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NOTE

Simplified model of low energy x-ray backscattering

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Abstract

The reflection of x-rays from a half space is studied within the framework of a model that assumes multiple isotropic scattering of photons without energy loss. An exactly solvable analytical expression for the angular distribution of reflected photons is derived. The range of validity of the model was determined by the Monte Carlo simulation thereby incorporating energy loss and angular dependences. For water as a scatterer, in the energy range from 10 to 60 keV, which is often used in x-ray diagnostics, the two approaches differ by at most 5%. The analytic results, confirmed by the Monte Carlo simulation, show that the angular distribution of reflected photons for energies greater than 30 keV—where multiple scattering events dominate—may be represented by a cosine law, within a few per cent of accuracy.

1. Introduction

The basic physical principles that govern the transport of radiation are well known, and recent advances in computer technologies have made possible numerical solutions of the corresponding transport equations with an acceptable accuracy. Powerful and fast general-purpose radiation transport codes based on numerical solutions of the underlying transport equations or on Monte Carlo simulations are available, and the usefulness and efficiency of these codes are generally acknowledged (Chilton *et al* 1984, Shultis and Faw 1996, Leipunovsky *et al* 1960). In such codes, the detailed treatment of a great variety of physical processes, described at an atomic level by accurate cross sections, is present so that the results may be considered as referent. However, often for particular applications such as radiation protection, qualitative insight including an overview of possible radiation damage may be enriched in a complementary way, by a modelling approach that can be treated analytically. Such an approach, while less accurate in terms of adherence to cross sections, is simpler and therefore more transparent, the results of which in order to be useful should not depart greatly from the referent one. Such an approach may also be useful when there is a need to recalculate some results for a range of variations of the parameters of the model, or to make possible rapid optimization such as would be difficult with the Monte Carlo based approach.

In the present note, starting from a couple of physically reasonable assumptions (section 2), an analytical solution for x-ray backscattering from the half space is obtained (section 3) and compared with the standard Monte Carlo simulation (MCNP code). The approaches may be considered to be somewhat complementary. In regions of space where one has good statistics, the results of the Monte Carlo simulations, due to accurate cross sections used in them, are more accurate than the analytic model calculations. However, when the scattering of a beam of particles is weak so that corresponding statistics is poor, the analytic solutions are of great help in correct descriptions of particle transport and reflection. Specially, in the treatment of reflection in regions of poor statistics, the Monte Carlo simulations often rely on the analytic solutions of the half-space problem (Leipunovsky *et al* 1960).

It is shown that in the energy range from 10 to 60 keV, the agreement of both approaches is satisfactory (section 4). Using these analytical results, the spatial distribution of the intensity of backscattered radiation from a patient during x-ray diagnostics is analysed. It is shown that the maximum of the potential irradiation of the head of the radiologist appears at the cone at a polar angle $\approx 55^\circ$ (section 5).

2. Basic physical assumptions

Among many different processes that may arise during the penetration of photons through matter, in the energy range 10–60 keV, that is considered in this work, only photoabsorption and scattering of photons are relevant (Chilton *et al* 1984, Shultis and Faw 1996, Leipunovsky *et al* 1960). For theoretical estimates of photon beam attenuation, the total cross section for the photoabsorption is needed and there is an abundance of such data (Shultis and Faw 1996).

For smaller energies photons are scattered coherently (Rayleigh scattering) and this process is energy preserving. For higher energies, Compton scattering is dominant, and the change of photon energy in this process may be obtained easily from the conservation laws of energy and momentum (Chilton *et al* 1984, Shultis and Faw 1996, Leipunovsky *et al* 1960). For intermediate energies, both types of scattering take place, and must be included in a correct description of photon scattering.

