Pairing instabilities of Dirac composite fermions

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Recently, a Dirac (particle-hole symmetric) description of composite fermions in the half-filled Landau level (LL) was proposed [D. T. Son, Phys. Rev. X **5**, 031027 (2015)], and we study its possible consequences on BCS (Cooper) pairing of composite fermions (CFs). One of the main consequences is the existence of anisotropic states in single-layer and bilayer systems, which was previously suggested in Jeong and Park [J. S. Jeong and K. Park, Phys. Rev. B **91**, 195119 (2015)]. We argue that in the half-filled LL in the single-layer case the gapped states may sustain anisotropy, because isotropic pairings may coexist with anisotropic ones. Furthermore, anisotropic pairings with the addition of a particle-hole symmetry-breaking mass term may evolve into rotationally symmetric states, i.e., Pfaffian states of Halperin-Lee-Read (HLR) ordinary CFs. On the basis of the Dirac formalism, we argue that in the quantum Hall bilayer at total filling factor 1, with decreasing distance between the layers, weak pairing of p-wave paired CFs is gradually transformed from Dirac to ordinary, HLR-like, with a concomitant decrease in the CF number. Global characterization of low-energy spectra based on the Dirac CFs agrees well with previous calculations performed by exact diagonalization on a torus. Finally, we discuss features of the Dirac formalism when applied in this context.

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I. INTRODUCTION

Composite fermions (CFs) [1] describe the physics of electrons in the fractional quantum Hall regime. At filling factor $\nu = 1/2$, essentially they absorb the external flux and make a metallic state [2] with its own Fermi surface—Fermi surface of CFs. By slightly modifying Read's dipole construction of composite (neutral) fermions in the half-filled lowest Landau level (LL) [3], an argument can be given for the accumulation of a Berry phase equal to π as a CF encircles its own Fermi surface [4]. This has motivated a description of the CFs in this setting in terms of Dirac fermions, which has been recently introduced in Ref. [5] and has attracted some interest [4,6–13]. The particle-hole (PH) symmetric description of the half-filled LL is given in terms of a Dirac system of composite quasiparticles—Dirac CFs at a finite chemical potential [4] and in the presence of a gauge field. However, the implied existence of singularity at zero momentum in the CF spectrum has been criticized [14,15]. We may add that, due to the requirement of gauge invariance in two dimensions, a small mass must be introduced into the Dirac theory ("parity anomaly") [16]. This may be a way to heal and complete in the high-energy domain ("UV completion" [17]) the Dirac description of CFs and avoid singularity.

Thus the description in terms of Dirac fermions may have the capacity to capture essential, at least qualitative, aspects of the CFs physics. To further examine this possibility in this work we consider BCS pairing of Dirac CFs. First, in the framework of the Dirac description of a single CF, we point out that, assuming Cooper pairing between spinor components, besides so-called PH symmetric Pfaffian, also anisotropic states can be realized. This is analogous to the ^{3}He system in which both B and A (anisotropic) phases are possible [18]. Next we discuss unconventional p-wave pairing of two kinds of Dirac CFs, motivated by the situation in the quantum Hall bilayer (QHB) at total filling factor 1, i.e., with each layer with a half-filled lowest LL. In this system p-wave pairing between two kinds of nonrelativistic Halperin-Lee-Read (HLR) composite fermions at intermediate interlayer distances was proposed in Ref. [19], and, recently, this scenario was further substantiated by the detection of the topological signatures of the p-wave system in the torus geometry [20]. Therefore, it is natural to ask how this picture may be modified if we take into account the description by two Dirac CFs of the two half-filled LL monolayers and consider their possible pairing.

One of the main conclusions that we can draw by applying the Dirac CF formalism in the context of BCS pairing is that due to the Dirac two-component nature, isotropic (gapped) pairing states may coexist with anisotropic ones, and this is in accordance with the results on PH symmetric, single-layer and bilayer fractional quantum Hall systems obtained by employing exact diagonalization [21,22], as well as with experimental findings [23,24], in which anisotropy is probed by an in-plane magnetic field. This may be a direct consequence of the dipole nature of CFs that is captured by Dirac formalism. Anisotropic pairing states may serve as seed states for Pfaffian and anti-Pfaffian state through a process in which PH asymmetry increases by introducing a mass term, while rotational symmetry gradually sets in. Furthermore, we find that the features, in particular, low-energy spectra, of the QHB at intermediate distances between the layers are better captured if we assume Dirac rather than HLR p-wave paired CFs at large distances (decoupled layers). Already at the effective field theory level, modeling the evolution with the distance between layers by Dirac CFs, we can detect the main feature of CF-composite boson (CB) mixed states [19,25]: the decrease in the number of CFs with decreasing distance.

The paper is organized as follows. In Sec. II, based on Dirac formalism, we discuss the single-layer case and its pairing instabilities, including the situation when the PH symmetry is spoiled by a mass term. In Sec. III we discuss the pairing instabilities in the bilayer system when the PH symmetry inside each layer is intact. In Sec. IV we examine the evolution of low-energy properties of the QHB with distance between layers, by including a mass term with an opposite sign in the two layers. The last section, Sec. V, is devoted to discussion and conclusions. Mean-field analysis of the coexistence of the isotropic and anisotropic pairings is presented in the Appendices.

