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Edge theory of ferromagnetic quantum Hall states

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We propose an effective low-energy theory for ferromagnetic Hall states. It describes the charge degrees of freedom, on the edge, by a (1 + 1) dimensional chiral boson theory, and the spin degrees of freedom by the (2 + 1) dimensional quantum ferromagnet theory in the spin-wave approximation. The usual chiral boson theory for spinless electrons is modified to include the charge degrees of freedom with spin. Our total, bulk plus edge, effective action is gauge invariant and we find a generalized ''chiral anomaly'' in this case. We describe two, charged and neutral, sets of edge spin-wave solutions. The spreading of these waves is much larger than the one for the charge (edge) waves and they have linear dispersion relations. [S0163-1829(98)02416-3]

I. INTRODUCTION

The bulk properties of the quantum Hall systems at filling fraction $1/m$, $m=$odd, in the presence of low magnetic fields, have been subject of many theoretical and experimental investigations in recent years. The spin degree of freedom plays an important role in these systems. Here we focus on properties of the boundary of these systems, which, in a special way, reflect bulk properties. In the spinless case this reflection was already described by Wen.[1] The low-energy (bulk) physics of these systems is identical to that of 2D quantum ferromagnets with spin waves as excitations. Due to exchange the spins of electrons in the ground state are all aligned in the same direction and the lowest-lying excitations are one-spin-flip (spin wave) excitations which leave the charge of the system unchanged. The lowest-lying charged excitations are topologically nontrivial skyrmion excitations[2] for which a local change in the charge density that characterizes them, is accompanied by a local change in the spin density. This scenario, in which a finite number of overturned spins follows the creation of the charged excitations, is supported by experimental findings.[3]

On the other hand, the physics of the boundary of quantum Hall systems without the spin degree of freedom is well understood.[1] In fact for any quantum Hall system, including the one which edge physics we would like to understand, it is expected that the charge dynamics is reduced to the edge of the system. This is maybe an oversimplification with respect to a general experimental situation where sharp edges (i.e., edges with steep confining potential) are not always present. Still, the effective one-space-dimensional edge theory–chiral boson theory of quantum Hall systems has received considerable experimental support in recent years.

Here we present a low-energy effective theory of quantum Hall ferromagnetic systems that describes the charge degrees of freedom restricted to live only on the edge and lowest-lying excitations of the bulk-neutral spin waves. We show that, under special conditions (and as solutions of the theory), edge spin waves exist, the characteristic width of which is smaller than that of bulk spin waves, which spread throughout the whole system. These excitations, edge spin waves, are characterized by gaps that are smaller than the Zeeman gap and linear dispersion relations. We find two classes of these waves, which we call, charged and neutral edge spin waves. One way to induce the charged edge waves is to subtract or add some charge to the edge. By a redistribution of the charge and, simultaneously, spin of the system on the edge (in the manner of spin textures as described first in Ref. 2) neutral edge spin waves are possible.

II. EFFECTIVE LOW-ENERGY FIELD THEORY WITH CHARGE DEGREES OF FREEDOM ON THE EDGE AND SPIN WAVES

In this section, we will first rederive the edge theory for spinless electrons[1] using the dual form of the Chern-Simons field-theory description of quantum Hall systems (at filling fractions $1/m$ where m is an odd integer. Then we will use the dual form of the Chern-Simons formulation of the ferromagnetic quantum Hall systems (at the same filling fractions with the spin degree of freedom taken into account) to derive a low-energy effective theory that describes not only the edge of these systems, but also the lowest-lying excitations in their bulk-spin waves. At the end we will demonstrate the gauge invariance of our total, bulk plus edge, action when perturbing external electromagnetic fields are present in it.

A. The edge theory of spinless quantum Hall systems

The systems that we consider are at filling fractions $1/m$ where m is an odd integer. We start with the dual formulation of the Chern-Simons effective description of these systems. In the Chern-Simons formulation the problem of the 2D electron system in magnetic field is mapped to a problem of Bose liquid with a long-range interaction described by a statistical gauge field. The vortex excitations of the Bose fluid correspond to the quasiparticle excitations of quantum Hall systems. In the dual form the vortex excitations (i.e., fluxes of the gauge field in the preceding formulation) are viewed as particles, and the charge-current density becomes the flux of some new gauge field. The first terms of the Lagrangian density in the dual form are

$$\mathcal{L}=-\frac{m}{4\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda+\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}A_\mu^{\rm ext}\partial_\nu A_\lambda-\mathcal{J}^v_\mu A^\mu-\frac{\lambda}{4}F_{\mu\nu}F^{\mu\nu}. \qquad (2.1)$$

The vector potential $A_\mu, \mu=0,1,2$ represents the newly introduced gauge field, which enters the definition of the charge current density $J_\mu, \mu=0,1,2$:

$$J_\mu=\epsilon_{\mu\nu\lambda}\frac{1}{2\pi}\partial^\nu A^\lambda. \qquad (2.2)$$

$F_{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu$. The vector potential $A_\mu^{\rm ext}, \mu=0,1,2$ describes the external electromagnetic field, and two first terms when A_μ is ''integrated out'' give the basic Hall response of the system, i.e., the Hall conductance equal to $(1/m)(e^2/h)$. The third term couples the vortex current density $\mathcal{J}^v_\mu, \mu=0,1,2$,

$$\mathcal{J}^v_\mu=\frac{1}{2\pi i}\epsilon_{\mu\nu\lambda}\partial^\nu\partial^\lambda\alpha, \qquad (2.3)$$

to the gauge field A_μ (according to the interchanged roles of fluxes and particles). In Eq. (2.3) α is the phase of the Bose field in the former (nondual) formulation and the vortex current density is nonzero only if α is a nonanalytic function of coordinates, i.e., $\partial^\nu\partial^\lambda\alpha\neq\partial^\lambda\partial^\nu\alpha$ for some λ and ν.

