Path Integral Measure via Schwinger-Dyson Equations

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Abstract

We present a way for calculating the Lagrangian path integral measure directly from the Hamiltonian Schwinger-Dyson equations. The method agrees with the usual way of deriving the measure, however it may be applied to all theories, even when the corresponding momentum integration is not Gaussian. For a general theory, the measure is given in terms of one quadrature. The method works even for the case of unstable theories.

1 Introduction

Schwinger-Dyson equations lie at the heart of the functional formalism of quantum theory, encoding the basic linearity of probability amplitudes. In their simplest form the Schwinger-Dyson equations are given in terms of the generating functional $Z[J]$. We have

$$\left(\frac{\delta S}{\delta q} \Big|_{q=\frac{\hbar}{i}\frac{\delta}{\delta J}} + J \right) Z[J] = 0 \ , \tag{1}$$

where $S[q]$ generates the Feynman rules: S'' is the inverse of the Feynman propagator, $S^{(n)}$ are the n-point vertices. Equation (1) is a linear (functional) differential equation for $Z[J]$. Note that the Fourier transform of this is just the Feynman path integral

$$Z[J] = \int [dq] \, \exp \frac{i}{\hbar} \left(S[q] + \int dt \, Jq \right) \ . \tag{2}$$

The $\hbar \to 0$ limit of $Z[J]$ is dominated by configurations near to the solutions of $\frac{\delta S}{\delta q} + J = 0$. On the other hand, $\hbar \to 0$ corresponds to classical physics given by $\frac{\delta I}{\delta q} + J = 0$, where I is the action. From this we see that

$$S[q] = I[q] + \frac{\hbar}{i} \, M[q] \ . \tag{3}$$

The additional term $M[q]$ is not determined in the functional formalism. It is common to write the integrand of (2) in terms of the action. The path integral measure is then $d\mu = [dq] \exp M[q]$. Lack of knowledge about M translates into lack of knowledge about the measure $d\mu$. So far, the only way to determine the measure has been to make contact with the operator formalism. From it we find the standard expression for the generating functional in terms of a Hamiltonian path integral

$$Z[J] = \int [dp \, dq] \exp \frac{i}{\hbar} \int dt \left(p\dot{q} - H(q,p) + Jq \right) \ . \tag{4}$$

Here the measure is trivial. The usual way of obtaining the corresponding Lagrangian expression is to integrate out all of the momentums.

In this letter we will demonstrate an alternate way for calculating the measure *inside* the functional formalism. To do this we shall use the Hamiltonian form of the Schwinger-Dyson equations. Using this we shall determine a differential equation that is satisfied by the measure, and solve it for various instructive models.

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2 Determining the Measure

The generating functional written as a Hamiltonian path integral is given by

$$Z[J,K] = \int [dp\,dq] \, \exp \frac{i}{\hbar} \int dt \, \left(p\dot{q} - H(q,p) + Jq + Kp\right) \, . \tag{5}$$

For later convenience we have added a source term for the momenta. The Schwinger-Dyson equations are easily derived from the identities

$$\int [dp\,dq] \, \frac{\delta}{\delta q} \, \exp \frac{i}{\hbar} \int dt \, \left(p\dot{q} - H(q,p) + Jq + Kp\right) = 0 \tag{6}$$

$$\int [dp\,dq] \, \frac{\delta}{\delta p} \, \exp \frac{i}{\hbar} \int dt \, \left(p\dot{q} - H(q,p) + Jq + Kp\right) = 0 \, . \tag{7}$$

This gives us

$$\left(\dot{P} + \frac{\partial H(Q,P)}{\partial Q} - J\right) Z[J,K] = 0 \tag{8}$$

$$\left(\dot{Q} - \frac{\partial H(Q,P)}{\partial P} + K\right) Z[J,K] = 0 \, , \tag{9}$$

where we have introduced $P = \frac{\hbar}{i}\frac{\delta}{\delta K}$, and $Q = \frac{\hbar}{i}\frac{\delta}{\delta J}$. The above Schwinger-Dyson equations look just like the classical Hamiltonian equations of motion. The only difference is that we have the following non-zero commutators