II. DIRAC COMPOSITE FERMION AND COOPER PAIRING

We begin by considering a single Dirac fermion which was proposed to effectively describe the half-filled lowest Landau level of electrons [5], with s-wave pairing between spinor components. The s-wave pairing suggested in Ref. [5] can be expressed by the following Bogoliubov-de Gennes Hamiltonian in the Nambu-Gorkov notation:

$$H = \frac{1}{2}\sum_{\mathbf{k}}[\Psi^\dagger(\mathbf{k}) \quad \tilde{\Psi}(-\mathbf{k})] \times \begin{bmatrix} \mathcal{D}(\mathbf{k}) & \mathcal{P}(\mathbf{k}) \\ \mathcal{P}^\dagger(\mathbf{k}) & -\mathcal{D}(-\mathbf{k}) \end{bmatrix} \begin{bmatrix} \Psi(\mathbf{k}) \\ \tilde{\Psi}^\dagger(-\mathbf{k}) \end{bmatrix}, \tag{1}$$

where $\Psi(\mathbf{k})$ denotes a two-component spinor with momentum $\mathbf{k}$,

$$\Psi(\mathbf{k}) = \begin{bmatrix} \Psi_a(\mathbf{k}) \\ \Psi_b(\mathbf{k}) \end{bmatrix}, \quad \tilde{\Psi}(\mathbf{k}) = \begin{bmatrix} \Psi_b(\mathbf{k}) \\ \Psi_a(\mathbf{k}) \end{bmatrix}, \tag{2}$$

and

$$\mathcal{D}(\mathbf{k}) = \begin{bmatrix} -\mu & k_x - ik_y \\ k_x + ik_y & -\mu \end{bmatrix} = -\mu\sigma_0 + k_x\sigma_x + k_y\sigma_y, \tag{3}$$

and the 2×2 matrix $\mathcal{P}(\mathbf{k})$ describes Cooper pairing between a and b spinor components,

$$\mathcal{P}(\mathbf{k}) = \begin{bmatrix} \Delta_s & 0 \\ 0 & -\Delta_s \end{bmatrix} = \Delta_s\sigma_z, \tag{4}$$

or more explicitly

$$\delta\mathcal{H} = \sum_{\mathbf{k}}\{-\Delta_s\Psi_a(\mathbf{k})\Psi_b(-\mathbf{k}) + \text{H.c.}\}. \tag{5}$$

Here, σ_0 is the 2×2 identity matrix, while $\boldsymbol{\sigma}$ are the standard Pauli matrices. Throughout the paper we set $\hbar = 1$ and the Fermi velocity $v_F = 1$. μ denotes a chemical potential equal to $\mu = \sqrt{B} = k_F$, where B and k_F are the external magnetic field and the Fermi vector, respectively.

Since the pairing matrix anticommutes with the free Dirac Hamiltonian at the zero chemical potential, the dispersion of Bogoliubons has the rotationally symmetric form

$$E_k^2 = (k \pm \mu)^2 + \Delta_s^2, \tag{6}$$

where $k \equiv |\mathbf{k}|$. This construction is considered in the literature as a basis for a PH symmetric Pfaffian system.

However, a different type of pairing is also possible with the pairing matrix

$$\mathcal{P}(\mathbf{k}) = \begin{bmatrix} 0 & \alpha k_x + \beta k_y \\ \alpha k_x - \beta k_y & 0 \end{bmatrix}, \tag{7}$$

or more explicitly

$$\delta\mathcal{H}' = \sum_{\mathbf{k}}\alpha k_x\{\Psi_a^\dagger(\mathbf{k})\Psi_a^\dagger(-\mathbf{k}) + \Psi_b^\dagger(\mathbf{k})\Psi_b^\dagger(-\mathbf{k})\} + \text{H.c.} + \sum_{\mathbf{k}}\beta k_y\{\Psi_a^\dagger(\mathbf{k})\Psi_a^\dagger(-\mathbf{k}) - \Psi_b^\dagger(\mathbf{k})\Psi_b^\dagger(-\mathbf{k})\} + \text{H.c.}, \tag{8}$$

where α and β are, in general, allowed to be complex coefficients. The overall form of $\delta\mathcal{H}'$ is fixed by the requirement of the CP symmetry, which, as emphasized in Ref. [5], is equivalent to the requirement of the PH symmetry in the real electron system. Namely, the CP symmetry is a product of the charge conjugation, C,

$$C\Psi(\mathbf{k})C^{-1} = \sigma_x\Psi^*(\mathbf{k}), \tag{9}$$

and a parity transformation, P,

$$P\Psi(\mathbf{k})P^{-1} = \Psi^*(\mathbf{k}'), \tag{10}$$

where $\mathbf{k} = (k_x,k_y) \to \mathbf{k}' = (k_x, -k_y)$ under the parity transformation. Thus,

$$CP\Psi(\mathbf{k})(CP)^{-1} = \sigma_x\Psi(\mathbf{k}'). \tag{11}$$

The starting Dirac Hamiltonian (1) with $\mathcal{P} = 0$ and $\delta\mathcal{H}'$ are both invariant under the CP transformation (11). Notice that Eq. (5) is invariant up to a sign change under the CP transformation. This is also a property of the small-mass term that seems necessary to ensure the gauge invariance of the theory and to avoid the singularity at $\mathbf{k} = 0$ [17]. The BCS pairing terms like the one in Ref. (5) may accommodate the sign change by gauge transformations [5]. Thus the theory is invariant under the CP transformation in a more general sense, allowing for terms that are invariant up to a change of the sign. This makes our choice for p wave not unique. Indeed, other p-wave pairing order parameters are also possible, including one analogous to the A phase of the ^{3}He system that features two (gapless) Fermi points. This case can be analyzed analogously to the one considered here, and the main conclusions remain. In the following, we restrict our discussion to the p-wave case (8) invariant under the CP transformation in the strict sense.