If our system has a boundary the action for a general gauge transformation $A_\mu\rightarrow A_\mu+\partial_\mu\Lambda$ for which Λ is not zero at the boundary is not gauge invariant (i.e., charge conserving). Any electric field along the boundary will produce a current normal to the boundary because of the nonzero value of the Hall conductance. But with restricted gauge transformations for which $\Lambda=0$ on the boundary we may use this action to derive, as was already done in Ref. 1, the kinetic term of the edge theory Lagrangian. These gauge transformations describe a well-defined boundary problem, in which some of the physics of the quantum Hall systems due to perturbing external electromagnetic fields is not present. Namely, by making all previously gauge-dependent quantities on the edge gauge independent, i.e., physical, we are deriving an effective edge theory that gives the right physics when charge exchange between the bulk and edge is absent.[4,5] (We defer description of the total action with perturbing electromagnetic fields, which is explicitly gauge invariant to Sec. II C.)

First we neglect the last term in Eq. (2.1) (as a higher order term in derivatives), regard the equation of motion for A_0 as a constraint, and take $A_0=0$. The constraint is

$$\frac{1}{2\pi}(m\vec{\nabla}\times\vec{A}-\vec{\nabla}\times\vec{A}^{\rm ext})=-\mathcal{J}^v_0, \qquad (2.4)$$

simply saying that any deviation in the charge density of the system is due to the creation of vortices. The solution of this equation, up to a gauge transformation is

$$\delta A_a=A_a-\frac{1}{m}A_a^{\rm ext}=-\frac{1}{m}\partial_a\alpha. \qquad (2.5)$$

We assume that there are no vortices in the bulk of the system so that α is analytic. When we plug in the solution (2.5) in the remaining term of the Lagrangian (we do not consider the term with $A_\mu^{\rm ext}$):

$$\Delta\mathcal{L}_{\rm eff}=-\frac{m}{4\pi}\epsilon_{a0b}A_a\partial_0 A_b \qquad (2.6)$$

we get, up to a total time derivative and with an assumption that $A_\mu^{\rm ext}$ is time independent, that

$$\mathcal{L}_{\rm eff}(x,y,t)=\frac{1}{4\pi m}\epsilon_{ab}\partial_b(\partial_a\alpha\partial_0\alpha). \qquad (2.7)$$

This total divergence can be translated into a surface term:

$$\mathcal{L}_{\rm eff}(x,t)=\frac{1}{4\pi m}\partial_x\alpha\partial_0\alpha, \qquad (2.8)$$

exactly the kinetic term of the chiral boson theory, if we consider the system to be defined in the lower half-plane with $y=0$ as a boundary. We get only the kinetic term of the edge theory because we started from the theory in which we neglected terms that bring dynamics. We might expect that the following term in small momentum expansion on the edge is

$$-\frac{v}{4\pi m}\partial_x\alpha\partial_x\alpha \qquad (2.9)$$

with a nonuniversal coupling v. This term gives dynamics to the edge theory and together with Eq. (2.8) makes the chiral boson theory Lagrangian density. Thus, we can conclude that the field α phase of the bosonic field in the standard Chern-Simons formulation on the boundary plays the role of the chiral boson field.

B. Effective low-energy field theory of ferromagnetic quantum Hall states

The Lagrangian density of the Chern-Simons theory for ferromagnetic quantum Hall states in the dual form is[6,7]

$$\mathcal{L}=-\frac{m}{4\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda+\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}A_\mu^{\rm ext}\partial_\nu A_\lambda-\mathcal{J}_\mu A^\mu-\frac{\rho_s}{2}(\vec{\nabla}\vec{n})^2+\frac{\Delta}{2}n_3-\frac{\lambda}{4}F^{\mu\nu}F_{\mu\nu}. \qquad (2.10)$$

Again the charge current density as a function of the statistical gauge field A^μ is given by the relation (2.2). Now quasiparticle current density consists of two contributions: vortex and skyrmion: $\mathcal{J}=\mathcal{J}^v_\mu+\mathcal{J}^s_\mu$. As before vortex excitations do not change the spin configuration of the ferromagnetic Hall states. Skyrmions, on the other hand, which lie lower in the energy spectrum, represent excitations followed by reversal of electron spins in the system. The vortex current density is given by Eq. (2.3), where again α is the phase of the bosonic field of the standard Chern-Simons formulation, and the skyrmion current density[6,7] is

$$\mathcal{J}^s_\mu=\frac{1}{2\pi i}\epsilon_{\mu\nu\lambda}\partial^\nu\overline{z}\partial^\lambda z. \qquad (2.11)$$

The field z is the two-component spinor of the standard Chern-Simons formulation,[6] in which the bosonic field is decomposed in an amplitude, phase, and spinor part:

$$\Psi_{\mathrm{bosonic}}=\rho\ \exp\{i\alpha\}z. \tag{2.12}$$

The spinor part describes the spin degree of freedom of the bosonic field associated with the original electron field.

