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Phase space theory of quantum–classical systems with nonlinear and stochastic dynamics

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HIGHLIGHTS

- A novel theory of quantum–classical systems is developed.
- Framework of quantum constrained dynamical systems is used.
- A dynamical description of the measurement induced collapse is obtained.

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ABSTRACT

A novel theory of hybrid quantum–classical systems is developed, utilizing the mathematical framework of constrained dynamical systems on the quantum–classical phase space. Both, the quantum and classical descriptions of the respective parts of the hybrid system are treated as fundamental. Therefore, the description of the quantum–classical interaction has to be postulated, and includes the effects of neglected degrees of freedom. Dynamical law of the theory is given in terms of nonlinear stochastic differential equations with Hamiltonian and gradient terms. The theory provides a successful dynamical description of the collapse during quantum measurement.

1. Introduction

Interaction of a quantum system with a classical one is in the standard formulation of quantum mechanics described by the collapse postulate, introduced by von Neumann [1]. However, a dynamical description of the postulate requires a consistent theory of systems which cannot be described by either quantum or classical mechanics alone. Such a description of interacting quantum–classical

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systems is commonly called a hybrid theory. The Schrödinger evolution of an isolated quantum system is linear and deterministic, and the evolution of classical systems is also deterministic, but is typically nonlinear. The collapse postulate requires the evolution of a quantum system interacting with the classical apparatus to be nonlinear and stochastic. The hybrid theory, developed in the present paper, incorporates both types of evolution into a single dynamical process.

Hybrid systems are interesting independently of their fundamental aspects (for a recent review see [2]). Despite "no go" theorems [3], several nonequivalent mathematically consistent hybrid theories have been constructed [4–8]. Formulation of the classical dynamics in terms of unitary transformations in an appropriate Hilbert space exists since long time ago [9]. Likewise, there is a formulation of quantum mechanics in terms of Hamiltonian dynamical systems with the appropriate symplectic phase space and the corresponding Hamiltonian dynamics [10,11]. However, the crucial difference between the two theories is not in the mathematical framework, but in the treatment of the interactions between subsystems.

Hybrid theories can be divided into two groups according to the conceptual status and aims. In the theories of the first group one considers all systems in Nature as described at the fundamental level by quantum theory and therefore the hybrid system is an approximation of two interacting quantum systems, where one of the systems is treated in the corresponding classical limit [4,8]. In the other approach, one assumes from the beginning that the classical and quantum mechanics are both fundamental theories with different domains of validity. The only restriction on the descriptions of the quantum–classical (QC) interaction is then given by the experiments involving micro–macro objects and the phenomenological collapse postulate. Of course, it is clear that a macro-object has many degrees of freedom which are not described by the macroscopic model of the classical theory. The effects of those degrees of freedom have to be somehow included into the manner a hybrid theory treats the QC interaction. The hybrid theory constructed in the present paper, and denoted by FHT (for "Fundamental Hybrid Theory" [12]), presents a particular way of doing this.

2. Mathematical framework

Mathematical framework of the hybrid theory to be developed is that of an abstract dynamical system $(\mathcal{M}, \Omega, G, H)$ on a differentiable manifold $\mathcal{M}$ with symplectic and Riemannian structures Ω and G respectively, with some preferred function, the Hamiltonian H. Let us stress right at the beginning that the dynamical law of the hybrid theory need not be of the Hamiltonian form, but will involve differential equations on $\mathcal{M}$ given in terms of Ω and G. The manifold is also assumed to possess a complex structure $J^2 = -I$, where I stands for identity, such that $G(x, y) = \Omega(x, Jy)$. Furthermore, the evolution law of the hybrid theory might be given in terms of a stochastic process, in which case the points from $\mathcal{M}$ are values of random variables on some probability space. The latter will not be explicitly referred.

Formulation of the classical mechanics of isolated conservative systems using $(\mathcal{M}, \Omega, H)$ is standard [13]. The formulation of quantum mechanics in terms of $(\mathcal{M}, \Omega, G, H)$ is perhaps less well known, but shall not be presented here in any detail since there exist excellent reviews [10,11,14] and brief accounts [15–19] which are sufficient for our purposes. Very briefly, the basic observation beyond the Hamiltonian formulation of quantum mechanics is that the evolution of a quantum pure state in a Hilbert space $\mathcal{H}$, as given by the Schrödinger equation, can be equivalently described by a Hamiltonian dynamical system on an Euclidean manifold $\mathcal{M}$. The manifold is just the Hilbert space considered as a real manifold, with the symplectic and Riemannian structures given by the real and imaginary parts of the Hilbert space scalar product. Representing a vector $|\psi\rangle \in \mathcal{H}$ in a basis $\{|k\rangle \mid k = 1, 2, \ldots, N\}$, where N is the dimension of the complex Hilbert space, by coefficients $\{c_k \mid k = 1, 2, \ldots, N\}$, one can introduce the canonical coordinates $x^k = (c_k^* + c_k)/\sqrt{2}$ and $y^k = i(c_k^* - c_k)/\sqrt{2}$, $k = 1, 2, \ldots, N$. Generic point from $\mathcal{M}$ is usually denoted by (x, y), X or X^a, where $a = 1, 2, \ldots, 2N$ is an abstract index. In what follows the symplectic and Riemannian structures on the quantum phase space are denoted by ω^{ab} and g^{ab}. The Hamilton's function $H(X)$ is given by the quantum expectation of the Hamiltonian $\hat{H}$ in the state $|\psi_X\rangle$ corresponding to a point X: $H(X) = \langle\psi_X|\hat{H}|\psi_X\rangle/\langle\psi_X|\psi_X\rangle$. In fact, all observables are represented by quadratic functions $A(X)$ on $\mathcal{M}$, and are the quantum mechanical ex-