We now consider the pairings given by Eq. (8), recently also discussed in Ref. [26], in light of the possibility of introducing an anisotropy. The choice of $\alpha = \Delta$ and $\beta = -i\Delta$ yields the pairing matrix $\mathcal{P}(\mathbf{k})$, proportional to the Dirac Hamiltonian $\mathcal{D}(\mathbf{k})$, at chemical potential $\mu = 0$, and thus explicitly rotationally invariant. (See also Sec. III for further analysis of the rotational symmetry.). In that case, the dispersion relation of Bogoliubons, $E_{\mathbf{k}}^2 = k^2(1 + \Delta^2) + \mu^2 \pm 2k\sqrt{\mu^2 + k^2\Delta^2}$, implies that the the pairing just renormalizes the chemical potential. On the other hand, by choosing $\alpha = \Delta$ and $\beta = +i\Delta$, we obtain

$$E_{\mathbf{k}}^2 = k^2(1 + \Delta^2) + \mu^2 \pm 2\sqrt{\mu^2k^2 + \Delta^2(k_x^2 - k_y^2)^2}. \tag{12}$$

This dispersion describes an *anisotropic* gapless system with four nodes at

$$k_x = \pm\frac{\mu}{\sqrt{1-\Delta^2}}, \quad \text{and} \quad k_y = 0, \tag{13}$$

and

$$k_y = \pm\frac{\mu}{\sqrt{1-\Delta^2}}, \quad \text{and} \quad k_x = 0. \tag{14}$$

The appearance of the four nodes related by the discrete C_4 symmetry is a consequence of the C_4 symmetry of the pairing (8) with $\alpha = \Delta$ and $\beta = +i\Delta$. In fact, Eq. (8) describes a whole family of gapless anisotropic solutions.

If we consider both the s-wave (5) and the p-wave (8) with $\alpha = \Delta$ and $\beta = +i\Delta$ pairings, the dispersion of the Bogoliubov quasiparticles is

$$\tilde{E}_{\mathbf{k}}^2 = \Delta_s^2 + k^2(1+\Delta^2) + \mu^2 \pm 2\sqrt{\mu^2k^2 + \Delta^2\left(k_x^2 - k_y^2\right)^2}, \tag{15}$$

i.e., the dispersion (12) simply acquired a shift of Δ_s^2 in the presence of the s-wave pairing. This is a consequence of the anticommutation of the matrices corresponding to the two pairings, similar to the situation in Refs. [27,28], which makes their coexistence likely at a finite chemical potential. Assuming a generic form of the two couplings driving the instabilities in the isotropic and anisotropic channels, in the presence of a small-mass term, we show in Appendix A that the low-energy description implies that the isotropic instability (5) may coexist with the anisotropic one. This is consistent with experimental [23,24], and theoretical [21,22] findings pointing out that gapped states at half-filled Landau levels can sustain and even harbor anisotropy.

In connection with the possible pairings given by Eq. (8) when $\alpha = \Delta$ and $\beta = +i\,\Delta$, we may notice that if we break CP (particle-hole symmetry) by a mass term of the form $\sim\Psi^\dagger(\mathbf{k})\sigma_3\Psi(\mathbf{k})$, one component, a or b, of the Dirac field will remain in the low-energy sector. The remaining fermion should correspond to the HLR (spinless) fermion which in turn pairs in the manner of the p wave. This should correspond to Pfaffian and anti-Pfaffian state [which comprise possible $(k_x \pm ik_y)$ states], in the absence of PH symmetry, but with an emergent rotational symmetry. A closely related proposal for the existence of the Pfaffian (Moore-Read) state in the presence of an excitonic instability already appeared in the context of Dirac CF physics in graphene [29].

To further understand the pairings in Eqs. (5) and (8), we now consider the chirality operator $\frac{\sigma\cdot\mathbf{k}}{|k|}$ and its eigenstates

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ \frac{k_+}{k}\end{bmatrix}, \quad |-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\ \frac{k_+}{k}\end{bmatrix}. \tag{16}$$

We can introduce Dirac operators with a definite chirality,

$$\Psi_+(\mathbf{k}) = \frac{1}{\sqrt{2}}\left[\Psi_a(\mathbf{k}) + \frac{k_-}{k}\Psi_b(\mathbf{k})\right] \tag{17}$$

and

$$\Psi_-(\mathbf{k}) = \frac{1}{\sqrt{2}}\left[-\Psi_a(\mathbf{k}) + \frac{k_-}{k}\Psi_b(\mathbf{k})\right], \tag{18}$$

to find that

$$\Psi_a(\mathbf{k})\Psi_b(-\mathbf{k}) = -\frac{1}{2}\frac{k_+}{k}[\Psi_+(\mathbf{k})\Psi_+(-\mathbf{k}) + \Psi_-(\mathbf{k})\Psi_-(-\mathbf{k})], \tag{19}$$

with $k_\pm \equiv k_x \pm ik_y$. We can clearly see from Eq. (19) that in the chirality basis, i.e., the eigenbasis of the noninteracting system, the pairing (5), in fact, describes a pairing in the odd (p-wave) channel. This can be understood as a consequence of the nontrivial Berry phase contributions, as discussed in Ref. [5]; see also Ref. [30] for the influence of the singularities (topological charges) on the vorticity of Cooper pairs. On the other hand, the anisotropic pairing (8) is a combination of odd-channel components in the chirality basis.