The fourth, extra term with respect to the spinless case in the Lagrangian density is the nonlinear σ model term with $\vec{n}=\overline{z}\vec{\tau}z$, where τ_x,τ_y, and τ_z are Pauli matrices. ρ_s is the stiffness constant. This term represents the cost in the exchange energy when the ground-state ferromagnetic configuration is modified.

The fifth term is the Zeeman term with constant $\Delta=(g\mu_B/2\pi)B$ where B is the external magnetic field.

To get the low-energy, effective theory that includes the edge physics, we repeat steps that we described in the previous subsection in the spinless case. The constraint equation in this case, as we vary A_0, is

$$\frac{1}{2\pi}(m\vec{\nabla}\times\vec{A}-\vec{\nabla}\times\vec{A}^{\mathrm{ext}})=-\mathcal{J}^{v}_{0}-\mathcal{J}^{s}_{0}. \tag{2.13}$$

The solution to this constraint is

$$\delta A_a=A_a-\frac{1}{m}A_a^{\mathrm{ext}}=-\frac{1}{m}\partial_a\alpha+\frac{i}{m}\overline{z}\partial_a z. \tag{2.14}$$

Now we assume that α is analytic so that no vortex excitations in the bulk are allowed. We do not make any restrictions on z. By plugging in the solution (2.14) in the beginning Lagrangian (2.10) with $A_0=0$ and without the second and the last term we get surface terms:

$$\mathcal{L}_{\mathrm{eff}}^{\mathrm{kin}}(x,t)=\frac{1}{4\pi m}[\partial_x\alpha\partial_0\alpha-2i\partial_x\alpha(\overline{z}\partial_0 z) +\text{terms without }\alpha]. \tag{2.15}$$

The field α cannot be found anymore in the bulk Lagrangian. Therefore pure charge degrees of freedom, i.e., those that are not followed by a change in the spin configuration are now restricted to live only on the boundary of the system. [This coincides with the microscopic physical picture that we have of these systems. The pure charge (quasihole) excitations, which lie higher in the energy spectrum than skyrmion excitations, in the bulk, can be found in the low-energy approximation on the edge of the system. There, on the edge, their excitation energy is smallest (in fact goes to zero).[8]] In Eq. (2.15) we see the most important result of this derivation, namely a nontrivial coupling between the spin and charge degrees of freedom on the edge of the system.

In the low-energy approximation that we make [i.e., neglecting the last term in Eq. (2.10)] we find that the charge current is equal to the sum of skyrmion and vortex currents. As we do not have vortex currents in the bulk of the system, the condition that the skyrmion current normal to the boundary is zero is equivalent to the demand that the charge current normal to the boundary is zero. In the absence of any external electromagnetic fields besides the uniform magnetic that defines the problem, we will require that no charge can leave the system, i.e., the skyrmion current normal to the boundary is zero. In that case

$$\partial_x(\overline{z}\partial_0 z)=\partial_0(\overline{z}\partial_x z) \tag{2.16}$$

and we may rewrite (after a partial integration) the second surface term in the $\mathcal{L}_{\mathrm{eff}}^{\mathrm{kin}}(x,t)$ as

$$\frac{1}{4\pi m}[-i\partial_x\alpha(\overline{z}\partial_0 z)-i\partial_0\alpha(\overline{z}\partial_x z)], \tag{2.17}$$

Besides, as we did not break the gauge invariance under the gauge transformation defined as

$$\alpha\rightarrow\alpha+\beta\quad\text{and}\quad z\rightarrow\exp\{-i\beta\}z \tag{2.18}$$

and present in the bosonic Chern-Simons formulation with field (2.12), we may also demand the same invariance on the edge of the system. This invariance is an expression of the confinement of spin and charge on electrons, and should exist also on the edge. As a result the surface gauge-invariant kinetic term that contains the field α is

$$\mathcal{L}_{\mathrm{eff}}^{\mathrm{kin}}==\frac{1}{4\pi m}(\partial_x\alpha-i\overline{z}\partial_x z)(\partial_0\alpha-i\overline{z}\partial_0 z). \tag{2.19}$$

We can conclude that, with respect to the spinless case instead of

$$\partial_\mu\alpha\mu=0,x, \tag{2.20}$$

in the case with spin we have

$$\partial_\mu\alpha-i\overline{z}\partial_\mu z\mu=0,x. \tag{2.21}$$

By considering couplings with external fields we may also conclude that these gauge-invariant expressions up to some appropriate constants represent charge density and current on the edge similarly to the spinless case. Then Eq. (2.21) formally expresses the physical fact that the charge current and density on the edge have contributions followed by changes in the spin configuration on the edge. Therefore we expect that under inclusion of a charge-density interaction term (which describes the dynamics on the edge) so that the Lagrangian density is

$$\mathcal{L}_{\mathrm{eff}}=\frac{1}{4\pi m}(\partial_0\alpha-i\overline{z}\partial_0 z)(\partial_x\alpha-i\overline{z}\partial_x z) -\frac{v}{4\pi m}(\partial_x\alpha-i\overline{z}\partial_x z)^2, \tag{2.22}$$

we have a complete low-energy description of the charge degree of freedom on the edge. The equation of motion for α on the edge simply tells us that the charge on the edge drifts with velocity v along the edge in only one direction as it should be in a quantum Hall system.