pectations of the corresponding quantum observables $A(X) = \langle\psi_X|\hat{A}|\psi_X\rangle/\langle\psi_X|\psi_X\rangle$. The Schrödinger dynamical law is that of Hamiltonian mechanics

$$\dot{X}^a = \omega^{ab}\nabla_b H. \tag{1}$$

The Hamiltonian formulation is also crucial in the formulation and applications of nonlinear constraints within quantum mechanics [15,18,19,16,17,8].

3. Construction of the hybrid theory

The total system is conceived as composed of a microscopic quantum system and a macro-system. It is the central assumption of the present hybrid theory that the macro-system has a distinguished set of degrees of freedom, described by classical mechanics. Usually, it is not claimed that macro-systems are composed of something other than microscopic parts well described by quantum theory. However, it is assumed that the dynamics of at least some of the observable degrees of freedom of a macroscopic system is correctly described by classical mechanics, and that the classical mechanical description need not be reduced or derived from quantum description of all the microscopic components.

3.1. Elements of the hybrid model

In the FHT the hybrid phase space $\mathcal{M}$ is assumed to be given by the Cartesian product $\mathcal{M} = \mathcal{M}_{qp} \times \mathcal{M}_{QP} \times \mathcal{M}_{xy}$. Local canonical coordinates are separated into three groups: (q, p), (Q, P) and (x, y). The first two groups $(q, p) \in \mathcal{M}_{qp}$ and $(Q, P) \in \mathcal{M}_{QP}$ correspond to the degrees of freedom of the macroscopic system, and the third $(x, y) \in \mathcal{M}_{xy}$ to the degrees of freedom of the microscopic quantum system, called quantum degrees of freedom (QDF). The coordinates (q, p) represent (usually a small number of) distinguished macroscopic degrees of freedom of the macroscopic object. They are supposed to be well described by classical mechanics and are called classical degrees of freedom (CDF).

The degrees of freedom denoted by (Q, P) describe the physical quantities that are not used in the characterization of the CDF of the macroscopic object nor of the QDF of the micro-system. Apart from the fact that there are many of these degrees of freedom, nothing else about their character is assumed in the hybrid theory. In other words, the FHT does not assume that (Q, P) are either classical or quantum. In the hybrid theory, it is assumed that the state of the system is completely described by the values of CDF and QDF, and the dynamical equations of the theory will be formulated in terms of (q, p, x, y) only, with no explicit reference to (Q, P). Particular physical interpretation of the (Q, P) degrees of freedom is not strictly a part of FHT. However, one could think of several different physical interpretations depending on the conceptual background and on the particular system. On the conceptual side, one could argue that the macroscopic system is composed of quantum microscopic components which interact and entangle with the micro-system. Therefore, the hybrid theory, with no possibility of explicit entanglement between CDF and QDF, must take the fact of entanglement due to the micro-system and micro-components of the macro-system into account in some manner. The influence of (Q, P) degrees of freedom on the CDF–QDF system might be interpreted partly as due to the entanglement between the micro and macro-systems, and partly due to the influence of the microscopic degrees of freedom of the macro-system on the CDF. This argument is expressed more formally as follows. The phase space of a bipartite quantum system, corresponding to the micro–macro system, is the real manifold $\mathcal{M}_{12}$ associated with the Hilbert space $\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are the Hilbert spaces of the micro and macro-systems, respectively. The phase space corresponding to the macro-system is denoted by $\mathcal{M}_2$. A submanifold, denoted by $\Gamma \subset \mathcal{M}_2$ corresponds to CDF of the macro-system. Local coordinates (x, y) of $\mathcal{M}_1$ correspond to QDF. The degrees of freedom (Q, P) are then the local coordinates of the complement of $\mathcal{M}_1 \times \Gamma$ in $\mathcal{M}_{12}$. Alternatively, one could just conceive (Q, P) degrees of freedom as a sufficiently general type of environment of the CDF–QDF degrees of freedom. Furthermore, the physical interpretation of (Q, P) degrees of freedom will depend on the physical picture of the particular macro-system. For example, the macro-system might be a large magnet, conceived as a large collection of spins, interacting via

the Heisenberg interaction. It is the main assumption of the hybrid theory that the interaction of such a magnet with a micro quantum system can be described by the selected degrees of freedom of the magnet, i.e. the macroscopic magnetization, which are well described by classical physics, provided that the effects of the unobserved degrees of freedom are somehow included in the hybrid theory.

