An Effective 1-D Model of Planetary Condensation

A. Balaž, A. Belić and A. Bogojević

Institute of Physics
P.O.B. 57, Belgrade 11001, Yugoslavia

Abstract

An effective 1-D model of planetary condensation through gravitational accretion is presented. The formation of planetary systems is is simulated starting from as many as $N = 10^{10}$ initial particles, and the spontaneous appearance of two distinct types of condensate is investigated. The two types of condensate, light and heavy, are distinguished by the way they scale with changes of N. The positions, mass and spin of the heavy condensates (planets) are found to be in very good agreement with Solar system data.

1 Introduction

There has been a large increase of the amount of available data on extra-solar planetary systems [1], and an even more dramatic increase is expected in the near future [2]. For this reason the long standing problem of the detailed understanding of planet formation [3] has received a lot of attention lately [4, 5]. The ultimate goal of such studies is to assess the likelihood of the formation of Earth-like planets. The exponential increase in computing power during the last two decades has made numerical simulation the most promising tool to achieve this goal. However, the brute force numerical simulation of the formation of whole planetary systems is still impossible due to the large number of initial particles ($N \gtrsim 10^6$) needed to resolve the resulting planets with reasonable accuracy. In order to deal with this problem we have constructed an effective model of planetary accretion. The simplifications in the model have made it effectively one dimensional, which not only makes gravitational condensation more transparent, but also makes possible the derivation of certain analytical results.

2 The Model

Our effective planetary condensation model starts from a given planar distribution of N initial particles, all of the same mass and with no spin. The particles have a uniform angular distribution, while the radial distribution is given by $\rho(r)$, which determines the initial conditions.

The dynamics is modeled by two independent processes—free propagation of particles and instantaneous interactions. Between interactions all particles move on circular trajectories according to Kepler's laws. The only interaction allowed is the merging of two particles into one. The merging happens if the two particles satisfy an interaction criterion given bellow. The result of the merging of two bodies with masses m_1 and m_2, at positions $\vec{r_1}$ and $\vec{r_2}$, and with spins S_1 and S_2, is a new body with mass $m_1 + m_2$, position $\vec{R}$, and spin $S = S_1 + S_2 + L_1 + L_2 - L$. L_1, L_2 and L are respectively the orbital angular momenta of the first, second and final particle. The point of joining follows from energy conservation (once we neglect heating due to accretion, energies due to the particles spin, as well as the potential energies between pairs of condensing particles). We find

$$\frac{m_1 + m_2}{R} = \frac{m_1}{r_1} + \frac{m_2}{r_2} \ . \tag{1}$$

The interaction criterion chosen is quite natural. We impose $F \triangle t \gtrsim |\triangle \vec{p}|$, where F is a mean value of the gravitational force between the bodies during the collision and $\triangle t \sim |\triangle \vec{r}|/|\triangle \vec{v}|$ is a characteristic time of the collision. For this criterion to be satisfied the two initial particles must be close in space. As a consequence, the angle between them must be small. In this paper we set $\theta = 0$, and disregard the $O(\theta^2)$ corrections. This makes the model one dimensional, and the criterion becomes

$$\frac{1}{m_1 m_2} \left| \frac{m_1 + m_2}{\sqrt{R}} - \frac{m_1}{\sqrt{r_1}} - \frac{m_2}{\sqrt{r_2}} \right| \left| \frac{1}{\sqrt{r_1}} - \frac{1}{\sqrt{r_2}} \right| |r_1 - r_2| \leq K \ . \tag{2}$$

As we see, the interaction is given in terms of a single parameter K. From the derivation we have $K \sim 1/M$, where M is the mass of the star. Note that Newton's gravitational constant G has cancelled by the use of Kepler's law. The above criterion is homogeneous with respect to changes of mass scale as well as of distance scale. We will find it convenient to fix the mass scale by setting the mass of the protoplanetary material M_P equal to one. By doing this, the mass of the initial particles becomes $1/N$, while K becomes dimensionless. Distance scales are fixed by our choice of initial mass density. In this paper we work with the triangular initial mass density

$$\rho(r) = \begin{cases} \frac{1}{5} r & \text{if } r \leq 1 \\ \frac{1}{45}(10 - r) & \text{if } 1 < r \leq 10 \ . \end{cases} \tag{3}$$

This is a rather simple density peaked at $r = 1$. It is convenient to work with reduced angular momentum quantities $\ell = L/\sqrt{MG}$, as well as $s = S/\sqrt{MG}$. In these units the orbital angular momentum of a body of mass m at a distance r from the star is equal to $\ell = m\sqrt{r}$. As a result of joining the spin of the new body becomes $s = s_1 + s_2 + m_1\sqrt{r_1} + m_2\sqrt{r_2} - (m_1 + m_2)\sqrt{R}$. It is easy to see that in this simple model spins are always positive, in accord with most of the planets in the Solar system.