For photon energies considered in this work, which are much smaller than the electron rest energy, the change of photon energy in the Compton scattering is small, while the scattering distribution (given by the Klein–Nishina formula) becomes nearly isotropic (Leipunovsky *et al* 1960, Shultis and Faw 1996). The same is true for the differential cross section of Rayleigh scattering; moreover, the photon in this process does not lose its energy at all. This is why at the energies considered the x-ray transport and reflection may be treated as classical monoenergetic particle transport with isotropic scattering. In this model, the energy fluence is simply the particle fluence multiplied by the incident photon energy. Therefore, we will restrict our consideration to particle fluence only. This model allows analytical treatment, which is the subject of the next section.

3. Reflection of photons from a half space assuming isotropic scattering

The integral equation which describes the reflection of photons from a half space assuming isotropic scattering and no energy loss is derived from first principles. We extract a differentially thin layer Δz just below the target surface. The layer is so thin that a photon may undergo at most one collision within it. Then we have the following five different cases shown in figure 1 which contribute to the angular distribution of reflected photons $R(\mu_0, \mu)$ and whose probabilities are linear functions in Δz. Here, μ_0 and μ are the directional cosines of

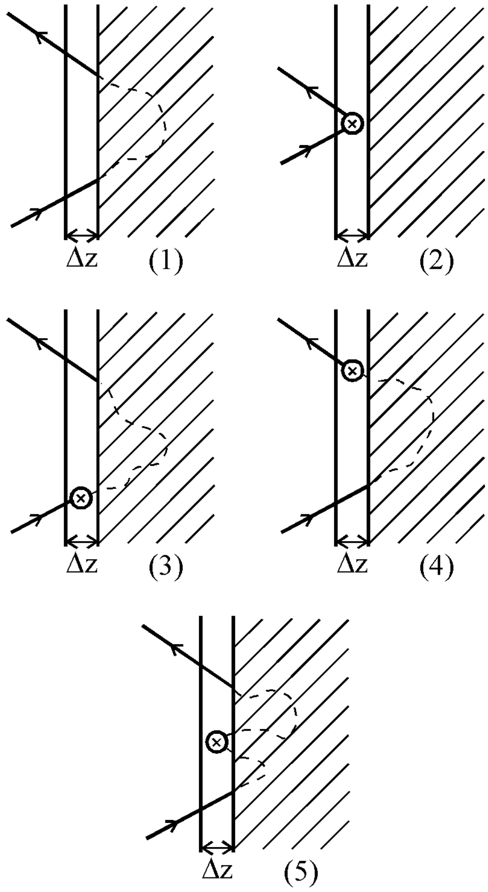

Figure 1. Relevant events in the thin layer during the reflection of a photon. The crosses denote single scattering in the layer while the broken lines represent backscattering in the infinite medium.

the incident and reflected photons with respect to the target surface normal, and $R(\mu_0, \mu)\,\mathrm{d}\mu$ gives the probability for a photon to be reflected with directional cosines between μ and $\mu + \mathrm{d}\mu$, irrespective of the azimuthal angle. All other cases are $O(\Delta z^2)$.

Let us characterize physically these five cases (1)–(5).

(1) A photon passes the thin layer without scattering, is reflected from the remaining half space and arrives back at the target surface without scattering in the thin layer, figure 1 (1). The probability of this event is given by

$$\left(1 - \frac{\Delta z}{\mu_0} n\sigma_T\right) R(\mu_0, \mu) \left(1 - \frac{\Delta z}{\mu} n\sigma_T\right)$$

where the factors are written in the order of the events whose probabilities they describe. Here σ_T is the total cross section for attenuation of the beam per atom, and n is the number of atoms per unit volume. To first order in Δz, the above expression reduces to

$$R(\mu_0, \mu) - \Delta z n\sigma_T \left(\frac{1}{\mu_0} + \frac{1}{\mu}\right) R(\mu_0, \mu).$$

(2) A backscattering event happens just in the thin layer, figure 1 (2). The probability of this event is

$$\frac{\Delta z}{\mu_0}\frac{\sigma_r}{4\pi}\cdot 2\pi n = \frac{\Delta z}{\mu_0} n \frac{\sigma_r}{2}$$

where σ_r is the total scattering cross section per atom, which includes both the Rayleigh and Compton scattering. The factor 4π in the denominator is a consequence of the assumption that the scattering is isotropic, while the factor 2π is the result of integration over azimuthal angle.