Interactions between various types of degrees of freedom might be of different nature. We shall assume that the interactions between (q, p) and (x, y) are conservative and described by the corresponding Hamiltonian. On the other hand, interactions between the unspecified degrees of freedom (Q, P) and the QDF (x, y) might be more general, and are described by a complex Hamiltonian of the form $H_{int}(x, y, Q, P) = F(Q, P)A(x, y)$ where $A(x, y)$ is a quadratic function of (x, y) corresponding to the operator $\hat{A}$ of the micro-system and $F(Q, P) = F_R(Q, P) + iF_I(Q, P)$ in terms of real functions $F_R(Q, P)$ and $F_I(Q, P)$. Of course, the equations of motion for the real coordinates (q, p, x, y) must be expressed only in terms of real quantities. We shall also suppose that the influence of the (Q, P) degrees of freedom on the macroscopic classical variables (q, p) is negligible. The dynamics of the total system is thus determined by the complex Hamiltonian of the following form

$$H = H_{cl}(q, p) + H_q(x, y) + H_{QP}(Q, P) + f(q, p)A(x, y) + F(Q, P)A(x, y). \tag{2}$$

The meaning of the first three terms is obvious, and the rest describes the interaction between the macroscopic system and the quantum system. In order to shorten the notation we have denoted the collection of all observables $\{A_n\}$, appearing in the interaction terms, by a single letter A. In the simplified version, presented here, all degrees of freedom of the macro-system are assumed to interact with the same quantum observables $A(x, y)$ which might, but need not, form canonical pairs. As pointed out the functions $F(Q, P)$ are complex. However, they do not enter into the part of the Hamiltonian that depends only on the (q, p, x, y) degrees of freedom

$$H_{phys}(q, p, x, y) = H_{cl}(q, p) + H_q(x, y) + f(q, p)A(x, y). \tag{3}$$

The equations of motion for the real quantities as functions of (q, p, x, y) must be real, but need not be Hamiltonian.

The main requirement on the hybrid theory of QDF evolution, based on the collapse model, is that if the state of the quantum system is a superposition of $\hat{A}$ eigenstates then, because of the interaction with the macro-system, the state must evolve towards one of the $\hat{A}$ eigenstates. However, such behavior is not obtained starting from the Hamiltonian dynamics with the Hamiltonian (2) of the hybrid. One is therefore forced to adopt different approaches in modeling the collapse requirements. One approach, adopted here, is to consider the collapse requirements as appropriate constraints onto the otherwise Hamiltonian dynamics and to derive the dynamical law as the constrained dynamics. The phase space formulation of quantum mechanics is specially suitable for the formulation and treatment of nonlinear constraints [15,18,19,16,17,8].

3.2. Constrained dynamics approach

The eigenstates of any observable $\hat{A}$ are characterized by the property that the dispersion $\Delta A = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2$ is equal to zero. In the case when all observables $\{\hat{A}_n\}$ interacting with the macro-system commute, the relevant constraint might be given in the form

$$\Gamma_A(x, y) = \sum_n \Delta A_n(x, y) = 0, \tag{4}$$

which corresponds to the common eigenstate of all the observables $\{\hat{A}_n\}$. However, if there are several non-commuting observables, then the relevant constraint assumes the form

$$\Gamma_A(x, y) = \sum_n \Delta A_n(x, y) - \Delta_{\min} = 0, \tag{5}$$

where $\Delta_{\min}$ is the minimal possible value of the sum of the relevant dispersions. If these observables generate a representation of a semi-simple Lie algebra, then the constraint submanifold given by (5) is in fact the manifold of coherent states of the algebra [20].

In order to satisfy the constraint, the component of the Hamiltonian vector field orthogonal to the constraint submanifold $\Gamma_A(x, y) = 0$ has to be removed, so that the QDF $X \equiv (x, y)$ evolve according to

$$\dot{X}^a = \omega^{ab}\nabla_b H - \lambda g^{ab}\nabla_b \Gamma_A, \tag{6}$$

where λ is a single Lagrange multiplier to be determined. Substitution of (6) in $\dot{\Gamma}_A(X(t))$ results in

$$\omega^{ab}\nabla_a \Gamma_A \nabla_b H = \lambda g^{ab}\nabla_a \Gamma_A \nabla_b \Gamma_A. \tag{7}$$

Substituting λ from (7) into (6) results in the constrained dynamical equations

$$\dot{X}^a = \omega^{ab}\nabla_b H - \frac{\{\Gamma_A, H\}}{\|\nabla \Gamma_A\|^2} g^{ab}\nabla_b \Gamma_A, \tag{8}$$

where $\{F_1, F_2\} = \omega^{ab}\nabla_a F_1 \nabla_b F_2$. The first term can be written more explicitly as

$$\begin{aligned} \omega^{ab}\nabla_b H &= \omega^{ab}\nabla_b H_{phys} + (F_R\omega^{ab} + F_I(J\omega)^{ab})\nabla_b A \\ &= \omega^{ab}\nabla_b H_{phys} + F_R\omega^{ab}\nabla_b A + F_I g^{ab}\nabla_b A. \end{aligned} \tag{9}$$

