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# Constrained event space and properties of the physical time observable

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**Abstract** – It is demonstrated that the important common property of operators representing various observable times in quantum mechanics, namely the fact that such an operator is always given by a nonorthogonal resolution of unity, can be obtained from a constraint on the possible physical events, *i.e.* on the extended space of the Hamiltonian formulation of the parametric dynamics. Operators which could generate meaningful probability distributions for various time measurements are suggested.



**Introduction.** – In classical and in quantum mechanics the time is treated in the same way as an evolution parameter. However, relativistic covariance on the one hand and the possibility to measure time of various events on the other, suggest that the time should have the same status as other measurable physical quantities, and should be represented by the same mathematical object as, for example, the space coordinates. These two motives have independently lead to different treatments of the problem of time observable in quantum mechanics: one related to the quantization of relativistically covariant theories (illustrated by the following selection of references [1–6]), and the other related to the problem of measurement of time of various quantum processes (for example [7–11], especially see [12] for a collection of relevant references and [13] for an insightful discussion). Time of arrival operator has been introduced and discussed also in relativistic [14] and quantum field [15] frameworks. In this letter we establish a relation between the two approaches by showing that the constraint on physical events, that appears in the treatment of re-parameterization invariant Hamiltonian theories, leads to the common mathematical property of the time operators representing various time measurements, described for example in [16,17].

The procedure to obtain an equivalent formulation of time and space variables is known as the parametric form of mechanics, and consists in extending the relevant state space. Standard references are [18] or [1] for the classical or the quantum case, respectively. The additional coordinate is to be interpreted as the time observable and its conjugated quantity is added linearly to the system's Hamiltonian in order to preserve the canonical evolution equations. However, general physical arguments indicate that the time and the space coordinate, and the corresponding mathematical representatives thereof, can not have the same properties. Physical interpretation of the additional degree of freedom as corresponding to the time sets up a relation between the time variable and the new evolution parameter, which represents a constraint on the possible physical events. Discussion of the extended-space formalism, presented in reference [19], provides qualitative arguments indicating that the constraint on the physical events is responsible for the peculiar properties of the time variable. In this letter it will be demonstrated that the physical intuition of the time observable, realized as a constraint on the space of physical events, leads to the properties of the quantum-mechanical time observable that have been obtained from the models of various time measurements. We shall see that the physical interpretation of the time coordinate in the extended space of events implies that the operator of the time observable can not be represented by a projection measure (PM) on $R$ but involves a positive operator-valued measure (POVM).

**Extended quantum system.** – Let us briefly recapitulate the well-known parametric formulation of

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classical or quantum conservative dynamics [1,18,19]. In this formulation the time is treated as the coordinate of an additional degree of freedom, and the evolution is parameterized by a new evolution parameter. In order that the dynamical equations of the original and the extended system describe the same evolution the Hamiltonian of the extended system is defined as a sum of the original Hamiltonian $H_s$ and the variable/observable $J_T$ conjugated to the time coordinate $T$, *i.e.* $H_{ex}=H_s-J_T$. In what follows we shall use the same symbol for the classical or the quantum object representing a physical measurable quantity. In the classical case $H_{ex}$ is a function on the extended phase space, which is for an example of a one-dimensional particle $M_{ex}=R^2\times R^2$, and in the quantum case $H_{ex}$ is a self-adjoint operator on the extended Hilbert space $\mathcal{H}_{ex}=\mathcal{H}_s\otimes\mathcal{H}_T$ where $\mathcal{H}_T$ provides the representation of the canonical commutation relations between $T$ and $J_T$. For the example of a one-dimensional particle $\mathcal{H}_{ex}=L_2(R)\otimes L_2(R)$. The extended space is called the space of events. Thus the term event is used here to denote an element from $\mathcal{H}_{ex}$ and has independent spatial and time coordinates.

**Physical events and the physical time observable.** – The physical meaning of the time variable is formalized as a constraint on the possible physical events, *i.e.* as a constraint on the extended phase space or the extended Hilbert space. In the classical case the constraint is given by $H_{ex}(q,p,J_t)=H_s(q,p)-J_t=0$, that is $J_t(q,p)=h_s(q,p)$ where $h_s(q,p)$ is the value of $H_s$ at $(q,p)$. In the quantum case the constraint on the physical events reads

$$H_{ex}|\psi\rangle=(H_s-J_t)|\psi\rangle=0. \tag{1}$$

The physical events form a submanifold, in the classical case, or a linear subspace in the quantum case, of the full space of events. The operator $J_t$ has a continuous spectrum so that the null eigenspace of $H_{ex}$ must be understood in the sense of generalized vectors. The issue will be addressed again shortly. In what follows we shall restrict our attention to the quantum case. It is possible to give a unified formulation of the classical and the quantum constrained dynamics, using the geometric Hamiltonian formulation of the constrained quantum systems [20,21].

