0\rangle\langle t_0|$ not just $\hat{B}\otimes 1$. The system's quantum state $|\psi_\mathrm{s}(t_0)\rangle$ is the state on the orbit determined by the dynamical vector $\hat{P}_\mathrm{dyn}|\psi\rangle$ that goes through $|\psi_\mathrm{s}(t_0)\rangle$ at time $t = t_0$. Explicitly and in the coordinate representation, $\psi_\mathrm{s}(x;t_0) = \langle x|\langle t_0|\hat{P}_\mathrm{dyn}|\psi\rangle$.

To proceed we ask the following general question: is there an operator $\hat{A}$ such that

$$\hat{P}_\mathrm{dyn}(\hat{B}\otimes|t_0\rangle\langle t_0|)\hat{P}_\mathrm{dyn} = \hat{P}_\mathrm{dyn}(\hat{A}\otimes\hat{T})\hat{P}_\mathrm{dyn}. \quad (25)$$

If the answer is positive, then Eq. (25) in combination with Eq. (24) can be considered as an implicit definition of operator $\hat{A}$ such that $\hat{P}_\mathrm{dyn}(\hat{A}\otimes\hat{T})\hat{P}_\mathrm{dyn}$ gives the statistics of the measurement of the system's observable $\hat{B}$ at time t_0. Such an operator $\hat{A}$ would obviously depend on time t_0. We shall analyze Eq. (25) for the specific case of $\hat{B} \equiv \hat{T}_H$.

In this case, the following relation between an operator of the form (23) and the free particle time $\hat{T}_H$ can be demonstrated:

$$\langle\psi(t_0)|\hat{T}_H|\psi(t_0)\rangle$$
$$= 2\pi\langle\psi|\hat{P}_\mathrm{dyn}(|-t_0^H\rangle\langle -t_0^H|\otimes\hat{T})\hat{P}_\mathrm{dyn}|\psi\rangle, \quad (26)$$

where $\hat{T}_H$ and $|t_0^H\rangle$ are the free particle time and its eigenvector as in Eq. (21), respectively. The relation (26) is obtained by explicit substitution in the left and the right sides of the relevant formulas. Indeed,

$$\langle\psi_\mathrm{dyn}|\hat{P}^H_{-t_0}\otimes|-t\rangle\langle -t||\psi_\mathrm{dyn}\rangle$$
$$= \left(\frac{1}{2\pi}\right)^2\sum_\alpha\int \mathrm{d}E\,\mathrm{d}E'\bar{\psi}(E,\alpha)\psi(E',\alpha)$$
$$\times \exp[\mathrm{i}(E-E')t_0^H]\exp[-\mathrm{i}(E-E')t], \quad (27)$$

which coincides with the explicit expression in the energy representation of

$$\langle\psi(t_0)||t^H\rangle\langle t^H||\psi(t_0)\rangle. \quad (28)$$

Integrating over $t^H\,\mathrm{d}t^H$ gives formula (26). According to the standard interpretation, the left-hand side of Eq. (26) represents the expectation of the system time $\hat{T}_H$ if measured in the moment of time t_0 for the free particle in the state $|\psi(t_0)\rangle$. Equation (26) expresses this quantity in terms of operator $|t_0^H\rangle\langle t_0^H|\otimes\hat{T}$ and dynamical vector $|\psi\rangle_{\rm dyn}$.

4. Probabilistic interpretation of dynamical vectors

In this section, we would like to clarify certain points about the probabilistic interpretation of the vectors from the extended Hilbert space and the dynamical vectors. Some probability interpretation is needed if the formulas, like formulas (26) and (15), are to be related with the standard quantum mechanical expressions.

Consider first the vectors from $\mathscr{H}_{\rm ex}$. Such a vector $|\psi\rangle$ has a finite norm, and in the coordinate representation, $\langle x|\langle t|\psi\rangle\equiv\psi(x,t)$, the function

$$\rho(x,t)=\frac{|\psi(x,t)|^2}{\int\int|\psi(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t}\tag{29}$$

has the properties of a joined probability distribution on $\mathbf{R}\times\mathbf{R}$.

From the joined probability $\rho(x,t)$, we can form conditional probabilities by using the standard procedure

$$\rho_{t_0}(x)=\frac{|\psi(x,t_0)|^2}{\int|\psi(x,t)|^2\,\mathrm{d}x},\tag{30}$$

$$\rho_{x_0}(t)=\frac{|\psi(x_0,t)|^2}{\int|\psi(x_0,t)|^2\,\mathrm{d}t}.\tag{31}$$

Notice that expression (30) is well defined even if $\psi(x,t)$ is not square integrable with respect to t, but is square integrable with respect to x for any fixed t_0.

The interpretations of $\rho_{t_0}(x)$ and $\rho_{x_0}(t)$, if they exist, could be as follows: a) $\rho_{t_0}(x)$ is the probability that the measurement of coordinate $\hat{Q}$ performed at time t_0 gives result x; b) $\rho_{x_0}(t)$ is the probability that if the measurement of $\hat{Q}$ gives x_0, then the measurement of $\hat{T}$ gives t.