The last two terms contain a large number of complicated functions of time $Q(t)$, $P(t)$. We shall suppose that these processes are well approximated by white noise. Consequently, functions $F_R(Q(t), P(t))$ and $F_I(Q(t), P(t))$ are also stochastic processes. The corresponding increments, denoted by dW_R and dW_I and understood in the Itô sense, are assumed to satisfy

$$\begin{aligned} &E[dW_{nR}] = 0, \qquad E[dW_{nI}] = 0, \\ &dW_{nR}dW_{mR} = dW_{nI}dW_{mI} = \delta_{nm}dt, \\ &dW_{nR}dW_{mI} = 0, \\ &dW_{nR}dt = dW_{nI}dt = 0, \end{aligned} \tag{10}$$

where $E[\,\cdot\,]$ denotes the expectation with respect to the stochastic process and n, m count up to the number of observables $\{\hat{A}_n\}$. This implies, among other things, that all $F_R(t)$, $F_I(t)$ satisfy the Markovian property. Finally, the dynamical equation of QDF in interaction with the macro-system is given by the stochastic differential equation of a non-autonomous diffusion process,

$$dX^a = \omega^{ab}\nabla_b H_{phys}dt - \frac{\{\Gamma_A, H_q\}}{\|\nabla \Gamma_A\|^2} g^{ab}\nabla_b \Gamma_A dt + \omega^{ab}\nabla_b A\, dW_R + g^{ab}\nabla_b A\, dW_I. \tag{11}$$

Equation (11) is the main dynamical equation of the QDF interacting with the macro-system of the FHT developed here. If all degrees of freedom of the system are described by quantum mechanics, then unitary quantum evolution applies and there is only the first term with $H_{phys} = H_q$. If there is an interaction of QDF and the macro-system, i.e. some of the degrees of freedom are a priori described by classical mechanics, then the full equation (11) applies. Notice that no unobservable degrees of freedom (Q, P) appear in the equation. The first part of the drift in (11) describes Hamiltonian evolution with the Hamiltonian $H_q(x, y) + f(q, p)A(x, y)$. The second term of the drift represents a gradient flow with the tendency to decrease the total dispersion $\Delta A = \sum_n \Delta A_n$. The joint effect of the Hamiltonian and gradient drift terms is to preserve constant the total dispersion. If there is only one observable $A(x, y)$, or a set of commuting observables, then the role of the gradient terms is to force the evolution towards the common eigenstates of $\{\hat{A}_n\}$. If the observables $\{\hat{A}_n\}$ do not commute, then there is a competition of tendencies due to the corresponding gradient terms. If these observables generate a representation of a semi-simple Lie algebra, then the gradient terms drive the system towards the invariant manifold of the coherent states of the algebra.

The stochastic terms are divided into two quite different groups. The Hamiltonian terms, which can be included as stochastic perturbations of the Hamiltonian H_{phys}, describe the Hamiltonian influence of the (Q, P) degrees of freedom on the motion of the quantum system. For example, this is like the influence of an external stochastic electromagnetic field. However, these terms do not contribute to the localization onto the constraint manifold. The gradient stochastic terms, on the other hand,

describe the influence of (Q, P) degrees of freedom which is not Hamiltonian. However, as opposed to the Hamiltonian stochastic terms, the gradient stochastic terms induce localization onto the constraint manifold. If all $\{\hat{A}_n\}$ are commuting, then the stochastic terms of both types are zero if $\nabla A_n(x, y) = 0$ for all observables. This means that the point (x, y) is a fixed point of the Hamiltonian evolution with each A_n as the Hamiltonian. Such a point corresponds to a common eigenstate of the nonlinear operators $\hat{A}_n - \langle\hat{A}_n\rangle$ with all eigenvalues being zero. The common eigenstates of these operators coincide with the common eigenstates of $\hat{A}_n$. Thus, the stochastic terms in Eq. (11) are equal to zero if only commuting quantum observables appear, and (x, y) corresponds to a common eigenstate of $\{\hat{A}_n\}$.

Dynamics of CDF

Classical degrees of freedom (q, p) satisfy the Hamiltonian evolution equations given by the Hamiltonian (3). The equations in terms of (q, p) are

$$\dot{q} = \frac{\partial H_{cl}(q,p)}{\partial p} + A(x,y)\frac{\partial f(q,p)}{\partial p}$$
$$\dot{p} = -\frac{\partial H_{cl}(q,p)}{\partial q} - A(x,y)\frac{\partial f(q,p)}{\partial q}. \tag{12}$$

The evolution of CDF is also stochastic because the quantum observables $A(x(t), y(t))$ evolve stochastically.

3.3. Quantum measurement process

Additional assumptions can be used in order to simplify the evolution equations (11) and (12) in the case of a quantum measurement process. One such approximation is based on the assumption that the dynamics of QDF is much faster than that of CDF. Consequently, one can replace in (11) the functions $(q(t), p(t))$ with their initial values (q_0, p_0). The equation for QDF becomes autonomous. The situation when QDF and CDF are coupled via only one observable $\hat{A}$ with the interaction term given by $H_{int} = pA(x, y)$, and when the gradient terms dominate the QDF dynamics, corresponds to the process of measurement of $\hat{A}$. QDF dynamics is approximately given by

$$dX^a = \omega^{ab}\nabla_b(H_q + p_0A(x,y))dt - \frac{\{\Gamma_A, H_q\}}{\|\nabla\Gamma_A\|^2}g^{ab}\nabla_b\Gamma_A dt$$
$$+ \omega^{ab}\nabla_b A(x,y)dW_R + g^{ab}\nabla_b A(x,y)dW_I. \tag{13}$$