We now analyze an alternative scenario for the coexistence with the p-wave pairing represented by the pairing matrix $\mathcal{P}(\mathbf{k}) = (\alpha k_x + \beta k_y)\sigma_x$ that features two Fermi points and does not require a mass for the coexistence with the isotropic state. In particular, as shown in Appendix B, a special anisotropic pairing with

$$\mathcal{P} \sim ik_y\sigma_x \tag{20}$$

can coexist with the isotropic pairing. Analogously, we can discuss pairing with $\mathcal{P}(\mathbf{k}) \sim (\gamma k_x + \delta k_y)\sigma_y$, where γ and δ are, in general, allowed to be complex coefficients. The ensuing pairing is then given by

$$\mathcal{P}(\mathbf{k}) \sim k_x\sigma_y. \tag{21}$$

Both these pairings are invariant up to a change of sign (up to a gauge transformation) under the CP transformation. Each pairing on its own features two Fermi points and is likely energetically advantageous over the pairing in Eq. (8) that has four Fermi points. As we explicitly show in Appendix B, these pairings do not need a mass term to coexist with the isotropic state. Furthermore, in the presence of a mass term, they develop new components and may thus evolve into the rotationally symmetric pairings of HLR fermions. These are the reasons that make states given by Eqs. (20) or (21) likely present when considering pairing instabilities in the half-filled LL, consistent with the exact diagonalization results of Refs. [21,22].

Finally, we point out that the Dirac-based microscopic wave functions of pairing instabilities have not been proposed and tested yet. The effective field theory approach seems currently to be the most efficient tool for treating the Dirac-based pairing instabilities and their properties. Once the microscopic description is provided, most importantly for the case of PH Pfaffian, anisotropic modifications may be induced in the manner described and discussed in Refs. [31,32].

III. DIRAC FERMIONS AND P-WAVE PAIRING

We consider the following general form of the Bogoliubov-de Gennes Hamiltonian, motivated by the situation in a QHB system with each of the two layers at half filling:

$$H = \sum_{\mathbf{k}}[\Psi_\uparrow^\dagger(\mathbf{k}) \quad \Psi_\downarrow(-\mathbf{k})] \times \begin{bmatrix}\mathcal{D}_\uparrow(\mathbf{k}) & \mathcal{P}(\mathbf{k})\\ \mathcal{P}^\dagger(\mathbf{k}) & -\mathcal{D}_\downarrow(-\mathbf{k})\end{bmatrix}\begin{bmatrix}\Psi_\uparrow(\mathbf{k})\\ \Psi_\downarrow^\dagger(-\mathbf{k})\end{bmatrix}, \tag{22}$$

where $\Psi_\uparrow(\mathbf{k})$ and $\Psi_\downarrow(\mathbf{k})$ are two component spinors,

$$\Psi_\uparrow(\mathbf{k}) = \begin{bmatrix} \Psi_{a\uparrow}(\mathbf{k}) \\ \Psi_{b\uparrow}(\mathbf{k}) \end{bmatrix}, \quad \Psi_\downarrow(\mathbf{k}) = \begin{bmatrix} \Psi_{b\downarrow}(\mathbf{k}) \\ \Psi_{a\downarrow}(\mathbf{k}) \end{bmatrix}. \tag{23}$$

Matrices $\mathcal{D}_\uparrow(\mathbf{k})$ and $\mathcal{D}_\downarrow(\mathbf{k})$ describe two identical Dirac systems, $\mathcal{D}_\uparrow(\mathbf{k}) = \mathcal{D}_\downarrow(\mathbf{k}) = \mathcal{D}(\mathbf{k})$, with $\mathcal{D}(\mathbf{k})$ given by Eq. (3), while the 2×2 matrix $\mathcal{P}(\mathbf{k})$ describes Cooper pairing between the two systems $\uparrow$ and $\downarrow$.

A triplet p-wave pairing between the same spinor components can be expressed as the following term in the Hamiltonian:

$$\delta\mathcal{H} = \sum_{\mathbf{k}} \{[\Delta_{\mathbf{k}}^* \Psi_{a\downarrow}(-\mathbf{k})\Psi_{a\uparrow}(\mathbf{k}) + \Delta_{\mathbf{k}}^* \Psi_{b\downarrow}(-\mathbf{k})\Psi_{b\uparrow}(\mathbf{k})] + \text{H.c.}\}, \tag{24}$$

with the pairing function $\Delta_{\mathbf{k}} = \Delta(k_x \pm i k_y)$. The corresponding pairing matrix in the Hamiltonian (22) is

$$\mathcal{P}(\mathbf{k}) = \begin{bmatrix} 0 & \Delta_{\mathbf{k}} \\ \Delta_{\mathbf{k}} & 0 \end{bmatrix} = \Delta_{\mathbf{k}}\sigma_x. \tag{25}$$