$$[P,K] = [Q,J] = \frac{\hbar}{i} \, . \tag{10}$$

Note that in this formalism P and Q commute. We will now use the above equations to derive the Lagrangian path integral measure. As an example let us look at a model whose Hamiltonian is simply

$$H(q,p) = \frac{1}{2}\, p^2 + V(q) \, . \tag{11}$$

In this case the Schwinger-Dyson equations read

$$\left(\dot{P} + V'(Q) - J\right) Z[J,K] = 0 \tag{12}$$

$$\left(\dot{Q} - P + K\right) Z[J,K] = 0 \, . \tag{13}$$

Differentiating the second of these equations with respect to time, and then adding it to the first, we get an equation for Q alone

$$\left(\ddot{Q} - V'(Q) - J\right) Z[J] = 0 \, , \tag{14}$$

where we have now turned off the source for momenta. In terms of the action $I[q] = \int dt \left(\frac{1}{2}\,\dot{q}^2 - V(q)\right)$, the Schwinger-Dyson equations are simply

$$\left(\frac{\delta I}{\delta Q} + J\right) Z[J] = 0 \, . \tag{15}$$

Fourier transforming this we obtain the usual Lagrangian path integral

$$Z[J] = \int [dq] \, \exp \frac{i}{\hbar} \int dt \, \left(\frac{1}{2}\,\dot{q}^2 - V(q) + Jq\right) \, . \tag{16}$$

We have just derived the well known result that the path integral measure is trivial for models whose Hamiltonian is of the simple form given in eq. (11).

Now let us look at a bit more complicated example. We consider a model with Hamiltonian given by

$$H(q,p) = \frac{1}{2}\, g^{-1}(q)p^2 + V(q) \ . \tag{17}$$

The Hamiltonian Schwinger-Dyson equations are now

$$\begin{aligned} \left(\dot{P} - \frac{1}{2}g^{-2}(Q)g'(Q)P^2 + V'(Q) - J\right) Z[J,K] &= 0 && (18)\\ \left(\dot{Q} - g^{-1}(Q)P + K\right) Z[J,K] &= 0 \ . && (19) \end{aligned}$$

We may write the eq. (19) as $PZ = g\,(\dot{Q} + K)Z$. From this we obtain

$$\begin{aligned} P^2 Z &= P(g\dot{Q} + gK)Z = \\ &= (g\dot{Q} + gK)PZ + [P,K]gZ = \left((g\dot{Q} + gK)^2 + \frac{\hbar}{i}\, g\right) Z \ , \end{aligned} \tag{20}$$

as well as

$$\dot{P}Z = \big(g'\dot{Q}(\dot{Q} + K) + g(\ddot{Q} + \dot{K})\big) Z \ . \tag{21}$$

Using this we can get rid of the P terms in eq. (18). Finally, setting $K = 0$ we find

$$\big(g\ddot{Q} + \frac{1}{2}g'\dot{Q}^2 + V' - \frac{1}{2}\,\frac{\hbar}{i}\,(\ln g)' - J\big) Z[J] = 0 \ . \tag{22}$$

This equation can again be written as $\left(\frac{\delta S}{\delta Q} + J\right) Z[J] = 0$, where we have $S = I + \frac{\hbar}{i}M$. The first term is just the action $I[q] = \int dt\,\big(\frac{1}{2}g(q)\dot{q}^2 - V(q)\big)$. The measure term equals $M = \int dt\, \ln \sqrt{g}$. Fourier transforming the last equation we find

$$Z[J] = \int \prod_t \left(dq(t)\sqrt{g(q)}\right) \exp \frac{i}{\hbar}\left(I + \int dt\, Jq\right) \ . \tag{23}$$

This agrees with the standard derivation of the Lagrangian path integral in which one performs the Gaussian momentum integration in the Hamiltonian path integral.