(3) A photon is scattered inwards in the thin layer, then it is reflected with no scattering when leaving the thin layer, figure 1 (3). The probability of this process is

$$\frac{\Delta z}{\mu_0} n \frac{\sigma_r}{2} \int R(\mu_x, \mu)\,\mathrm{d}\mu_x .$$

The integration extends over all inwards directions μ_x.

(4) A photon entering passes Δz without scattering is then reflected and undergoes scattering in the thin layer just before leaving the target surface, figure 1 (4). The total probability is now

$$n \frac{\sigma_r}{2} \int R(\mu_0, \mu_y) \frac{\Delta z}{\mu_y}\,\mathrm{d}\mu_y .$$

The integration here extends over all outward directions μ_y.

(5) A photon entering passes Δz without scattering, is then reflected and undergoes scattering in the thin layer but this time inwards, is again reflected, and leaves the target without further scattering, figure 1 (5). For the total probability of this process, integration over all allowed directions yields

$$\int R(\mu_0, \mu_y) \frac{\Delta z}{\mu_y}\,\mathrm{d}\mu_y n \frac{\sigma_r}{2} \cdot \int R(\mu_x, \mu)\,\mathrm{d}\mu_x .$$

The probabilities for all other cases conceivable are of second or higher order in Δz. Neglecting them, the angular distribution of reflected photons may be expressed as a sum of the five probabilities just described (1)–(5). The term $R(\mu_0, \mu)$ appears on both sides of this expression and may be cancelled. Cancelling, further, the common linear factor $\Delta z n$ in all remaining terms, after simple rearrangements, the following integral equation for $R(\mu_0, \mu)$ may be formed

$$R(\mu_0, \mu) = \frac{1}{2}\omega \frac{\mu}{\mu + \mu_0}\left[1 + \int R(\mu', \mu)\,\mathrm{d}\mu'\right] \times \left[1 + \mu_0 \int \frac{R(\mu_0, \mu'')}{\mu''}\,\mathrm{d}\mu''\right]. \quad (1)$$

Note that the scattering medium is characterized only through the parameter

$$\omega = \frac{\sigma_r}{\sigma_T} = \frac{\sigma_r}{\sigma_r + \sigma_a}. \quad (2)$$

This parameter represents the ratio of the total cross section for scattering σ_r, which includes both Compton and Rayleigh scattering, to the total attenuation coefficient which is the sum of σ_r and the total cross section for photoabsorption σ_a. Both cross sections are calculated per atom.

Writing $R(\mu_0, \mu)$in the form

$$R(\mu_0, \mu) = \frac{\omega}{2}\frac{\mu}{\mu + \mu_0} H(\omega, \mu_0) H(\omega, \mu) \quad (3)$$

one obtains, from equation (1), the following integral equation for $H(\omega, \mu)$

$$H(\omega, \mu) = 1 + \frac{\omega}{2}\mu H(\omega, \mu) \int \frac{H(\omega, \mu')}{\mu + \mu'}\,\mathrm{d}\mu'. \quad (4)$$