The physical time is given by the restriction of $1\otimes T$ onto the space of physical events. Thus, in the quantum case

$$T_{phys}=P(1\otimes T)P, \tag{2}$$

where $P$ is projection on the subspace of physical events, *i.e.* the vectors satisfying (1). We shall see that $T_{phys}$ does not define a PM on $R$ but is given by a nonortogonal resolution of unity by positive operators *i.e.* by a POVM.

Let us denote by $\int|E\rangle\langle E|\mathrm{d}E$ the spectral resolution of unity associated with the Hamiltonian operator $H_s$. The integral is understood as a sum or as an integral over the spectra of $H_s$, depending on the type of the spectra. Similarly $\int_R|J_t\rangle\langle J_t|\mathrm{d}J_t$ and $\int_R|t\rangle\langle t|\mathrm{d}t$ denote the continuous orthogonal resolutions of unity corresponding to the operators $J_t$ and $T$ on $\mathcal{H}_t$. Thus $|J_t\rangle$ and $|t\rangle$ are the (generalized) eigenvectors of $J_t$ and $T$, respectively. Operators $H_{ex}$ and $T$ inherit the canonical commutation relation valid for $T$ and $J_t$. On the other hand, the system's operators and in particular the Hamiltonian $H_s$ commute with the time $T$ operator.

The constraint (1) demands that the operators $H_s\otimes 1$ and $1\otimes J_t$ are equal when restricted on the subspace of the physical events. A vector $|E_0\rangle\otimes|J_t\rangle$ represents a physical event if and only if $J_t=E_0$, *i.e.* if the "eigenvector" $|J_t\rangle$ of $J_T$ has the "eigenvalue" of $J_t$ numerically equal to the eigenvalue $E_0$ of the Hamiltonian. We denote such vectors by $|E\rangle\otimes|E\rangle\equiv|E,E\rangle$. A general vector from the space of events is given by

$$|\psi\rangle=\int\int\mathrm{d}E\mathrm{d}J_t f(E,J_t)|E\rangle\otimes|J_t\rangle, \tag{3}$$

and for the physical events $f(E,J_t)=\delta(E-J_t)$ leading to

$$|\psi\rangle_{phys}=\int\mathrm{d}Ef(E)|E\rangle\otimes|E\rangle. \tag{4}$$

The projector on the subspace of the physical events $\mathcal{H}_{phys}$ is written as

$$P_{phys}=\int\mathrm{d}E|E,E\rangle\langle E,E|. \tag{5}$$

Obviously, the subspace of physical events is invariant under the Schroedinger evolution generated by $H_{ex}$.

Notice that the second component of the elementary physical state $|E\rangle\otimes|E\rangle$ is a generalized eigenvector of the operator $J_t$. This operator has only the continuous spectrum and its "eigenvectors" are not normalizable and do not belong to the Hilbert space $\mathcal{H}_T$, but are properly speaking elements of the corresponding rigged Hilbert space. In the same way the "eigenspace" $\mathcal{H}_{phys}$ of the extended Hamiltonian $H_{ex}$ must be understood it the sense of generalized vectors. The expression $P_{phys}$ given by (4) represents a projection operator only formally and in this generalized sense. Despite this fact we shall continue to address $P_{phys}$ as the projection operator baring on mind the mathematical subtleties involved in its definition. The physical meaning of $P_{phys}$ is seen in the fact that it associates with an arbitrary vector of the form $|\psi_0\rangle|t_0\rangle$ its image $P_{phys}|\psi_0\rangle|t_0\rangle\in\mathcal{H}_{phys}$, and the latter can be identified with a solution of the Schroedinger equation in $\mathcal{H}_s$, *i.e.* the system's orbit, that goes through the state $|\psi_0\rangle\in\mathcal{H}_s$ at time $t_0$. Many events of the form $|\psi\rangle|t\rangle$ are associated, by projecting on a single $|\psi\rangle_{phys}$, with the single orbit which goes through states $|\psi\rangle$ at times $t$.

Let us now demonstrate that the operator of the physical time (2) gives a nonorthogonal resolution of unity. Positivity of $P_{phys}|t\rangle\langle t|P_{phys}$ is obvious.