3 Basic Results

The first physical quantity that we look at is $\Omega \equiv n/N$, the ratio of the number of final condensates and initial particles. Ω is a monotonous function of K that decreases from 1 (small K) to 0 (large K) and distinguishes between two phases, dominated by light and heavy condensates respectively. The behavior of the system is most complicated in the intermediate regime, in which Ω differs significantly from 0 or 1, i.e. when there is a significant mixture of both light and heavy condensates. It is rather easy to fit the obtained data to a simple law. We find

$$\frac{1}{\Omega} - 1 = A N^{\alpha} K^{\beta} \ , \tag{4}$$

where $\alpha = 0.737(6)$, $\beta = 0.251(2)$, while $A = 2.10(5)$. Since $n_{light} \gg n_{heavy}$, we see that Ω is just the relative number of light condensates. Ω is a global property of light condensates. A more detailed understanding of their structure is achieved by studying their distributions in mass and distance from the star. The mass distribution, i.e. the fraction of condensates at mass m, can be fitted by a simple power law

$$\Delta(m) = \begin{cases} 0 & \text{if } m < 1/N \\ \tau N^{-\tau} m^{-\tau-1} & \text{if } 1/N \leq m < m^* \ , \end{cases} \tag{5}$$

where $\tau = 1.20(5)$. The mass scale m^* is the dividing line distinguishing between light and heavy condensates. More details about this and other properties of light condensates will given in a forthcoming publication. Condensates with masses greater than m^* are designated as heavy. Planets belong to this category. We have studied their masses, locations and spins. Unlike the light condensates, the properties of the heavy condensates do not scale with N. The K dependence of the masses of the four heaviest condensates is shown in Figure 1. The choice of $K = 0.1$ follows from equating m_2/m_1 with the ratio of the masses of Saturn and Jupiter. By fixing K we have completely determined our model. The predicted masses of the other giants also agree with Solar system data up to 5%.

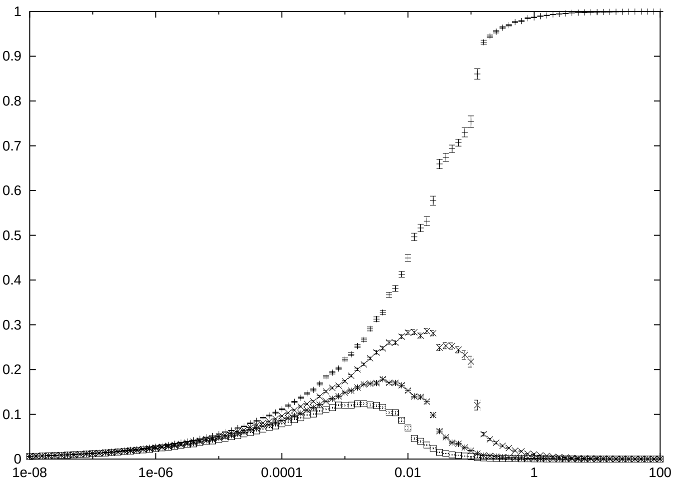

Figure 1: The mass of the four heaviest condensates as functions of K.