At this stage of deriving the low-energy, effective theory, the bulk Lagrangian density is

$$\Delta\mathcal{L}_{\mathrm{eff}}(x,y,t)=-\mathcal{J}^{s}_{a}\left(\frac{1}{m}A^{a}_{\mathrm{ext}}+\frac{i}{m}\overline{z}\partial^{a}z\right)-\frac{\rho_s}{2}(\vec{\nabla}\vec{n})^2+\frac{\Delta}{2}n_3, \tag{2.23}$$

where $a=x,y$. Now we will apply the spin-wave approximation in which field z is decomposed in the following way:

$$z=\begin{pmatrix} 1-\frac{1}{2}|\Psi|^2 \\ \Psi \end{pmatrix}. \tag{2.24}$$

The complex bosonic field Ψ expresses fluctuations with respect to the ground-state configuration and is to be considered small and slowly varying, and therefore, we will keep only terms quadratic in Ψ. Then, the $2+1$ dimensional Lagrangian density (2.23) of the bulk becomes

$$\mathcal{L}_{\rm eff}(x,y,t)=i\rho_0\bar{\Psi}\partial_0\Psi-2\rho_s\vec{\nabla}\bar{\Psi}\vec{\nabla}\Psi-\Delta|\Psi|^2. \tag{2.25}$$

Now, we will again assume that our system is in the lower half-plane. After a partial integration inside the expression for the system Lagrangian we get an extra (to the chiral boson Lagrangian) surface term:

$$\Delta\mathcal{L}_{\rm surf}(x,t)=-2\rho_s\bar{\Psi}\partial_y\Psi, \tag{2.26}$$

and the $2+1$ dimensional Lagrangian density of the bulk,

$$\mathcal{L}^{\rm bulk}_{\rm eff}(x,y,t)=i\rho_0\bar{\Psi}\partial_0\Psi+2\rho_s\bar{\Psi}\vec{\nabla}^2\Psi-\Delta|\Psi|^2. \tag{2.27}$$

For a moment we will discuss only the problem of 2D ferromagnet described by the Lagrangian densities (2.26) and (2.27) (in the spin-wave approximation) in order to explain the meaning of the surface term (2.26). In that case, the corresponding Euler-Lagrange equations for the field Ψ is

$$i\rho_0\partial_0\Psi+2\rho_s\vec{\nabla}^2\Psi-\Delta\Psi=0 \tag{2.28}$$

in the bulk, and

$$\partial_y\Psi|_{y=0}=0 \tag{2.29}$$

on the edge. The normal mode solutions of Eq. (2.28) in the bulk are spin waves,

$$\Psi=A\ \exp\{i\vec{k}\vec{r}\}\exp\{-iwt\} \tag{2.30}$$

with the dispersion relation:

$$w=\frac{\Delta}{\rho_0}+\frac{2\rho_s}{\rho_0}k^2. \tag{2.31}$$

To satisfy boundary condition (2.29), the class of solutions is further reduced to the form:

$$\Psi=A\cos\{k_y y\}\exp\{ik_x x\}\exp\{-iwt\}. \tag{2.32}$$

The condition (2.29) ensures that spin currents normal to the boundary are zero. Namely, if we use the Noether expression for them,

$$\mathcal{J}^a_\mu=-\frac{i}{2}\frac{\delta\mathcal{L}_{\rm QFM}}{\delta(\partial^\mu z_i)}\tau^a_{ij}z_j+\text{H.c.}, \tag{2.33}$$

where $\tau^a, a=1,2,3$ are Pauli matrices and $\mathcal{L}_{\rm QFM}$ (a Lagrangian density of a quantum ferromagnet) is

$$\mathcal{L}_{\rm QFM}=i\rho_0\bar{z}\partial_0 z-\frac{\rho_s}{2}(\vec{\nabla}\vec{n})^2+\frac{\Delta}{2}\bar{z}\tau_3 z, \tag{2.34}$$

we may find in the spin-wave approximation the following expressions for them:

$$\mathcal{J}^3_y=i\rho_s(\bar{\Psi}\partial_y\Psi-\partial_y\bar{\Psi}\Psi),$$
$$\mathcal{J}^1_y=i\rho_s(\partial_y\bar{\Psi}-\partial_y\Psi),$$
$$\mathcal{J}^2_y=i\rho_s(\partial_y\bar{\Psi}+\partial_y\Psi). \tag{2.35}$$

We neglected the terms of order higher than two in Ψ; the single condition (2.29) ensures that all of them are zero at the boundary.

The edge spin-wave solutions of the form

$$\Psi=B\ \exp\{\beta y\}\exp\{ik_x x\}\exp\{-iwt\}, \tag{2.36}$$

β positive, satisfy the equation in the bulk, but the condition (2.29) forces β to be zero. Therefore, in the case of a pure quantum ferromagnet, the constraint of the spin conservation on the boundary, does not allow the existence of the edge spin excitations.