A rotation around the z axis by the angle ϕ in both subsystems $\uparrow$ and $\downarrow$ is represented by the matrix $R = \exp(i\sigma_z\phi/2)$ so that

$$R\sigma_x R^{-1} = \sigma_x \cos\phi - \sigma_y \sin\phi,$$
$$R\sigma_y R^{-1} = \sigma_x \sin\phi + \sigma_y \cos\phi. \tag{26}$$

It can be readily seen that $\tilde{R}H(\mathbf{k})\tilde{R}^{-1} \neq H(\mathbf{k}')$, where $k_x^{'} = k_x\cos\phi - k_y\sin\phi$ and $k_y^{'} = k_x\sin\phi + k_y\cos\phi$, and $\tilde{R} = \tau_0 \otimes R$, with τ_0 as the 2×2 unity matrix in the subsystem space. Therefore, the system with the pairing matrix $\mathcal{P}(\mathbf{k}) = \Delta_{\mathbf{k}}\sigma_x$ is not rotationally invariant and may lead to anisotropic behavior. In fact, the system is gapless and supports two anisotropic Dirac cones at $k_x^2 = \mu^2/(1-\Delta^2) = k_0^2$ and $k_y = 0$. Expanding around $\pm k_0$ we obtain for $\Delta \ll 1$, $E^2 \approx (1 - 2\Delta^2)(\delta k_x)^2 + \Delta^2(\delta k_y)^2$. We find similar results if we choose

$$\mathcal{P}(\mathbf{k}) = \begin{bmatrix} 0 & \Delta_{\mathbf{k}} \\ -\Delta_{\mathbf{k}} & 0 \end{bmatrix}. \tag{27}$$

Therefore the systems that we have considered by now do not possess the quantum spin Hall effect; due to anisotropy they are likely to be fragile under disorder and certainly cannot represent stable phases in realistic circumstances.

On the other hand, the system with the pairing matrix

$$\mathcal{P}(\mathbf{k}) = \begin{bmatrix} \Delta_{\mathbf{k}} & 0 \\ 0 & -\Delta_{\mathbf{k}} \end{bmatrix} = \Delta_{\mathbf{k}}\sigma_z \tag{28}$$

yields the dispersion relation of the Bogoliubons

$$E_\pm = \sqrt{(k \pm \mu)^2 + |\Delta_{\mathbf{k}}|^2} \tag{29}$$

and therefore resembles very closely the p-wave pairing of ordinary fermions. We now express the pairing in the chirality basis to obtain

$$\Delta_{\mathbf{k}}^*[\Psi_{b\downarrow}(-\mathbf{k})\Psi_{a\uparrow}(\mathbf{k}) - \Psi_{a\downarrow}(-\mathbf{k})\Psi_{b\uparrow}(\mathbf{k})] = -\Delta_{\mathbf{k}}^* \frac{1}{2}\frac{k_+}{k}[\Psi_{+\downarrow}(-\mathbf{k})\Psi_{+\uparrow}(\mathbf{k}) - \Psi_{-\downarrow}(-\mathbf{k})\Psi_{-\uparrow}(\mathbf{k})]. \tag{30}$$

Thus depending whether $\Delta_{\mathbf{k}} = \Delta(k_x + i k_y)$ or $\Delta_{\mathbf{k}} = \Delta(k_x - i k_y)$, we obtain s-wave or d-wave pairing, respectively, in the chirality basis. In this sense there is no surprise to find that the pairing matrix (28) gives rise to a singlet state for $\uparrow$ and $\downarrow$ electrons. The choice for the pairing without the minus sign in Eq. (28), i.e., $\mathcal{P}(\mathbf{k}) = \Delta_{\mathbf{k}}\sigma_0$, is not energetically favorable, since pairing just renormalizes the chemical potential in that case.

We now provide a topological characterization of pairing in Eq. (28) through the (pseudo)spin Chern number, C_s. In fact we find that in this case $C_s = 1$, if we use the low-energy theory with Eq. (28) and $\Delta_{\mathbf{k}} = \Delta(k_x + i k_y)$. We calculated the Chern number by taking the eigenvectors of the two lower Bogoliubov bands, $|v_-(\mathbf{k})\rangle$ and $|v_+(\mathbf{k})\rangle$, corresponding to the eigenvalues $-E_-(\mathbf{k})$ and $-E_+(\mathbf{k})$, respectively. We first computed the Berry curvature of each vector,

$$F_{xy}^\sigma(\mathbf{k}) = i[\partial_x\langle v_\sigma(\mathbf{k})|\partial_y|v_\sigma(\mathbf{k})\rangle - \partial_y\langle v_\sigma(\mathbf{k})|\partial_x|v_\sigma(\mathbf{k})\rangle], \tag{31}$$

and then the Chern number,

$$C_s = \frac{1}{2\pi}\sum_\sigma \int d\mathbf{k} F_{xy}^\sigma(\mathbf{k}), \tag{32}$$

where the sum in Eq. (32) is over the two lowest bands. Nevertheless, as discussed in the previous paragraph, and also due to the form of eigenvectors below, we expect that the real winding number is 0 or 2 if a complete description is taken into account.