The situation with spin is even more interesting. We find that

$$s \propto K^{\epsilon} m^{\omega} \ , \tag{6}$$

where $\omega = 1.75(3)$, and $\epsilon = 0.40(2)$. This is shown in Figure 2. Except for the very lightest particles whose mass is near the $1/N$ cut-off all the points lie on the law given in equation (6). The corresponding data

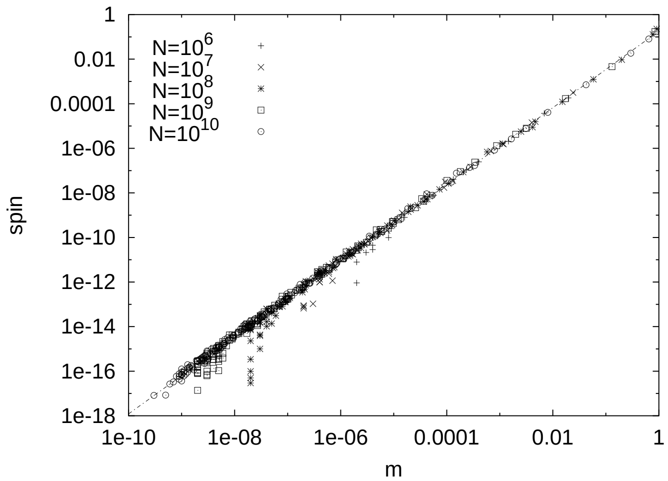

Figure 2: The spin of condensates as a function of their mass for $K = 0.1$ and $N = 10^6, 10^7, \ldots, 10^{10}$. The data fits to $s \propto m^{1.75}$.

for the planets in the Solar system is given in Figure 3. We see that our simplified accretion model agrees quite well with phenomenology, giving rise to a scaling exponent for spin of $\omega = 1.75(3)$, which is close to the measured value of $\omega = 2.00(4)$. The only two planets that do not satisfy the above spin-mass relation are Mercury and Venus. This is not surprising, as these are the two planets nearest to the sun, where additional tidal lock effects play an important role. Of the above properties, the exponents governing Ω and Δ have been found to be independent of the choice of initial mass density. The same is true of spin. The masses of the heavy condensates do depend on ρ, but the general form of this dependence is quite similar for all ρ's. The properties of the condensates that depend *very strongly* on the initial mass density are the positions of the planets.

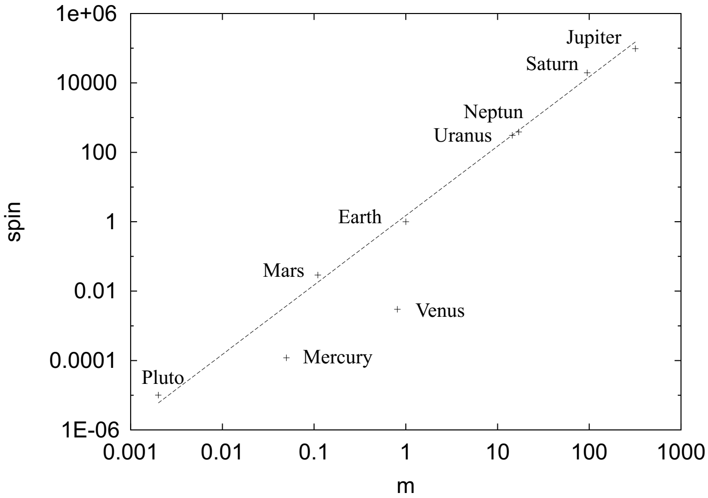

Figure 3: In the Solar system the planet's spin fits a $s \propto m^2$ law, that is, the exponent is $\omega = 2$.

In conclusion, the presented one dimensional model of planetary formation leads to simple power law behaviour that agrees well with Solar system data. The detailed study of the universality of these laws with respect to changes of initial mass distribution $\rho(r)$ is in progress, and will be given in a forthcoming publication.

The numerical simulations presented in this contribution have been performed on an SGI Origin 2000 super-computer. We would like to acknowledge the help of the staff of the IPCF (Institute of Physics Computing Facilities). This research was supported in part by the Serbian Ministry of Science and Technology under research projects 01M01 and 01E15.

References

[1] G. W. Marcy, and R. P. Butler, *Annu. Rev. Astron. Astrophys.* **36** 57-97 (1998).

[2] M. G. Lattanzi, *et. al.* Hipparcos Venice '97, ESA SP-402 (1997).

[3] R. Isaackman and C. Sagan, *Icarus* **31**, 510-533 (1977).

[4] S. Ida and J. Makino, *Icarus* **96** 107-120. (1992); *Icarus* **98** 28-37. (1992).

[5] E. Kokubo and S. Ida, *Icarus* **114** 247-257. (1995); *Icarus* **123** 180-191. (1996); *Icarus* **133** 171-178. (1998).