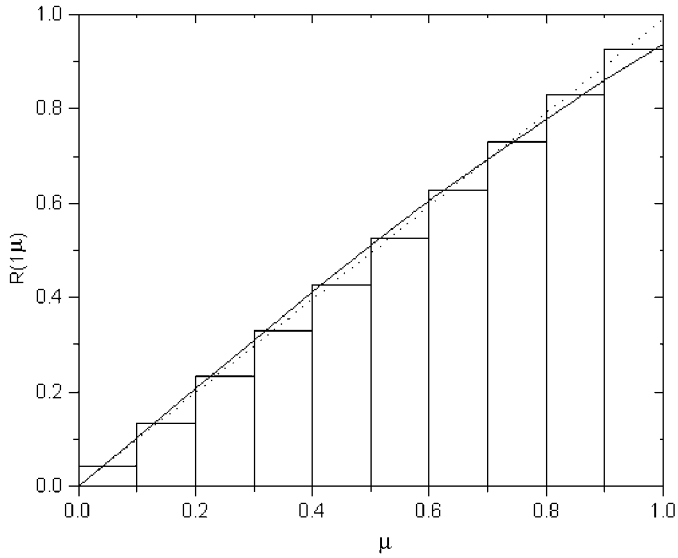

Figure 2. Angular distribution of backscattered photons at normal incidence for photon energy $E = 60$ keV. (———) Results from isotropic model and (······) results from cosine distribution. The histogram is from the Monte Carlo simulation.

The solution of the integral equation (4) is the well-known H-function (Chandrasekhar 1960). The H-function is tabulated for various values of the variable μ and the parameter ω (Chandrasekhar 1960, Shultis and Faw 1996). Different analytic approximations of the H-functions are also available (Simović and Vukanić 1999).

In this way the final solution for the angular distribution of reflected photons in an analytic form, expressed through the H-function, is given by equation (3).

The total reflection coefficient, i.e. the total probability of backscattering for an incident photon, follows by integrating the angular distribution from equation (3) over all exit directions and is given by

$$R_N(\mu_0) = 1 - \sqrt{1-\omega}H(\mu_0, \omega). \tag{5}$$

For large values of the parameter ω, the complicated analytic expression (3) can be, using the features of the H-function, represented with good accuracy as a cosine distribution

$$R(\mu_0, \mu) \approx 2R_N(\mu_0)\mu. \tag{6}$$

An analogous conclusion holds for the angular distribution of low-energy ions reflected from solids (Vukanić and Simović 1996, Simović and Vukanić 1997).

4. Comparison of the results from the present model and the Monte Carlo simulation

We have compared the model results obtained from equation (3) with the results of Monte Carlo simulation for water as a scatterer. We have chosen for discussion two extreme points of our energy interval: 60 keV and 10 keV. In figure 2 the angular distribution of reflected photons is presented for the photon energy of 60 keV. It is seen from the figure that analytical results from equation (3) and the results of the Monte Carlo simulation agree within a few per cent. It is also evident that the cosine distribution approximates both the analytic theory and the Monte Carlo simulation with similar accuracy.

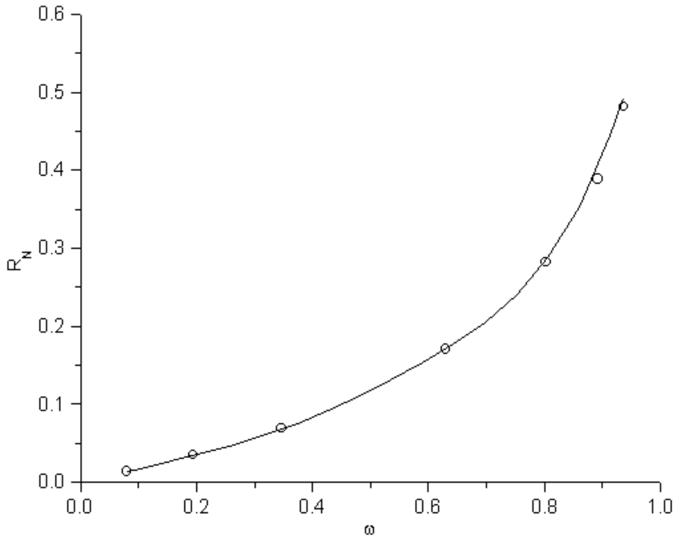

Figure 3. Total reflection coefficient for normal photon incidence as a function of the parameter ω. Comparison of the Monte Carlo results (circles) with analytic theory (full line).