To further characterize the pairing state, let us consider the four-component vectors of the Bogoliubov bands with positive energy, $E_-(\mathbf{k})$ and $E_+(\mathbf{k})$,

$$u_-(k) = \frac{1}{2\sqrt{E_-}}\left\{-\sqrt{E_- - (\mu - k)}\left(1, \frac{k_+}{k}\right), \frac{\Delta \cdot k}{\sqrt{E_- - (\mu - k)}}\left(-\frac{k_-}{k}, 1\right)\right\}, \tag{33}$$

and

$$u_+(k) = \frac{1}{2\sqrt{E_+}}\left\{\sqrt{E_+ - (\mu + k)}\left(1, -\frac{k_+}{k}\right), \frac{\Delta \cdot k}{\sqrt{E_+ - (\mu + k)}}\left(\frac{k_-}{k}, 1\right)\right\}, \tag{34}$$

where we regrouped components to appear with common factors. In each Bogoliubov eigenstate, the first two-component spinor, (,), is an eigenstate of the chirality operator, given by Eq. (16), while the second one is the eigenstate that is complex conjugated and with inverted components due to the ordering in the Nambu-Gorkov representation, and we fix $\Delta_{\mathbf{k}} = \Delta(k_x + i k_y)$. From the coefficients in front of the fixed chirality states, we find the long-distance behavior of the pairing function ($g_{\mathbf{k}} \sim v_{\mathbf{k}}/u_{\mathbf{k}}$ in the usual BCS problem) in each band,

$$g(z) \sim 1/|z|, \tag{35}$$

where $g(z)$ is the pairing function in the real space and $z = x + iy$. Thus the pairing function has the characteristic s-wave feature.

In this case the lowest gap is at the Fermi surface, $\Delta E \sim \Delta \cdot k_F$, in contrast with ordinary p-wave pairing where the lowest gap is at zero momentum, and it is equal to $\Delta E \sim k_F$ [33].

IV. QUANTUM HALL BILAYER AND P-WAVE PAIRED COMPOSITE FERMIONS

In light of recent advance in understanding of each (isolated PH symmetric) half-filled quantum Hall monolayer based on Dirac CFs it is quite natural to consider the physics of the bilayer, especially at the intermediate distances, in the same framework. It is important to take into account the p-wave pairing [19] which was initially expressed in terms of ordinary HLR CFs. The picture based on the ordinary CFs does not have a clear answer for the lowest-lying spectrum which appears nearly gapless (with small gap) or gapless when the system is put on a torus, while the topological p-wave pairing of ordinary fermions [33] would likely produce a clear gap of the order μ. However, even if we neglect possible insufficiencies with p-wave pairing of ordinary fermions, it is fundamentally important to address the problem of the QHB in terms of Dirac CFs.

First we may notice that the presence of the interlayer Coulomb interaction, which increases with decreasing distance between layers, spoils the PH symmetry inside a layer. We incorporate this breaking of the PH symmetry by introducing a mass, r, in the Dirac matrices $\mathcal{D}_\uparrow(\mathbf{k})$ and $\mathcal{D}_\downarrow(\mathbf{k})$, with opposite signs in each layer,

$$\mathcal{D}_\uparrow(\mathbf{k}) = \sigma_x k_x + \sigma_y k_y - \mu + r\sigma_z = \mathcal{D}_\downarrow(\mathbf{k}). \tag{36}$$

Second, the components of the spinors in different layers are inverted with respect to each other, and thus the mass term of the opposite sign in the two layers enters with the same sign in the matrices $\mathcal{D}_\uparrow(\mathbf{k})$ and $\mathcal{D}_\downarrow(\mathbf{k})$. The dispersion relation in this case acquires the form

$$E_\pm = \sqrt{(\sqrt{k^2 + r^2} \pm \mu)^2 + |\Delta_{\mathbf{k}}|^2}. \tag{37}$$

The masses in the two layers are of the opposite sign, due to the requirement of the PH symmetry of the whole system. Namely, under the transformation in each layer masses change sign [5], and if we, in addition, exchange the layer index we recover the original Hamiltonian.

There are two important things to notice regarding the evolution of the CF state with the increasing mass r.

(a) The minimum of the lower Bogoliubov band shifts from a finite value at $k_F^2 = \mu^2/(1+\Delta^2)^2 - r^2$ to $k = 0$, and this transition—without closing of the gap—occurs at $r = \mu/(1+\Delta^2)$.

(b) Because $k_F^2 = \mu^2/(1+\Delta^2)^2 - r^2$, the Fermi momentum decreases with the mass, and therefore the number of CFs reduces as the distance between the layers decreases.

Therefore the most important consequence of the assumed Dirac description of individual layers at large distances is that the number of CFs decreases as the distance between the layers decreases. For large distances we may assume that the pairing is weak, the order parameter is small, and the pairing cannot be detected then due to finite temperature effects, for instance. In any case we may choose $\Delta_{\mathbf{k}} = \Delta(k_x + ik_y)$, so that there is no Hall drag (pseudospin Hall effect) at large distances, but it develops gradually as the interlayer distance decreases and reaches the quantized value in agreement with experiments [34]. This choice of the order parameter agrees with Refs. [19,20]. For smaller distances ($r \sim \mu$ but $r < \mu$) we may assume that the upper Bogoliubov band is pushed to high energies and an effective description in terms of quadratically dispersing CFs paired via weak p-wave pairing emerges, implying an algebraically decaying Cooper pair wave function [33]. The description of the system within this scenario then implies that at intermediate distances a CB-CF mixture accounts for the total number of electrons [25,19]. As a consequence, composite bosons cannot have long-range order and likely have critical, algebraic pairwise correlations [20].