However the integral in formula (31) is divergent for dynamical vectors $|\psi\rangle_{\rm dyn}\in\mathscr{H}_{\rm dyn}$. Thus, for such vectors, the presented joined probability (29) and the conditional probability (31) are not defined. Furthermore, the would be interpretation of $\rho_{x_0}(t)$ conflicts with the physical intuition. Indeed, there could be many instants t or no instant t at all when measurement of $\hat{Q}$ gives x_0. Thus, for the physically plausible vectors from the extended space, the interpretation of $\rho_{x_0}(t)$ as the conditional probability is not viable. This is formally confirmed by the fact that $\psi(x_0,t)$ is not square integrable for any dynamical vector $|\psi\rangle_{\rm dyn}\in\mathscr{H}_{\rm dyn}$ satisfying the constraint (5). On the other hand, the conditional probability interpretation for $\rho_{t_0}(x)$ given by Eq. (30), where $\psi(x,t)\in\mathscr{H}_{\rm dyn}$ is a dynamical vector, is perfectly consistent. In fact, it is reduced to the standard probability interpretation for the vectors from $\mathscr{H}_{\rm s}$.

5. Conclusion

We have studied properties of the dynamical time operator as it appears in the extended formulation of reparameterization invariant theory of Hamiltonian quantum mechanics. This is the time operator that corresponds to the parameter time of the Schrödinger dynamical equation. The construction of the extended Hilbert space and the constraint corresponding to the reparametrization invariance have been recapitulated.

Vectors from the extended Hilbert space that satisfy the constraints are called the dynamical vectors, and each dynamical vector represents an entire orbit in the Hilbert space of the system's states. Obviously, the time that can be associated with such dynamical vectors, i.e., the entire orbits, is in fact rather trivial. Mathematical properties of the dynamical time operator (11) are consistent with its trivial physical character when considered as a dynamical observable. In particular, it is proportional to the unit operator on the space of dynamical vectors, and it generates a POVM on $\mathbf{R}$. The later property of the dynamical time operator appears as a consequence of the reparametrization invariance of the Hamiltonian formalism, and does not depend on the properties of a particular Hamilton operator. On the other hand, the similar property obtained for the few operators representing different measurable time in terms of the system's dynamical observables is always a consequence of the semi-boundedness of the spectrum. The latter is a property of the Hamiltonian operator of certain type of system and not that of the Hamiltonian formalism.

After studying the properties of the operator of the dynamical time, we have explored the relation between the most well-known system time, i.e., the case of a free particle, and the corresponding dynamical time. It turns out that the statistics of the system time observable measurement at some time can be expressed entirely in terms of operators and vectors of the extended Hilbert space involving the dynamical vectors and particularly the modified dynamical time operator. In particular, the fact that the system time generates a POVM can be obtained from the POVM generated by the operator related with the dynamical time of the extended formalism.

Appendix A

It is well known that a quantum system with the Hilbert space $\mathscr{H}$ can be considered as a Hamiltonian dynamical system with the projective Hilbert space $P\mathscr{H}$ as the symplectic phase space.[15–17] In the Hamiltonian formulation, the real and the imaginary parts of the Hermitian scalar product, reduced on $P\mathscr{H}$, generate the Riemannian and the simplectic structures on the phase space, respectively. Pure quantum states are represented by points in the phase space, and the observables $\hat{A}$ by functions $\langle\hat{A}\rangle$, whose Hamiltonian vector fields generate the isometries. The Schrödinger evolution equation is reproduced by the Hamilton dynamical equations with the Hamilton's function $H = \langle\hat{H}\rangle$, where $\hat{H}$ is the Hamiltonian. The function representing the commutator between two observables is given by the Poisson bracket of the corresponding functions. The geometric formulation of quantum mechanics has been used to study the constrained quantum dynamics.[18,19]

Consider now a quantum system with the time as an additional degree of freedom, i.e., with the extended Hilbert space $\mathscr{H}_{\rm ex} = \mathscr{H}_{\rm s} \otimes \mathscr{H}_T$. In the geometric Hamiltonian formulation, the system with spacial degrees of freedom and the time are associated with the corresponding real manifolds $\mathscr{M}_{\rm s}$ and $\mathscr{M}_{\rm t}$. The extended system is associated with the product manifold $\mathscr{M}_{\rm ex} = \mathscr{M}_{\rm s} \times \mathscr{M}_{\rm t}$. Here $\mathscr{M}_{\rm ex}$ is obviously different from the real manifold associated with $\mathscr{H}_{\rm ex}$, but we shall nevertheless use the somewhat misleading notation $\mathscr{M}_{\rm ex}$. The constraint (5), equivalent to constraint (6), is imposed with the functional constraint on $\mathscr{M}_{\rm ex}$

$$\Phi(x_\psi) \equiv \langle\psi|\hat{H}^2_{\rm ex}|\psi\rangle \equiv H^2_{\rm ex}(x_\psi) = 0, \qquad \text{(A1)}$$

which is well defined. No problem with the constrained eigenspace composed of non-normalizable vectors occurs in the geometric formulation.

We can see that the quantum problem with the Hamiltonian $\hat{H}$ and the constraint $\hat{H}|\psi\rangle = 0$ is replaced by a classical infinite-dimensional Hamiltonian system with the Hamilton function $H_{\rm ex} = \langle\hat{H}_{\rm ex}\rangle$ and the constraint $H^2_{\rm ex} = 0$. The constraint $H^2_{\rm ex} = 0$ is irregular and can be replaced by the regular one $H_{\rm ex} = 0$. The total Hamilton function is

$$\begin{aligned} H_{\rm tot}(x) &= H_{\rm ex}(x) + \lambda(x) H_{\rm ex}(x) \\ &= k(x) H_{\rm ex}(x). \end{aligned} \qquad \text{(A2)}$$

The reparameterization invariance is seen as a gauge freedom associated with the first class constraint $H_{\rm ex} = 0$.

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