The total reflection coefficients for photon backscattering from water, obtained analytically from equation (5) and by the Monte Carlo simulation, are shown in figure 3. The energy interval in which the calculations were performed was from 10 to 60 keV. As in this energy range the absorption coefficient for water varies greatly (Fitzgerald *et al* 1967), so does the parameter ω. In figure 3 the reflection coefficient is presented as a function of ω so that these results, although obtained for a water target, are universal. It is visible that the analytical and simulation results agree again within a few per cent. This is a surprisingly good agreement. The agreement shows that the simplifying assumptions of the analytical model leading to equation (3) are justified. The nature of the result in equation (3) is essentially of statistical character so that it includes the possibility of any number of collisions—theoretically even infinite. Practically, the average number of scattering events before reflection depends on the parameter ω whose values, as seen from equation (2), lie in the interval $0 \leqslant \omega \leqslant 1$. For high values of the parameter ω ($\omega \approx 1$), the absorption is small, and the average number of collision events which happen in the target material before reflection is high. This high number of collision events leads to isotropization of the distribution of photons inside the target even when the cross section of individual events is not very close to the isotropic case. This fact is already noticed for the case of ion transport (Vukanić and Simović 1996, Simović and Vukanić 1997).

If ω is small compared to unity ($\omega \ll 1$), which is the case in the lower end of our energy range (10 keV incident photon energy), the probability of absorption in water is high, the probability of scattering is small compared with it, so that the reflection is essentially determined by single backscattering events and depends now on the details of the differential cross section that are not included in our model. This explains the discrepancy between analytical results and the Monte Carlo histogram for the angular distribution of reflected photons, visible in figure 4. The full line is the analytical theory. The histogram is the MC simulation. The dotted line in figure 4 is obtained by replacing in equation (3) the isotropic cross section with the Klein–Nishina cross section only for single scattering events, whereupon the agreement is greatly improved. This agreement further substantiates the claim

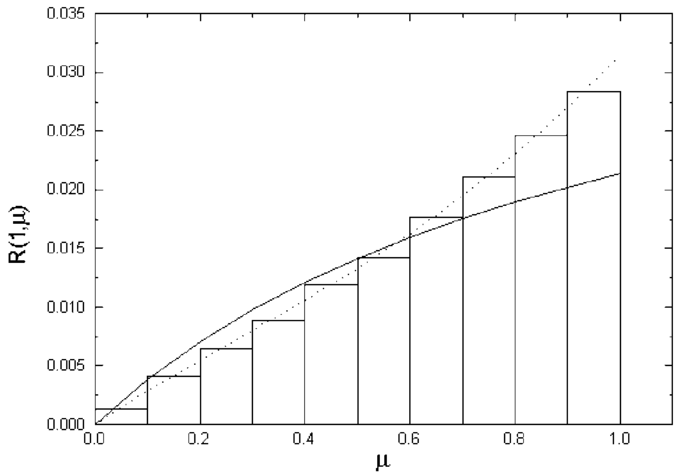

Figure 4. Angular distribution of backscattered photons for photon incident energy E = 10 keV. (———) Results from isotropic model. (······) Results with the single backscattering anisotropy included. Histogram from the Monte Carlo simulation.

that single backscatter events are the predominant mode of scatter at lower energies. The relative contribution of multiple backscattering events increases with increasing ω, and in the limiting case, $\omega = 1$ (scattering without absorption), the contribution of single collisions in photon reflection is only about 15%.

Altogether, our results show that for energies 10 to 60 keV often used in x-ray diagnostics, results obtained from the exactly solvable analytical model agree within several per cent with the Monte Carlo simulation data. (Near the lower limit of our energy range, such agreement is achieved by including the analytic correction which takes into account the anisotropy of the cross section in single backscattering events.) Therefore, the results of the model may be considered as reliable and sufficiently accurate in this energy range. Moreover, the angular distribution expressed through the H-function may be with a good accuracy represented by a much simpler cosine distribution, for energies greater than 30 keV.