Due to the gradient terms, the state approaches one of the eigenstates of $\hat{A}$, denoted by $(x_\alpha, y_\alpha) \equiv |\alpha\rangle$, with the eigenvalue $A(x_\alpha, y_\alpha) = \alpha$. The stochastic term introduces fluctuations, and the probability of the asymptotic eigenstate (x_α, y_α) depends on its distance from the initial state $(x, y)_{init} \equiv |\psi\rangle_{init}$, i.e. on $\|\langle\psi_{init}|\alpha\rangle\|^2$. These facts can be demonstrated numerically as we shall do shortly. The asymptotic dynamics of (13), or of (11) and (12), can also be analyzed using methods of stochastic stability analysis [21], in particular the stochastic generalization of the first Lyapunov method with the constraint Γ_A playing the role of the Lyapunov function, as will be illustrated elsewhere. Using the same assumption about different time scales and assuming that H_{cl} is negligible, the CDF dynamics of the coordinate of the apparatus pointer is approximated by

$$\dot{q} = \alpha \tag{14}$$

and reads the eigenvalue of $\hat{A}$. Thus, the approximate equations describe well the dynamics and the results of the measurement process.

3.4. Numerical example

We shall illustrate the hybrid dynamics modeling the measurement as given by (11) and (12) using the simplest example where the quantum system is a single 1/2-spin and the classical system is an

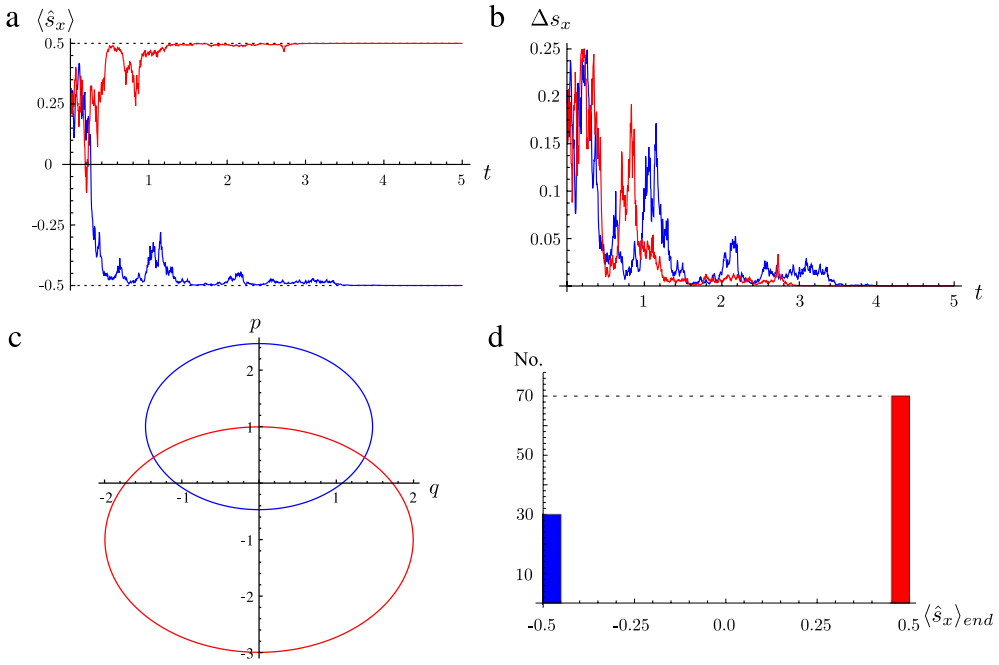

Fig. 1. (Color online) (a) $\langle \hat{s}_x \rangle(t)$ and (b) $\Delta s_x(t)$ for two typical stochastic paths. (c) Classical orbits corresponding to the stochastic paths in parts (a) and (b). (d) Histogram of the number of paths converged to $+1/2$ or $-1/2$ eigenstate of $\hat{s}_x$.

oscillator. The phase space of the quantum part corresponding to the Hilbert space $\mathbb{C}^2$ is $\mathbb{R}^4$, with the canonical coordinates (x_1, x_2, y_1, y_2). The relations between the real canonical coordinates and the complex expansion coefficients, (c_1, c_2) in the computational basis, of a normalized vector from $\mathbb{C}^2$ are given by the following formulas

$$c_k = \frac{x_k + iy_k}{\sqrt{2}}, \qquad c_k^* = \frac{x_k - iy_k}{\sqrt{2}}, \quad k = 1, 2. \tag{15}$$

The quantum Hamiltonian of a single spin is $\hat{H}_q = \omega \hat{s}_z$, the classical Hamiltonian of the oscillator is $H_{cl} = p^2/2m + m\Omega^2 q^2/2$ and the interaction $\hat{H}_{int} = \mu p \hat{s}_x$ corresponds to the measurement of $\hat{s}_x$. The functions on the QC phase space corresponding to $\hat{H}_q$ and $\hat{H}_{int}$ are

$$H_q(x, y) = \frac{\omega}{2} \frac{x_1^2 + y_1^2 - x_2^2 - y_2^2}{x_1^2 + y_1^2 + x_2^2 + y_2^2} \tag{16}$$

$$H_{int}(q, p, x, y) = \mu p \frac{x_1 x_2 + y_1 y_2}{x_1^2 + y_1^2 + x_2^2 + y_2^2}. \tag{17}$$