The generalization of the previous example to more variables gives us the σ-model

$$L = \frac{1}{2}\, g_{\alpha\beta}(q)\dot{q}^\alpha \dot{q}^\beta \ . \tag{24}$$

The Hamiltonian is given in terms of the inverse metric $g^{\alpha\beta}$, and equals $H = \frac{1}{2}\, g^{\alpha\beta} p_\alpha p_\beta$. The Schwinger-Dyson equations become

$$\begin{aligned} \left(\dot{P}_\alpha + \frac{1}{2}g^{\gamma\delta}_{\ \ ,\alpha}P_\gamma P_\delta - J_\alpha\right) Z[J,K] &= 0 && (25)\\ \left(\dot{Q}^\alpha - g^{\alpha\beta}P_\beta + K^\alpha\right) Z[J,K] &= 0 \ . && (26) \end{aligned}$$

Just as in the previous example it is a simple exercise to get rid of the P terms and derive the Lagrangian Schwinger-Dyson equation. It may be compactly written as

$$\left(\frac{\delta I}{\delta Q^\alpha} - i\hbar\,\frac{1}{\sqrt{g}}\,\partial_\alpha \sqrt{g} + J_\alpha\right) Z[J] = 0 \ , \tag{27}$$

where $g = \det\, g_{\alpha\beta}$. The corresponding path integral has the familiar form

$$Z[J] = \int \prod_t \left(dq(t)\sqrt{g(q)}\right) \exp\left(\frac{i}{\hbar}\big(I + \int dt\, J_\alpha q^\alpha\big)\right) \ . \tag{28}$$

From these examples it is obvious that the generalization from 1-dimensional field theory, i.e. quantum mechanics, to d-dimensional field theory is trivial. The d-dimensional expressions just contain more dummy labels.

3 Unstable Theories

The above prescription for deriving the measure is always applicable. In this section we will apply it to a simple model whose Hamiltonian is not quadratic in p. Take

$$H = \frac{1}{3}\,p^3 \ . \tag{29}$$

Obviously the energy will not be bounded from bellow, i.e. we are dealing with an unstable theory. Unstable theories can be found in many important areas of physics, for example in the treatments of vacuum polarization [1, 2, 3], critical droplets [4], instantons [5] and matrix models [6, 7], to name a few.

At this moment we do not wish to focus on questions of stability, but rather simply to apply our method for the derivation of the measure to the model given in eq. (29). At the end of this section we will see that the Schwinger-Dyson equations will in fact be able to make a sensible theory out of this. The Hamiltonian Schwinger-Dyson equations are now

$$\begin{aligned} (\dot{P} - J)Z &= 0 & (30)\\ (\dot{Q} - P^2 + K)Z &= 0 \ . & (31) \end{aligned}$$

The second of these equations may be written as $P^2 Z = (\dot{Q} + K)Z$. Now we are faced with a problem. In order to get rid of the P dependence of the first Schwinger-Dyson equation we need to know how P acts on the generating functional. Instead of this we are given how P^2 acts on Z. If P and K commuted then the answer would be simply $PZ = \sqrt{\dot{Q} + K}\ Z$. In fact, as we shall see, this indeed holds when we take $\hbar \to 0$. From its definition we have $P = \frac{\hbar}{i}\,\frac{\delta}{\delta K}$, so that what we have in eq. (31) is actually

$$\frac{\delta^2}{\delta K^2} Z[J,K] = -\frac{1}{\hbar^2}\,(K + \dot{Q}) Z[J,K] \ . \tag{32}$$

Let us note that this is an ordinary differential equation: J is a label, and $\dot{Q}$ is just a constant as far as K differentiation is concerned. We have

$$\frac{d^2}{dK^2} Z = -\frac{1}{\hbar^2}\,(K + \dot{Q}) Z \ . \tag{33}$$

We don't realy need to solve this — all we need is to find PZ. Because of this we impose

$$\frac{dZ}{dK} = \frac{i}{\hbar}\,F(K) Z \ . \tag{34}$$

Differentiating this and using eq. (33) we find that F satisfies the Riccati equation

$$-i\hbar \frac{dF}{dK} + F^2 = K + \dot{Q} \ . \tag{35}$$

There are two general ways for dealing with Riccati equations. The first is to write $F \propto \frac{W'}{W}$ and choose the constant of proportionality in such a way that W obeys a linear differential equation of second order. This is however just our starting equation (33), so this doesn't help us. The second way to solve Riccati equations leads to the general solution when any particular solution is known. Again this is of no use since we know no obvious particular solution of eq. (35). Equation (35), however, does have a natural small parameter in it, and we can find perturbative solutions, i.e. solutions written in terms of a power series in $\hbar$. Writting $F = F_0 + \hbar F_1 + \hbar^2 F_2 + \ldots$ we find