5. Direction of the highest irradiation by x-rays reflected from a patient

Radiation reflected from a patient during x-ray diagnostics is usually the main source of irradiation for the radiologist. We consider a situation of backscattered radiation from a patient schematically shown in figure 4. In this figure the head of the radiologist is represented by a sphere of radius r_0. S denotes the irradiated surface of the patient. The position of the head is defined by the horizontal distance x and the height z. Now we have for the backscattered radiation intensity reaching the head of the radiologist ΔI

$$\Delta I(\vec{\Omega}) = I_0 S \cdot R(1, \mu)\Delta\Omega/2\pi. \tag{7}$$

Here I_0 is the initial radiation intensity and $\Delta\Omega = \pi r_0^2 / r^2$ is the elementary solid angle corresponding to the radiologist's head. The factor $1/(2\pi)$ appears due to the azimuthal symmetry of the backscattered radiation. With the help of equations (6) and (5), taking into account the geometric relations represented in figure 5, one obtains

$$\Delta I = I_0 r_0^2 S(1 - \sqrt{1-\omega} H(1, \omega))z/(x^2 + z^2)^{3/2}. \tag{8}$$

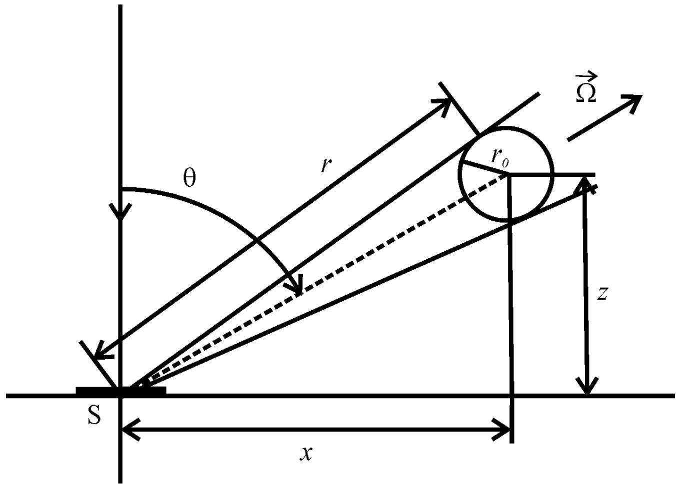

Figure 5. The scheme of the scattering geometry.

As in our notation, $\cos\theta = \mu$, the last equation may be transformed to the form

$$\Delta I = K\mu(1-\mu^2) \tag{9}$$

where the constant K is easily identified from equation (8). This function is obviously peaked and has a maximum for $\mu = 1/\sqrt{3}$ which gives the angle of the highest radiation $\theta_{\max} \approx 54^\circ\,44'$. This direction defined by $\theta_{\max} \approx 55^\circ$ should be avoided by a medical team. On a given radiological site, the constant K may be determined by a single measurement, and then equation (9) determines completely the spatial distribution of reflected radiation.

6. Concluding remarks

The main new insights and results of the present work are the following:

1. Reflection of x-rays from water in the energy range from 10 to 60 keV can be with sufficient accuracy described as isotropic particle scattering without energy loss.
2. The derivation of the main transport equation describing the process of reflection (equation (3)) is given in a very simple and physically transparent way.
3. It is shown that in the energy range from 30 to 60 keV, the angular distribution of the reflected photons can be, with a sufficient accuracy, approximated with a cosine distribution.
4. The results of the Monte Carlo simulation agree within several per cent with those of the exactly solvable model.
5. The cosine angular distribution of photons backscattered from a patient made possible the determination of the direction of the maximum of potential irradiation of the radiologist's head.

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