The constraint Γ_{s_x}, corresponding to the measurement of $\hat{s}_x$, is $\Delta s_x = \langle \hat{s}_x^2 \rangle - \langle \hat{s}_x \rangle^2 = 0$, and is given in terms of the canonical coordinates (x, y) by a slightly more complicated expression

$$\Gamma_{s_x} = \frac{((x_1 - x_2)^2 + (y_1 - y_2)^2)((x_1 + x_2)^2 + (y_1 + y_2)^2)}{(x_1^2 + y_1^2 + x_2^2 + y_2^2)^2}. \tag{18}$$

The Poisson bracket $\{\Gamma_{s_x}(x, y), H_q(x, y)\}_{x,y}$, the gradients $\nabla \Gamma_{s_x}(x, y)$ and $\nabla s_x(x, y)$ are easily computed and shall not be presented. These expressions are used to write down the dynamical equations (11) and (12), which are solved using the appropriate code for numerical solutions of SDE. Results are illustrated in Fig. 1(a)–(d). Each of 100 sample stochastic paths after some time converges to either

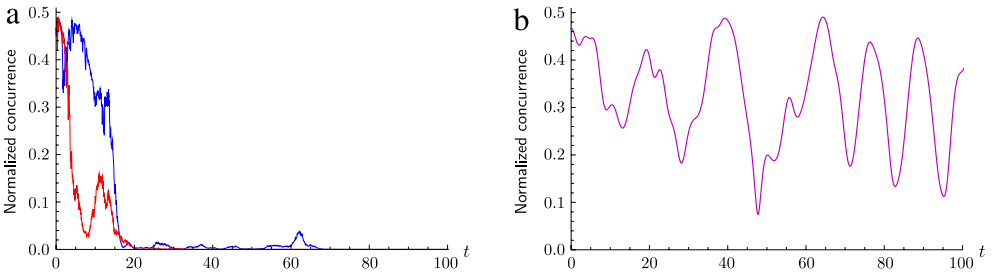

Fig. 2. (Color online) Normalized concurrence for two typical sample paths of FHT evolution (a) and for purely Hamiltonian evolution (b) starting from the same initial state (see text for details).

$-1/2$ or $1/2$ eigenstate of $\hat{s}_x$, denoted by $|1/2, -1/2\rangle$ and $|1/2, 1/2\rangle$, respectively. Fig. 1(a) and (b) show $\langle\hat{s}_x\rangle(t)$ and $\Delta s_x(t)$ for two typical realizations of the stochastic process starting from the same initial state and converging to the state $|1/2, 1/2\rangle$ (red curves) and the state $|1/2, -1/2\rangle$ (blue curves), respectively. The initial state is determined by $|\psi\rangle_{init} \equiv (x_1, x_2, y_1, y_2)_{init} = \sqrt{2}(2, 4, -2, 1)/5$ and $(q, p) = (1, 1)$, which yield $|\langle 1/2, -1/2|\psi_{init}\rangle|^2 = 0.26$ and $|\langle 1/2, 1/2|\psi_{init}\rangle|^2 = 0.74$. Fig. 1(c) illustrates the evolution of CDF (q, p) for the two stochastic sample trajectories related to Fig. 1(a) and (b). The two classical orbits are obviously different. The percentage of stochastic paths converging to either of the eigenstates is illustrated in Fig. 1(d) and is proportional to the distance of the initial state form the eigenstates. Qualitatively the same results are obtained for all different initial states that we have tested.

Let us point out that in the described numerical example the full system of equations (11) and (12) was used, and the sufficiently fast convergence of the QDF and the inertial properties of the CDF are obtained by the appropriate choice of the parameter values.

4. Remarks

(1) *Dynamics of entanglement* in a quantum system coupled to a classical one, as described in FHT, can be studied using, for example, a pair of qubits interacting with a classical oscillator. The relevant part of the Hamiltonian is given by

$$\begin{aligned} \hat{H}_q &= \omega\hat{s}_z^1 + \omega\hat{s}_z^2 + c\hat{s}_x^1\hat{s}_x^2, \\ H_{cl} &= \frac{p^2}{2m} + \frac{m\Omega^2 q^2}{2}, \\ \hat{H}_{int}(q, p) &= \mu p\hat{s}_z^1. \end{aligned} \tag{19}$$

The complex coefficients of an arbitrary two spin state $|\psi\rangle \in \mathbb{C}^4$ in the computational basis are denoted by c_1, c_2, c_3, c_4 and their real and imaginary parts are the canonical coordinates given by $(x_k, y_k) = \sqrt{2}(\mathrm{Re}(c_k), \mathrm{Im}(c_k))$, $k = 1, 2, 3, 4$. The total Hamilton's function is $H(x, y, q, p) = H_q(x, y) + H_{int}(x, y, q, p) + H_{cl}(q, p)$ where $H_q(x, y) = \langle\psi|\hat{H}_q|\psi\rangle/\langle\psi|\psi\rangle$ and $H_{int}(x, y) = \langle\psi|\hat{H}_{int}|\psi\rangle/\langle\psi|\psi\rangle$. The constraint corresponding to $\hat{H}_{int}$ in (19) is $\Delta s_z^1 = 0$.