$$\begin{aligned} F_0 &= \sqrt{K + \dot{Q}} & (36)\\ F_1 &= \frac{i}{4}\,(K + \dot{Q})^{-1} & (37)\\ F_2 &= \frac{5}{32}\,(K + \dot{Q})^{-5/2} \ , & (38) \end{aligned}$$

and so on. This gives us

$$PZ = \left((K+\dot{Q})^{1/2} + \frac{i\hbar}{4}\,(K+\dot{Q})^{-1} + \frac{5\hbar^2}{32}\,(K+\dot{Q})^{-5/2} + \ldots \right) Z \ . \tag{39}$$

Differentiating, substituting into the first Schwinger-Dyson equation and setting $K = 0$ we get

$$\left(\frac{1}{2}\dot{Q}^{-1/2}\ddot{Q} - \frac{i}{4}\hbar\dot{Q}^{-2}\ddot{Q} - \frac{25}{64}\hbar^2\dot{Q}^{-7/2}\ddot{Q} + \ldots - J \right) Z[J] = 0 \ . \tag{40}$$

This is again of the form $\left(\frac{\delta S}{\delta Q} + J\right) Z[J] = 0$, where $S = I + \frac{\hbar}{i}\,M$. The measure term is now

$$M = -\frac{1}{4}\int dt\ \ln\dot{q} - \frac{5i}{48}\hbar\int dt\,\dot{q}^{-3/2} + \ldots \tag{41}$$

Perturbative solutions like this are nice, if there is nothing better arround. However, for this model, we know that the standard treatment of the theory does not work since H is not bounded from bellow. Let us therefore look at equation (33) again. If we introduce $x = -\hbar^{-2/3}(K+\dot{Q})$ then the equation simplifies to

$$\frac{d^2 Z}{dx^2} = x\,Z \ . \tag{42}$$

This is Airy's differential equation — the *Escherichia coli* in the field of asyptotic expansions [10, 11, 12]. The general solution for $x \in \mathbb{C}$ can be written as the Airy integral

$$f(x) = \frac{1}{2\pi i}\int_C dt\ \exp\left(tx - \frac{1}{3}t^3\right) \ . \tag{43}$$

For $f(x)$ to converge, the integrand must vanish at the end-points. The contour can't be closed because the integral of an analytic function over such a contour vanishes. We thus have three topologicaly distinct contours availible corresponding to the end points at infinity with phases $-\frac{2\pi}{3}$, $\frac{2\pi}{3}$, and 0. If we label these points as x_1, x_2, x_3 then contour C_{ij} goes from x_i to x_j. In addition we also have $C_{12} + C_{23} + C_{31} = 0$,

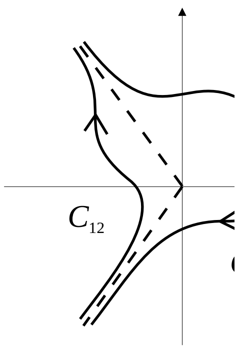

Figure 1: Contours of integration for Airy functions

so that this gives us just two independent solution. This is as it should be since the Airy equation is of second order. The standard choices are the two real independent solutions

$$\mathrm{Ai}(x) = f_{12}(x) \tag{44}$$
$$\mathrm{Bi}(x) = i\big(f_{23}(x) - f_{31}(x)\big) \ . \tag{45}$$

f_{ij} corresponds to the function f given by contour C_{ij}. Note that eq. (42) is the Schwinger-Dyson equation for the 0-dimensional path integral of a theory with Euclidian action $I = \frac{1}{3}t^3 - tx$. The distinction between Euclidian and Minkowski theories is trivial in 0-dimensions: By writing $s = it$ we get $f(x) = \frac{1}{2\pi}\int_C ds\, e^{i(\frac{1}{3}s^3+sx)}$, which is the corresponding Minkowski expression.