If at intermediate distances solely a collection of p-wave paired composite fermions, quadratically dispersing as in Ref. [33], were present, signals of a topological phase with a large gap, $\Delta E \sim \mu$, would appear. Instead, as detected on a torus in Ref. [20], there is an abundance of various low-energy excitations. This is in accordance with the above physical picture that implies a small portion of CFs at intermediate distances in a topological phase with a small gap, $\Delta E \sim \mu - r$, and $\mu \simeq r$.

As in the single-layer case, an anisotropic gapless solution, Eq. (25), is possible also for a bilayer. In the presence of the mass term $\sim r$ and in the case of the pairing (25) we obtain two anisotropic Dirac cones at $k_x^2 = \mu^2/(1-\Delta^2) = k_0^2$ and $k_y = 0$. Expanding around $\pm k_0$ with $r \ll \mu$ we obtain $E^2 \approx (1 - 2\Delta^2 - \frac{r^2}{\mu^2})(\delta k_x)^2 + \Delta^2(\delta k_y)^2$. The absence of a gap suggests a nontopological behavior. On the other hand, topological signatures were detected at intermediate distances in Ref. [20], in agreement with the characterization of isotropic weak p-wave pairing. Thus the presence of the isotropic pairing, which may be accompanied by anisotropic ones, seems crucial for the explanation of the properties at intermediate distances.

V. DISCUSSION AND CONCLUSIONS

The existence of anisotropic candidates for BCS paired states, in the case of monolayers (Sec. II) and bilayers (Secs. III and IV), is in agreement with the results in Ref. [21]. In that paper, the physics of the PH symmetric case of a half-filled second Landau level is studied by exact diagonalization on a torus. The main result of this numerical study is that the paired quantum Hall state in that case, as well as the closely related (by antisymmetrization) bilayer state, made of two kinds of electrons that each occupy a quarter of the available single-particle states in the second Landau level, are susceptible to anisotropic instabilities. By using the Dirac description of the dipole nature of CFs, we can identify the paired quantum Hall state of Ref. [21] with PH Pfaffian and its closeness to anisotropy as a sign of the relevance of anisotropic solutions discussed in Sec. II. On the other hand, the relevance of the anisotropy for the bilayer state at the effective $\nu = 1/2 = 1/4 + 1/4$ total filling factor [21] may be again due to the composite, dipole nature of the CFs at filling factor $\nu = 1/4$. The Dirac description could be the easiest way to capture the dipole nature of a CF, despite the doubling of the fermionic degrees of freedom. In other words, we need particles and holes to describe dipoles [35], and the Dirac formalism could be a way to achieve that even in the cases when CFs have a Berry phase equal to $\pi/2$ (at quarter filling), with appropriate mass and chemical potential. If the Diracness is the cause of the anisotropic behavior, we can conclude that the Dirac formalism is equally applicable at $\nu = 1/2$ and $\nu = 1/4$. In this sense "nothing is special at $\nu = 1/2$" (Ref. [14]) since only PH symmetry singles out the Dirac description. The PH symmetry is sufficient but not necessary to cause the Diracness at the filling equal to one half.

If we restrict our discussion only to the case when CFs possess a Berry phase equal to π, and thus Dirac formalism seems appropriate for the bilayer case at total filling factor 1, we demonstrated that the description by Dirac fermions is justified due to a global appearance and characterization of low-energy spectrum from the exact diagonalization on a torus [20]. In fact, the Dirac CF in the bilayer changes its Berry phase from value π at large distances, to a value of ~ 0, at small distances (HLR fermion), while retaining its fermionic character. The second important consequence, due to the use of the Dirac formalism, is that the number of CFs is decreasing with the decreasing distance between the layers. This is in agreement with the necessity to use CF-CB mixed states to describe the bilayer at intermediate distances [19].

Thus we can conclude that the Dirac formalism can capture the basic phenomenology of the bilayer at $\nu = 1$ and the nature of the gapped paired states in the single-layer quantum Hall systems with half-filled LLs. We therefore expect it to become an indispensable tool for further understanding of the paired states in this context.

Note added in proof. Recently, Ref. [26] appeared. It is a study of possible pairings, based on the Dirac formalism, and their realization in the case of a single layer with a half-filled LL. Wang and Chakravarty [26] considered pairings in the low-energy subspace of the Dirac spectrum in the context of a specific pairing mechanism. In our work the low-energy projection is in place after the consideration of the pairing instabilities within the Dirac formalism. In this way we are able to account for the anisotropic pairings, with the consequences consistent with theoretical and experimental findings, as we already emphasized.