It can be shown, by numerical computations, that the entanglement of an initial entangled state of the qubits evolves to zero for sufficiently large ratio μ/c. The entanglement dynamics is most easily studied by monitoring the normalized concurrence of the pure state of QDF given by $C = |c_1c_4 - c_2c_3|/(|c_1|^2 + |c_2|^2 + |c_3|^2 + |c_4|^2)$. A pure state of the qubit pair is separable iff the concurrence is zero. The asymptotic QDF state of the evolution for μ/c sufficiently large has zero concurrence. This fact is illustrated by the time series $C(t)$ with full FHT equations in Fig. 2(a) and with the purely Hamiltonian dynamics discussed in the remark (2) in Fig. 2(b) starting from the same initial state. The asymptotic state of QDF is a product state of the form $|1/2, \pm 1/2\rangle_1 \otimes |\psi\rangle_2$, where $|1/2, \pm 1/2\rangle_1$ are the eigenstates of $\hat{s}_z^1$ and $|\psi\rangle_2$ is a state of the second qubit. Two sample paths in Fig. 2(a) correspond to the concurrence in these two cases.

(2) The constraint (5) was introduced so as to obtain a hybrid system such that the selected observables of the quantum part behave as almost classical. This is admittedly an *ad hoc* assumption. Alternatively one might study the Hamiltonian system (2) *with no additional constraints*, and analyze it as a purely Hamiltonian system with possibly complicated interactions. This is the approach adopted for example in [2], where it was supposed that there are no (Q, P) degrees of freedom so that the evolution is given by the Hamiltonian system on $\mathcal{M}_{qp} \times \mathcal{M}_{xy}$ with $H = H_q(x, y) + H_{cl}(q, p) + H_{int}(q, p, x, y)$. The result is mathematically consistent purely Hamiltonian theory of a hybrid system. However, application of the theory to the measurement situation shows that classical pointer variable is in general coupled to the expectation $\langle\hat{A}\rangle$ of the measured observable $\hat{A}$ and not to its eigenvalues [22,23]. Furthermore, the theory in its exact form predicts some features of QDF which might imply possibility of superluminal communication [24]. The evolution of QDF can be presented in the form of the Schrödinger equation with the Hamiltonian that depends on the total system state. Also, different initial convex representations of a mixed state $\hat{\rho}$ might evolve into different $\hat{\rho}(t)$. Furthermore, investigations of entanglement dynamics, like in the remark (1), show that the entanglement between qubits oscillates with large amplitudes forever and for any values of the parameters. It is well known that the possibility of entanglement and nonlinear evolution, or the dependence of a density matrix evolution on its initial convex representation, might be used for superluminal communication [25,26]. In the FHT this nonphysical effect might be prevented by the stochastic terms in the evolution.

In short, the purely Hamiltonian theory predicts properties of QDF, interacting with CDF, that are not displayed by physical systems. The way to remedy the theory might be to include the influence of the internal degrees of freedom $(Q(t), P(t))$, perhaps in the form of stochastic perturbations. This has not been done in full generality. Some results [2], where the CDF are treated as an environment and are supposed to introduce stochastic perturbations, indicate that such an approach might be successful. In conclusion, purely Hamiltonian theory with the Hamiltonian (2) must be supplemented by an analysis of complicated classical systems with complex CDF dynamics, and only after physically plausible approximations might explain the observed behavior.

(3) Instead of imposing the main effects of the collapse process as the general requirements on the dynamical equation for QDF, and realizing those requirements as a minimal but adequate constraint, one can postulate that the dynamical equations of QDF are given by some of the existing *dynamical collapse models*, reviewed recently in [27] or open quantum system dynamics [28,29] or models of continuous measurements [30]. Such equations usually assume some properties, and specific form, that are not necessary for the most general description of the hybrid dynamics. The most well known dynamical collapse models are given as nonlinear and stochastic modifications of the Schrödinger equation, and contain the Schrödinger term, the nonlinear gradient term and the stochastic term. Similarly, the master equation for the density operator $\hat{\rho}(t)$ of an open quantum system under the Markovian assumption is of the Lindblad form, and can be written as a stochastic diffusion equation for the individual quantum systems in pure states [29,28], with terms of the similar form and the same effect on the evolution as in the explicit collapse models. One such equation, with minimal appropriate generalization, can be postulated for the QDF dynamics of the hybrid and coupled with the Hamiltonian equations (11) for the CDF. An example of such approach is studied in [30]. The result is a set of stochastic differential equations of the form similar to those of *FHT*. Nevertheless, conceptual differences should be stressed. The theories of explicit collapse do not make an a priori distinction between quantum and classical systems. Instead, unique nonlinear and stochastic dynamics for micro and macro systems is postulated, the only difference being in the values of the relevant parameters. If there is a micro-system coupled to a macro-system, then the micro-system dynamics is indistinguishable from the linear Schrödinger evolution, and the collapse occurs in the macroscopic part of the system. This collapse is a consequence of the macroscopic size of the macro-system. In *FHT*, classical behavior of CDF of the macro-systems is assumed from the beginning, and in this respect the theory is conceptually similar to the hybrid theory in [30]. The collapse occurs directly in the quantum part and is a consequence of the interaction between the quantum system and the macro-system, where the latter is conceived as a system with some degrees of freedom described by classical mechanics.