Airy functions can be asymptotically expanded by using the steepest descent method. The saddle points ($I' = 0$) are at $t = \pm\sqrt{x}$. Paths of steepest descent are given by $Im(I) =$ const. If we write $t = u + iv$, and look at x real and positive, then the paths of steepest descent passing through the saddle points are $v = 0$ and $v^2 = 3u^2 - 3x$. We have $I''(\pm\sqrt{x}) = \pm 2\sqrt{x}$, so that the left saddle point contributes when going through it along $v^2 = 3u^2 - 3x$, while the right saddle point contributes when we pass through it along $v = 0$. We thus get the asymptotic formulas

$$\mathrm{Ai}(x) \quad \sim \quad \frac{1}{2\sqrt{\pi}}\, x^{-1/4}\, \exp -\frac{2}{3}x^{3/2} \tag{46}$$

$$\mathrm{Bi}(x) \quad \sim \quad \frac{1}{\sqrt{\pi}}\, x^{-1/4}\, \exp \frac{2}{3}x^{3/2}\ . \tag{47}$$

If we choose $Z = \mathrm{Bi}(x)$ then $Z \propto (K + \dot{Q})^{-1/4} e^{\frac{2}{3}(-)^{3/2}\frac{1}{\hbar}(K+\dot{Q})^{3/2}}$. We have $\arg x \in [-\pi, \pi)$, so that $(-)^{3/2} = i$. Therefore, we find

$$\frac{i}{Z}\frac{dZ}{dK} = -\frac{1}{4}(K + \dot{Q})^{-1} + \frac{i}{\hbar}(K + \dot{Q})^{1/2}\ , \tag{48}$$

hence

$$PZ = \left((K + \dot{Q})^{1/2} + \frac{1}{4}i\hbar(K + \dot{Q})^{-1}\right) Z\ . \tag{49}$$

This is in agreement with our perturbative result. The choice of the $\mathrm{Ai}(x)$ solution gives us a similar result, but with a wrong sign in front of the classical part of eq. (49). On the other hand, if we choose the solution $Z = \mathrm{Ai}(x) + \frac{1}{2}\mathrm{Bi}(x)$ then we find

$$PZ = \left((K + \dot{Q})^{1/2}\tanh\left(\frac{2}{3}\frac{i}{\hbar}(K + \dot{Q})^{3/2}\right) + \frac{1}{4}i\hbar(K + \dot{Q})^{-1}\right) Z\ , \tag{50}$$

which doesn't look at all like our perturbative solution. The above solution differs from the perturbative one by pieces that are smaller than any power of $\hbar$.

We have seen that the naive expansion in $\hbar$ automatically picks out one solution of eq. (33). The correct procedure is thus to solve eq. (42). To this we need to add additional physical input that tells us which initial conditions to choose (or in the language of the path integral which contour to choose). This multitude of solutions to the Schwinger-Dyson equations is *always* present. For a theory whose action is for example $I = \int dx\, \left(\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{n!}g\phi^n\right)$, the Schwinger-Dyson equation is a linear (functional) differential equation of $(n-1)^{\mathrm{st}}$ order. There are thus $n - 1$ independent solutions. We naively solve the Schwinger-Dyson equation by a (functional) Fourier transform. However, in this way we choose a specific contour — the real axis. For n even this is indeed one of the possible solutions. In fact it is the correct one as we know from the operator formalism. For n odd the real axis is not one of the allowed contours and we seem to have a problem. As we have seen in the simple example of Airy functions there in fact is no problem — we just have to be careful in choosing the correct contours. What is the problem in such theories is that the standard operator formalism does not work, so we seem to lack a criterion that will tell us which of the allowed contours to choose.

There is a rather natural way around this obstacle. We propose that the correct contour is the unique one that has the correct semi-classical limit. Said another way — we should choose the contour that has the correct physics up to one loop. Let us see what this means on the example of Airy functions. The naive contour would be the real axis. It is wrong since the path integral doesn't converge. However, one can still formally calculate its asymptotic expansion. What we find is that it only gets a contribution from the right saddle point $t = \sqrt{x}$. The left saddle point doesn't contribute because in going along the real axis it represents a maximum of the action, not a minimum. Now let us look at the true solutions.