Now, we will summarize our effective theory for a quantum Hall system with a boundary, having in mind the specific geometry in which the system is in the lower half of the plane. The bulk Lagrangian density (in the spin-wave approximation) is given by

$$\mathcal{L}_{\rm bulk}=i\rho_0\bar{\Psi}\partial_0\Psi+2\rho_s\bar{\Psi}\vec{\nabla}^2\Psi-\Delta|\Psi|^2 + \text{higher order terms}, \tag{2.37}$$

and the edge Lagrangian density is

$$\mathcal{L}_{\rm edge}(x,t)=-2\rho_s\bar{\Psi}\partial_y\Psi+\frac{1}{4\pi m}(\partial_x\alpha^f\partial_0\alpha^f-v\,\partial_x\alpha^f\partial_x\alpha^f) + \text{higher order terms}, \tag{2.38}$$

where

$$\partial_\mu\alpha^f=\partial_\mu\alpha-\frac{i}{2}(\bar{\Psi}\partial_\mu\Psi-\partial_\mu\bar{\Psi}\Psi). \tag{2.39}$$

C. The proof of gauge invariance of the effective field theory

In the spinless case the effective action of the bulk,

$$\mathcal{L}_{\rm bulk}=-\frac{m}{4\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda+\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}A^{\rm ext}_\mu\partial_\nu A_\lambda \tag{2.40}$$

when the field A_μ is integrated out, is

$$\mathcal{L}^{\rm eff}_{\rm bulk}=\frac{1}{4\pi m}\epsilon^{\mu\nu\lambda}A^{\rm ext}_\mu\partial_\nu A^{\rm ext}_\lambda. \tag{2.41}$$

On the other hand the edge action with external electromagnetic field included is[1,4]

$$\mathcal{L}_{\rm edge}=\frac{1}{4\pi m}[\partial_x\alpha\partial_0\alpha-v(\partial_x\alpha)^2]$$
$$\times\frac{1}{2\pi m}(vA_x^{\rm ext}-A_0^{\rm ext})\partial_x\alpha$$
$$-\frac{1}{4\pi m}(vA_x^{\rm ext}-A_0^{\rm ext})A_x^{\rm ext}. \qquad (2.42)$$

Under the gauge transformation $\alpha\to\alpha+\Lambda$ and $A_\mu^{\rm ext}\to A_\mu^{\rm ext}+\partial_\mu\Lambda$ the total action, $\mathcal{L}_{\rm bulk}+\mathcal{L}_{\rm edge}$ or $\mathcal{L}_{\rm bulk}^{\rm eff}+\mathcal{L}_{\rm edge}$, is invariant. Namely, the chiral anomaly[4] term

$$\frac{1}{4\pi m}\Lambda(\partial_0A_x^{\rm ext}-\partial_xA_0^{\rm ext}), \qquad (2.43)$$

which we get by the gauge transformation of the bulk action is of the same absolute value but of opposite sign as the term that we get gauge transforming the edge action (2.42).

In our previous derivation we assumed that only the vector potential of the constant magnetic field is present in (2.40) and, therefore, the field A^μ did not have any dynamics; by the equation of motion of the action it was constrained to satisfy $m\epsilon^{ab}\partial_aA_b=\epsilon^{ab}\partial_aA_b^{\rm ext}$. Because of the absence of the perturbing external electromagnetic fields the whole dynamics of the system was on the edge described by the chiral boson theory [the first two terms in Eq. (2.42)].

Similarly, in the case with spin, more general effective bulk action with external perturbing electromagnetic fields is

$$\mathcal{L}_{\rm bulk}=-\frac{m}{4\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda+\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}A_\mu^{\rm ext}\partial_\nu A_\lambda-\mathcal{J}^s_{\ \mu}A_\mu$$
$$-\frac{\rho_s}{2}(\vec{\nabla}\vec{n})^2+\frac{\Delta}{2}n_3. \qquad (2.44)$$

Compare with Eq. (2.10) and note the absence of the vortex current $\mathcal{J}^v_{\ \mu}$ in the action. As we pointed out before this signifies the fact that the charge excitations without spin changes are to be found on the edge of the system in the low-energy approximation.

If we integrate out A_μ field we get

$$\mathcal{L}_{\rm bulk}^{\rm eff}=\frac{1}{4\pi m}\epsilon^{\mu\nu\lambda}A_\mu^{\rm ext}\partial_\nu A_\lambda^{\rm ext}+\frac{1}{4\pi m}\epsilon^{\mu\nu\lambda}A_\mu^{\rm ext}\partial_\nu(i\bar{z}\partial_\lambda z)$$
$$+\text{terms without } A_\mu^{\rm ext}. \qquad (2.45)$$

For the edge the complete action [see Eq. (2.22)] with external electromagnetic fields is

$$\mathcal{L}_{\rm edge}=\frac{1}{4\pi m}[(\partial_x\alpha-i\bar{z}\partial_xz)(\partial_0\alpha-i\bar{z}\partial_0z)$$
$$-v(\partial_x\alpha-i\bar{z}\partial_xz)^2]$$
$$+\frac{1}{2\pi m}(vA_x^{\rm ext}-A_0^{\rm ext})(\partial_x\alpha-i\bar{z}\partial_xz)$$
$$-\frac{1}{4\pi m}(vA_x^{\rm ext}-A_0^{\rm ext})A_x^{\rm ext}. \qquad (2.46)$$

Again, in this case, the extra term that we get by setting $A_\mu^{\rm ext}\to A_\mu^{\rm ext}+\partial_\mu\Lambda$ in Eq. (2.45),

$$\frac{1}{4\pi m}\Lambda[\partial_0(A_x^{\rm ext}+i\bar{z}\partial_xz)-\partial_x(A_0^{\rm ext}+i\bar{z}\partial_0z)] \qquad (2.47)$$

is canceled by the term that comes out from taking $A_\mu^{\rm ext}\to A_\mu^{\rm ext}+\partial_\mu\Lambda$ and $\alpha\to\alpha+\Lambda$ in Eq. (2.46). Therefore the gauge invariance of the total action, $\mathcal{L}_{\rm bulk}+\mathcal{L}_{\rm edge}$ [or $\mathcal{L}_{\rm bulk}^{\rm eff}+\mathcal{L}_{\rm edge}$, Eqs. (2.44) and (2.46)], is proved.