*Resolution of unity in $\mathcal{H}_{phys}$*
Using the explicit form of $P_{phys}$ given by (4) we have

$$\begin{aligned}\int_R P_{phys}(1\otimes|t\rangle\langle t|)P_{phys}\mathrm{d}t = \\ \int \mathrm{d}t\mathrm{d}E\mathrm{d}E'|E,E\rangle\langle E,E|(1\otimes|t\rangle\langle t|)|E',E'\rangle\langle E',E'| \\ = \int \mathrm{d}t\mathrm{d}E\mathrm{d}E'\delta(E'-E)\exp i(E-E')t|E,E\rangle\langle E',E'| \\ = \int \mathrm{d}t\mathrm{d}E|E,E\rangle\langle E,E|, \qquad (6)\end{aligned}$$

which is, up to a normalization, the unity in $\mathcal{H}_{phys}$.

*Nonorthogonality*
Consider the product of $P_{phys}(1\otimes|t\rangle\langle t|)P_{phys}$ for two different values of $t=t_1$, $t_2$:

$$P_{phys}(1\otimes|t_1\rangle\langle t_1|)P_{phys}(1\otimes|t_2\rangle\langle t_2|)P_{phys}. \qquad (7)$$

Substitution of formula (4) for $P_{phys}$ gives

$$\begin{aligned} &= \int \mathrm{d}E\mathrm{d}E'\mathrm{d}E''|E,E\rangle\langle E,E|(1+\otimes|t_1\rangle\langle t_1|) \\ &\quad\times|E',E'\rangle\langle E',E'|(1\otimes|t_2\rangle\langle t_2|)|E'',E''\rangle\langle E'',E''| \\ &= \int \mathrm{d}E\mathrm{d}E'\mathrm{d}E''\delta(E-E')\delta(E'-E'') \\ &\quad\times\exp[it_1(E-E')+it_2(E'-E'')]|E,E\rangle\langle E'',E''| \\ &= \int \mathrm{d}E|E,E\rangle\langle E,E|\neq 0. \qquad (8)\end{aligned}$$

Thus we conclude that the operator $T_{phys}$ is not given by a PM but to a POVM. This is the crucial mathematical property of the operators representing various time observables that has been obtained from particular examples [12,16,17]. We have seen that the properties of $T_{phys}$ do not depend on the particular properties of the spectra of the Hamiltonian, but are induced solely by the constraints satisfied by the physical events, *i.e.* by the definition of $T_{phys}=P_{phys}TP_{phys}$. The constraint is of an essentially classical origin. This is the main result of this letter.

The crucial properties (5) and (8) easily follow from the observation that the operator $T_{phys}$ acts trivially on the space of physical states $\mathcal{H}_{phys}=P_{phys}\mathcal{H}_{ex}$ where $P_{phys}$ is given by (4). Trivial action of $T_{phys}$ is a consequence of the constraint (1) and the definition (2). Physical interpretation of this fact is that the physical time $T_{phys}$ is not a proper dynamical variable, and it does not corresponds to an independent degree of freedom. Proper physical degrees of freedom of the extended parametric formulation, represented by $Q\otimes 1$ or $P\otimes 1$ do not trivialize on the physical states $\mathcal{H}_{phys}$.

The physical time operator $P_{phys}(1\otimes T)P_{phys}$ has peculiar properties, consistent with the redundant nature of the physical time considered as a dynamical variable on $\mathcal{H}_{phys}$. For example, it is easily seen that $\langle T_{phys}\rangle=\delta'(0)$ for any event $|\psi\rangle$, which can be informally read as $\langle T_{phys}\rangle=0$. Furthermore, $\langle P_{phys}(1\otimes\int_{t_1}^{t_2}t|t\rangle\langle t|\mathrm{d}tP_{phys}\rangle=(t_2^2-t_1^2)/2$ independently of $|\psi\rangle$.

Despite the fact that the physical time $T_{phys}$ is trivial variable on $\mathcal{H}_{phys}$ the expression (2) can be modified so that it represents measurements of more meaningful quantities. Nontrivial distributions with interesting physical interpretation could be obtained by replacing the unit operator in (2) by some nontrivial system's observable. It can be demonstrated that such type of operators can have a finite dispersion which depends on the the vector $|\psi\rangle\in H_{ex}$. Thus, such operators generates probability distributions with nontrivial properties. We are currently studying relations of such operators with internal phase of the system and measurements of time conditioned on results of some other measurement. The results will be presented elsewhere.

**Summary. –** In summary: We have shown that the Hamiltonian formulation of evolution and a quite general physical intuition, formalized as a constraint on physical events, is sufficient to derive the main mathematical property of the operator that represents the time observable in quantum mechanics, namely the fact that such an operator is necessarily given by a POVM on $R$. In the derivation no reference is made to a particular process of time measurement or a particular physical system. It would be interesting to use the time observable $T_{phys}$, and its modifications, to simulate time distributions obtained in particular time measurements. Such an analyzes should establish relations between $T_{phys}$ and the POVM's related to the particular measurements of different time observables.

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