We shall illustrate a possible hybrid theory based on an explicit collapse model, given basically by Hughston [31], since it has been formulated using the quantum phase space. We present the equations in the case when there is only one observable $\hat{A}$, and in terms of evolution on $\mathcal{M}$. A hybrid theory with

typical collapse equation for the QDF would then be of the form

$$dX^a = 2\omega^{ab}\nabla H(X, q, p)dt - \frac{\mu^2}{4}g^{ab}\nabla_b(\Delta A(X))dt + \mu\nabla A(X)dW \tag{20}$$

where $X \equiv (x, y)$ and dW are the stochastic increments of the Wiener process. Eq. (20) for QDF should be supplemented by Eq. (12) for the CDF. Other models of continuous collapse or individual open system dynamics might be written in forms quite similar to (20) with real or complex noise. In Hughston [31] and QMUPL [27] equations dW are real, while in the QSD equation [29] dW are increments of a complex Wiener process. The Hamiltonian $H = \langle\hat{H}\rangle$ is modified to include the interaction with CDF given by $\mu f(p, q)A(x, y)$. Together with the corresponding equations (12) for the CDF dynamics the system represents a model of an individual hybrid system evolution, which has not been investigated in the literature (to the best of our knowledge). Eq. (20) is similar with (11) in that it has a deterministic gradient term, given by the gradient of the relevant dispersion, and the gradient stochastic term given by the gradient of the relevant observable. However, the dynamics of a single quantum open system, for example in QSD [29], is equivalent to the Lindblad equation which is physically justified using weak coupling approximation, and no such approximation is assumed in (11). The major technical difference between (20) and (11) is that the latter has a pre-factor multiplying the deterministic gradient term. A further and deeper comparison of the hybrid theories with Eq. (11) or (20) for the QDF part will certainly be of some interest.

5. Summary

In summary, we have constructed a novel theory of hybrid quantum–classical systems of the type where the quantum and the classical mechanics are both treated as fundamental theories. We have started from the observation that if all degrees of freedom of the system are considered as quantum then the evolution is given by the Schrödinger law, while if there are some degrees of freedom which behave as described by classical mechanics then the collapse postulate should be added to the Schrödinger evolution of the quantum degrees of freedom. Our goal was to derive a theory that provides a dynamical description of the Schrödinger evolution supplemented with the collapse postulate. It is assumed that such a theory would provide a unified dynamical description of the system with quantum and classical degrees of freedom. The basic requirement imposed on the theory is to obtain dynamical equations of the hybrid systems such that the sum of dispersions of the quantum observables that figure in the quantum–classical interaction are constrained to be minimal during the evolution. The crucial assumption that was used to simplify the constrained equations is that the dynamics of the unobserved degrees of freedom is to be replaced by white noise. Furthermore, it was assumed that part of the interaction with the unobserved degrees of freedom is described by complex Hamiltonian, but the equations for the real canonical coordinates (q, p, x, y) are real. The resulting evolution of the hybrid system is nonlinear and stochastic. Some of the stochastic terms are multiplied by the gradients of expectations of the chosen quantum observables, and together with the deterministic gradient terms lead to localization onto the constraint manifold. If the hybrid system is intended as a model of the measurement process of one observable, then the constraint gives the dynamics with eigenstates as attractors, and the stochastic term describes the stochastic nature of the process with the correct probabilities for different asymptotic eigenstates. At the same time, interaction establishes the necessary correlations between the states of the quantum and classical parts.

The hybrid theory derived here has been considered at an abstract level, with the primary goal of demonstrating that consistent hybrid theories, formulated within the specific mathematical framework, are possible. Validity of the theory was tested only with reference to the simplified description of the measurement process as summarized by quantum mechanics with the collapse postulate. There are several immediate questions that are interesting and should be analyzed. On the theoretical side, one should analyze if the hybrid dynamics given by FHT can be used for superluminal communication between entangled quantum systems in interaction with the corresponding macroscopic objects. To this end, one should analyze the FHT dynamics of ensembles of hybrid systems with the corresponding

master equation for the QDF. Because of the stochastic terms, and perhaps under physically justified assumptions, one expects that the evolution of the suitably defined density matrix pertaining to QDF can be expressed with no reference to particular convex representations of the density matrix. However, the Fokker–Planck equation for general hybrid densities implied by the stochastic FHT dynamics (11) of pure states is rather complicated, and we are not presently able to obtain from it a closed form equation for the mixed states of the quantum system. This question will certainly be thoroughly analyzed. Such analysis will also help to clarify the relation of FHT with the hybrid theories based on models of explicit collapse, as discussed in the remark (3). Another theoretical task is to analyze in detail, using suitable examples, the form of the theory where the quantum and macroscopic systems interact via several non-commuting observables. This would pave the way to apply the theory onto realistic physical systems, other than the rudimentary measurement setting, which are expected to be in the domains of hybrid theories.

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