As we take $A_\mu^{\rm ext}$ to be the vector potential of the constant magnetic field and apply the spin-wave approximation the action (2.44) is transformed into Eq. (2.25) and the edge action becomes Eq. (2.22).

III. SOLUTIONS OF THE FIELD THEORY

As we vary the surface terms of the low-energy effective Lagrangian [Eqs. (2.37) and (2.38)] with respect to α and $\bar{\Psi}$ (or Ψ) we get two equations. As we vary α, we get

$$\partial_x\partial_0\alpha^f=v_c\partial_x^2\alpha^f \qquad (3.1)$$

and, by varying $\bar{\Psi}$ and using the previous equation,

$$2\rho_s\partial_y\Psi+\frac{1}{4\pi m}(-i)[\partial_x\Psi\partial_0\alpha^f-\partial_0\Psi\partial_x\alpha^f$$
$$-v_c2\partial_x\Psi\partial_x\Psi\partial_x\alpha^f]=0. \qquad (3.2)$$

When $\partial_\mu\alpha^f=0,\mu=0,x$, i.e., there are no charge excitations on the boundary, the spin waves (2.32) are solutions of the bulk and surface equations. When $\Psi=0$, i.e., there are no spin excitations in the system; the only solutions of the equations are charge-density waves of the chiral boson theory.

From the bulk equation (2.28) we get the following dispersion relation for the edge spin waves (2.36):

$$w=\frac{\Delta}{\rho_0}+\frac{2\rho_s}{\rho_0}(k_x^2-\beta^2). \qquad (3.3)$$

The coefficient β, which characterize the extension of the edge spin waves into the bulk of the system, comes from the boundary equation (3.2). The one-dimensional charge density and current are given by

$$j_0=\frac{1}{2\pi m}\partial_x\alpha^f \quad\text{and}\quad j_x=\frac{1}{2\pi m}\partial_0\alpha^f, \qquad (3.4)$$

respectively. For $\alpha^f=\alpha^f(x+v_ct)$, the general solution of Eq. (3.1), we have

$$v_c\partial_x\alpha^f=\partial_0\alpha^f, \quad\text{i.e.,}\quad v_cj_0=j_x, \qquad (3.5)$$

and we may rewrite Eq. (3.2) as

$$-4\rho_s\partial_x\Psi+ij_0\partial_0\Psi+ij_0v_c\partial_x\Psi=0. \qquad (3.6)$$

A. Charged edge spin-wave solutions

The solution of the form (2.36) of Eq. (3.6) with $\beta=$ const exists only if j_0 is, by itself, a constant, i.e., if there is a constant charge density along the boundary of the system. For the ground state $j_0=0$, and the previous condition (with

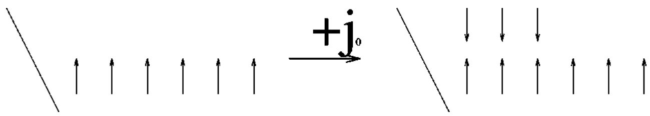

FIG. 1. Adding charge to the edge of the $\nu=1$ filling fraction in the single-particle picture.

$j_0\neq 0$) means that some (extra) charge is added to ($j_0>0$) or subtracted from the system ($j_0<0$). Plugging in the form (2.36) into Eq. (3.6), and using Eq. (3.3), we obtain the following candidates for solutions with β equal to

$$\beta_{1,2}=-\frac{\rho_0}{j_0}\pm\frac{\rho_0}{|j_0|}\sqrt{1+\frac{j_0^2}{2\rho_s\rho_0^2}(\Delta+2\rho_s k_x^2-v_c\rho_0 k_x)}, \tag{3.7}$$

from which only those with β positive can describe the edge spin waves. In the approximations in which the second term under the square root in Eq. (3.7) is small,

$$\beta\approx\frac{j_0}{4\rho_s\rho_0}(\Delta+2\rho_s k_x^2-v_c\rho_0 k_x) \quad \text{for } j_0>0$$

$$\text{and } \beta\approx 2\frac{\rho_0}{|j_0|} \quad \text{for } j_0<0. \tag{3.8}$$

For $j_0\rightarrow 0$ the second solution is unacceptable because our effective theory assumes (spatially) slowly varying quantities, which is not the case with the solution with β large.

Also, we may notice looking at Eq. (3.8) asymmetry between the $j_0>0$ and $j_0<0$ cases. One way to understand the asymmetry is to first consider the single-particle picture of adding and subtracting charge to the edge of these systems. In this picture of systems without spin, adding or subtracting charge is always equivalent to simple shifts (translations) of the boundary. In the case of the systems with spin there is an additional possibility to add charge as shown in Fig. 1.