Ai(x) only sees the left saddle point. In the direction of its contour this saddle point is a minimum of the action, so everything is OK, however, this doesn't agree with the imposed semi-classical results. On the other hand Bi(x) only sees the right saddle point. The contributions from the left saddle point cancel for the two contours C_{23} and $-C_{31}$. Therefore, Bi(x) has precisely the correct semi-classical behaviour. It is easy to see that it is the unique such solution of the Airy differential equation.

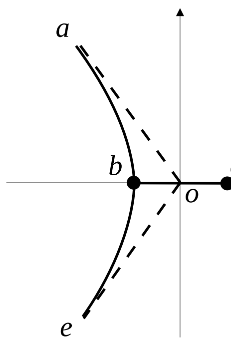

Figure 2: Semi-classical physics picks out the unique contour. In this case the correct contour is $1/2\big((abcd) + (ebcd)\big)$. This is the appropriate contour of steepest descent. By Cauchy's theorem, the same answer follows from using $1/2\big((aocd) + (eocd)\big)$.

We have calculated the measure for our model using the Schwinger-Dyson equations. The way the measure is usually calculated is by performing the momentum integration in the Hamiltonian path integral. Thus, what we have in fact solved is

$$\int [dp] \exp \frac{i}{\hbar} \int dt \left(p\dot{q} - \frac{1}{3}p^3 \right) \ . \tag{51}$$

As we have seen the solution was given in terms of the Airy differential equation. This is not surprising. The Airy integral is simply the 0-dimensional version of eq. (51). In fact, the relation is stronger since eq. (51) is an integration over p of an expression that doesn't contain derivatives. Therefore

$$\int [dp] \exp \frac{i}{\hbar} \int dt \left(p\dot{q} - \frac{1}{3}p^3 \right) = \prod_t f\big(-\dot{q}(t) \big) \ . \tag{52}$$

Now we come to an important point — the choice of contour of the 0-dimensional integral completely determines the path integral (51). We therefore need to use the Bi(x) Airy function in eq. (52). We have now determined the measure term exactly. We have

$$M[q] = \int dt \, \ln \mathrm{Bi}(-\dot{q}) \ . \tag{53}$$

The seemingly simple model that we have considered in this section has in fact taught us how to determine the path integral measure of a general model. The measure is given in terms of an single definite integral — the associated 0-dimensional path integral. In the above case it was the Airy function Bi(x). The Lagrangian path integral measure of a general model is thus given in quadratures. As we have seen, physics at one loop uniquely picks out the integration contour, i.e. *uniquely* determines the measure.

4 Conclusion

Schwinger-Dyson equations offer us a new way to calculate the measure for the Lagrangian path integral. Using them we have determined the measure of a general theory in terms of one quadrature. The approach also works for unstable theories: Euclidean theories whose action is not bounded from bellow, or conversely Minkowski theories whose energy is not bounded from bellow. A prototypical unstable theory is Einstein gravity in Euclidean space. Several authors have looked at unstable field theories [1]-[9], and found that the answer is an analytic extension of the path integral, where one deforms the contour of integration of the path integral. The choice of contour was dictated by the specific model. For example, in [6] David determined the contour for his matrix model approach to strings from the requirement that his non-perturbative results match the well-known perturbative string results. We meet unstable theories all the time — for example in instanton calculations when we do Gaussian integrations over negative modes. Again, the way to get the correct tunneling results is to rotate the contours of integration for the corresponding modes. The nice thing about the Schwinger-Dyson approach to the measure is that it uniquely picks out which contour one should use.

All of our examples concerned quantum mechanical systems, but the generalization to field theory in more then one dimension is trivial. What is not trivial, when one tackles full-fledged field theory, is how to deal with gauge symetries and anomalies. Therefore, it will be interesting to extend this work to the treatment of gauge theories, and re-derive the measures obtained by Faddeev-Popov and Batalin-Vilkovisky. Another interesting thing to try is to find a differential equation satisfied by the measure term. The fact that we now have the general measure written in quadrature should make it possible to determine this equation solely in terms of the Lagrangian. Doing this would enable us to complete what Dirac and Feynman started: to define a complete quantum theory in terms of the Lagrangian.

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