We believe that this single-particle picture is behind the many-body state–charged edge spin wave for $j_0>0$. This can be supported by the following consideration. First, we may try to interpret the solution as a solution of a generalized pure-spin (quantum-ferromagnet) problem, in which the second and third term in Eq. (3.6) correspond to some surface terms in that problem. For a finite Zeeman coupling and in the small-momentum approximation, the solution with $j_0>0$ corresponds to the following term:

$$\frac{j_0}{\rho_0}\Delta\bar{\Psi}\Psi \tag{3.9}$$

with an effective magnetic field $B_{\rm eff}=-(j_0/\rho_0)B$ on the boundary. As we add electrons, we effectively create a magnetic field in the opposite direction of the external magnetic field. This magnetic field makes possible the creation of the spin edge solution. As we add more electrons (j_0 larger) the effective magnetic field increases in its magnitude and favors edge spin waves localized near the boundary (β larger). Therefore the edge-spin-wave, one-spin flip will be more localized on the boundary if there are more pairs (see Fig. 1) each of them energetically unfavorable because each of them consists of two electrons in the same orbital.

The corresponding term (in the pure-spin problem) for the second solution (with $j_0<0$) is

$$-\frac{2\rho_s j_0}{\rho_0}\partial_y\bar{\Psi}\partial_y\Psi. \tag{3.10}$$

It is of the opposite sign from the gradient term in the bulk and, therefore, favors the solution that disorders the spin configuration of the ground state and, we expect, as it is usual in these systems, that it is also followed by a redistribution of the charge on the boundary.

We also expect that these charged edge spin waves can be created without any change in the total charge of the system, i.e., by simple redistributions of the ground-state charge on the boundary where j_0 is a parameter that characterizes them. Therefore, when considering the total energy of excitations we will neglect the energy term of the chiral boson Lagrangian. Then the surface contribution is

$$E_{\rm surface}=2\rho_s\beta^2 \tag{3.11}$$

and the contribution from the bulk is given by Eq. (3.3) to get Eq. (3.11) we normalized the wave to describe one electron spin flip:

$$\Psi=\sqrt{\frac{\beta}{\pi}}\exp\{\beta y+ikx-iwt\}. \tag{3.12}$$

As a consequence, we may write the total energy of excitations as

$$E_{\rm tot}=\frac{\Delta}{\rho_0}+\frac{2\rho_s}{\rho_0}k^2+\beta^2\left(4\rho_s-\frac{2\rho_s}{\rho_0}\right). \tag{3.13}$$

Because $\rho_0=1/2\pi m$ the last term in Eq. (3.13) is always negative, and at $k=0$ $E_{\rm tot}$ is always lower than the constant Zeeman term. When Eq. (3.8) is substituted for β in the expression (3.13), the energy (3.13) is of the following form:

$$E_{\rm tot}=c_0+c_1 k+c_2 k^2+\cdots, \tag{3.14}$$

(where c's do not depend on k). By having a linear term in k this dispersion relation is asymmetric around $k=0$.

B. Neutral edge spin-wave solutions

We define the neutral edge spin waves by requiring that the field α, which lives on the boundary, associated with pure (not carrying also spin) charge degrees of freedom, is zero ($\alpha=0$). As a consequence there is one less surface equation to be satisfied, because the constraint equations (3.1) and (3.2) cannot be satisfied simultaneously. And, as we will see, the constraint also implies losing one of the two (charge and spin) local conservation laws. Namely, local spin currents normal to the boundary will be nonzero in general. It will be also shown that these excitations are neutral, i.e., the total change in the charge of the system, when these excitations occur, is zero.

With the condition $\alpha=0$ the charge current and density on the boundary are proportional to

$$\partial_\mu\alpha^f=-\frac{i}{2}(\bar{\Psi}\partial_\mu\Psi-\partial_\mu\bar{\Psi}\Psi)\,\mu=0,x \tag{3.15}$$

These charge degrees of freedom must satisfy Eq. (3.1), and from that it follows that the dispersion relation for the waves is $w=v_c k$. This dispersion relation follows when a general solution that is a superposition of the waves of the form (2.36) is considered. The frequency of the wave is also equal to Eq. (3.3). This enables us to find β in this case. It is given by the following expression:

$$\beta^2=\frac{\Delta}{2\rho_s}+k^2-\frac{v_c\rho_0}{2\rho_s}k, \tag{3.16}$$

i.e., in the small-momentum approximation $\beta\approx\sqrt{\Delta/2\rho_s}$. But the waves do not satisfy Eq. (3.2), which, in the spin-wave approximation, is equivalent to the condition (2.29). This means that the spin currents normal to the boundary of the system are, in general, nonzero and exchange of the spin of the system with outside is allowed. Nevertheless, $\mathcal{J}_y^3=0$ [see Eq. (2.35)] everywhere on the boundary and, also, the total spin of the system is conserved, i.e.,

$$\int_{-\infty}^{+\infty}\mathcal{J}_y^2 dx=0 \quad \text{and} \quad \int_{-\infty}^{+\infty}\mathcal{J}_y^1 dx=0. \tag{3.17}$$

Although the bulk excitation energy ($w=v_c k$) is gapless, these excitations have a gap. To see this we have to calculate their total energy and include the surface contribution (3.11), with β given by Eq. (3.16). As a final result we have

$$E_{\text{tot}}=2\Delta+v_c k(1-2\rho_s)+4\rho_s k^2. \tag{3.18}$$

The requirement $E_{\text{tot}}>0$ for each k fixes the allowed range of the parameters in Eq. (3.18) and in the theory. The excitations are allowed to propagate in both directions, though the spectrum is asymmetric around $k=0$ because of the presence of the linear term in E_{tot}. The gap (the smallest excitation energy for certain k) of these excitations is always smaller than the Zeeman gap.

To prove the neutrality of these excitations we start with a general edge spin-wave solution:

$$\Psi(x,y,t)=\sum_k \sqrt{\frac{\beta}{\pi}}\exp\{\beta y+ikx-iwt\}a(k), \tag{3.19}$$

where $a(k)$ are arbitrary (complex) coefficients in this expansion. The density of charge on the boundary of the system can be expressed as

$$\rho_{\text{surface}}(x,t)=\frac{\partial_x\alpha^f}{2\pi m}=\frac{1}{2\pi m}\left(-\frac{i}{2}\right)(\overline{\Psi}\partial_x\Psi-\partial_x\overline{\Psi}\Psi)$$
$$=\frac{\beta}{2\pi^2 m}\left(-\frac{i}{2}\right)\sum_{k,k'}\exp\{-i(k-k')x\}$$
$$\times\exp\{i(w_k-w_{k'})t\}i(k'+k)a(k)^*a(k'). \tag{3.20}$$

The same quantity in the bulk is proportional to the topological density[6] and equal to

$$\rho_{\text{bulk}}(x,y,t)=\frac{(-i)}{2\pi m}(\partial_x\overline{z}\partial_y z-\partial_y\overline{z}\partial_x z). \tag{3.21}$$

In the spin-wave approximation we have

$$\rho_{\text{bulk}}(x,y,t)=\frac{(-i)}{2\pi m}(\partial_x\overline{\Psi}\partial_y\Psi-\partial_y\overline{\Psi}\partial_x\Psi)$$
$$=\frac{-i\beta^2}{2\pi^2 m}\sum_{k,k'}\exp\{2\beta y\}\exp\{-i(k-k')x\}$$
$$\times\exp\{i(w_k-w_{k'})t\}$$
$$\times(-i)(k+k')a(k)^*a(k'). \tag{3.22}$$

Clearly, the total change in the charge of the boundary,

$$Q_{\text{surface}}=\int_{-\infty}^{+\infty}\rho_{\text{surface}}(x,t)dx=\frac{\beta}{\pi m}\sum_k ka(k)^*a(k), \tag{3.23}$$

is of the same amount, but of opposite sign than the one in the bulk $[Q_{\text{bulk}}=\int_{-\infty}^{+\infty}dx\int_{-\infty}^{0}dy\,\rho_{\text{bulk}}(x,y,t)]$.

At the end of this section we would like to comment on the nature of these neutral edge spin waves. As they are completely skyrmionic, their charge is fully specified by their spin configuration. Their spread increases with the decreasing of the Zeeman energy, which is a well-known skyrmion property. As they carry fixed (unit) amount of spin they vanish when the Zeeman coupling is zero. Due to the close relationship between their charge and spin and the linearity of the chiral boson dispersion relation, the dispersion relation of these spin waves is also linear.

IV. CONCLUSIONS AND DISCUSSION

In conclusion, we proposed an effective edge theory for ferromagnetic quantum Hall states. It describes their (1 + 1) dimensional charge degrees of freedom by a chiral boson theory and their (2 + 1) dimensional spin degrees of freedom by the effective theory of a quantum ferromagnet in the spin-wave approximation. We found two classes of the edge spin-wave solutions. The class with the charged edge spin waves is obtained by removing or adding electrons to the edge of the system. The second class of the neutral edge spin waves does not require any change in the total charge of the system. All these edge spin excitations are characterized by linear dispersion relations and gaps in the excitation energy.

We did not consider most-general surface terms for the spin degrees of freedom, which would be present in a most-general theory, because we wanted to emphasize and examine the influence of the terms that describe the charge degrees of freedom. The latter terms, collected in the chiral boson theory, give a complete effective description of the charge degrees of freedom on the edge.

Reference 9 pointed out that for small Zeeman energies and soft confining potentials a reconstruction (from narrow and spin-polarized edge) to the edge with spin textures should occur. Because of our assumption of the edge with the same polarization as that of the bulk, our theory is valid only for steep enough (see for estimates in Ref. 9) confining potentials.

In conclusion, we would like to comment on the relationship between our work and the recent work concerning edge excitations and edge reconstruction at $\nu=1$.[10] There, the Hartree-Fock procedure was used to determine the energy spectrum of some proposed edge spin-flip excitations (in fact, a special form of these excitations was first suggested in Ref. 11). These excitations reconstruct the edge when the Zeeman energy is small and the confining potential softened. They are followed by an outward movement of charge and exist even at zero Zeeman energy. The latter property is not shared by our neutral edge excitations and charged edge excitations for $j_0 > 0$ and, therefore, they should not be confused with those of Ref. 10 (the neutral edge waves even require, as we described, special boundary conditions to exist). On the other hand, the detailed description of the charged edge excitations for $j_0 < 0$ (which can be associated with the outward movement of charge) is impossible in the scope of our theory. The theory gives only an inkling of the possible instability of the ground state with respect to their creation.

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