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APPENDIX A: COEXISTENCE OF THE *CP* INVARIANT *P*-WAVE AND *S*-WAVE PAIRINGS: MEAN-FIELD ANALYSIS

The lower Bogoliubov band of the quadratic Hamiltonian, Eqs. (1)–(4) with the additional pairing in Eq. (8) with $\alpha = \Delta$ and $\beta = +i\Delta$, and the mass term $r\Psi^\dagger(\mathbf{k})\sigma_3\Psi(\mathbf{k})$ is

$$E^2 = \mu^2 + \Delta_s^2 + r^2 + k^2 + \Delta^2 k^2 - 2\sqrt{\mu^2(k^2+r^2) + \left[\Delta\left(k_x^2 - k_y^2\right) + \Delta_s r\right]^2}.$$

We analyze the pairing instabilities in the low-energy theory by introducing the cutoff Λ, so that relevant momenta from the interval around Fermi energy are defined by Λ, $k \in (\mu - \Lambda, \mu + \Lambda)$. Also we assume that $\mu\Delta \ll \Delta_s \ll \Lambda \ll \mu$ and, at zero temperature, estimate the free energy when both isotropic (Δ_s) and anisotropic (Δ) pairings are present. From the BCS mean-field decoupling of effective attractive interactions we obtain terms proportional to the order parameters Δ^2 and Δ_s^2 (condensate energy) besides the contribution arising from the quasiparticles in the lower Bogoliubov band. (The upper band is assumed effectively to be a constant due to the constraint on the momenta.) The free-energy density $\mathcal{F}/A$ then reads

$$\frac{\mathcal{F}}{A} = g_1\Delta_s^2 + g_2\Delta^2 - \frac{\mu}{4\pi}\Lambda^2 - \frac{\mu}{4\pi}\left\{\left(1 + \ln\frac{4\Lambda^2}{\Delta_s^2}\right)\mathcal{M}\right\}, \tag{A1}$$

where

$$\mathcal{M} = \Delta_s^2 + \frac{\Delta^2\mu^2}{4} - \frac{r}{2}\Delta_s\Delta, \tag{A2}$$

with g_1 and g_2 as positive coupling constants which drive the instabilities in the respective channels. Here, we assume $r \ll \frac{\Delta_s}{\Lambda}(\mu\Delta)$.

We derive Eq. (A1) with Eq. (A2) by expanding the square root for large μ and then performing the integral over k (i.e., radial component of vector $\mathbf{k}$). Before the final angular integration, we further simplify the result of the k integration by assuming the stated ordering of scales.

In the BCS weak-coupling limit, by minimizing the free energy, i.e., the total ground state energy, we obtain

$$\Delta_s \approx 2\Lambda \exp\left\{-\frac{2\pi g_1}{\mu}\right\},$$
$$\Delta \approx \frac{r\tilde{g}_1}{\tilde{g}_1\mu^2 - 4g_2}\Delta_s, \tag{A3}$$

where $\tilde{g}_1 = \frac{\mu}{4\pi} + g_1$. Thus we can conclude that for $\mu\Delta \ll \Delta_s \ll \Lambda \ll \mu$, and in the presence of the small mass r, the isotropic instability can be accompanied by the anisotropic pairing. This is due to the cross term in $\mathcal{F}$ with Δ_s and Δ—see Eqs. (A2) and (A1). This may also be understood from the fact that the matrices corresponding to the isotropic and anisotropic pairings anticommute.

APPENDIX B: COEXISTENCE OF THE *CP* ASYMMETRIC *P*-WAVE AND *S*-WAVE PAIRINGS: MEAN-FIELD ANALYSIS

Here we discuss a pairing defined by

$$\mathcal{P}(\mathbf{k}) = \begin{bmatrix} 0 & \alpha k_x + \beta k_y \\ \alpha k_x + \beta k_y & 0 \end{bmatrix} = (\alpha k_x + \beta k_y)\sigma_x, \tag{B1}$$

or in terms of the spinors, as a part of the complete Hamiltonian,

$$\sum_{\mathbf{k}}(\alpha k_x + \beta k_y)\{\Psi_a(\mathbf{k})\Psi_a(-\mathbf{k}) + \Psi_b(\mathbf{k})\Psi_b(-\mathbf{k})\} + \text{H.c.}, \tag{B2}$$

where α and β are, in general, allowed to be complex coefficients.

The lower Bogoliubov band of the quadratic Hamiltonian, Eqs. (1)–(4) with the additional pairing in Eq. (B2), is

$$E^2 = \mu^2 + \Delta_s^2 + k^2\left(1 + f_1^2 + f_2^2\right) - 2k\sqrt{\mu^2 + \Delta_s^2 f_2^2 + k_x^2\left(f_1^2 + f_2^2\right) - 2\Delta_s f_2 \frac{k_y}{k}\mu}.$$

Here, $\alpha k_x + \beta k_y = k(f_1 + i f_2)$, where $f_i = \alpha_i \cos\phi + \beta_i \sin\phi$ [$i = 1$ and 2; α_1, α_2, β_1, and β_2 are real; and ϕ is the polar angle of the momentum vector].

As in Appendix A, here we also analyze the pairing instabilities in the low-energy theory by introducing a cutoff Λ, so that relevant momenta from the interval around the Fermi energy are defined by $\Lambda, k \in (\mu - \Lambda, \mu + \Lambda)$. Also we assume that $\mu\omega \ll \Delta_s \ll \Lambda \ll \mu$, where ω can be α_1, α_2, β_1, or β_2, and, at zero temperature, estimate the free energy when both isotropic (Δ_s) and anisotropic (f_1, f_2) pairings are present. From the BCS mean-field decoupling of effective attractive interactions we have terms proportional to f_1^2 and f_2^2 (averaged over angles) and Δ_s^2 next to the contribution from the lower Bogoliubov band. (The upper band is assumed effectively to be a constant due to the constraint on the momenta.) The free-energy density $\mathcal{F}/A$ then reads

$$\frac{\mathcal{F}}{A} = g_1\Delta_s^2 + g_2\left(\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2\right) - \frac{1}{2}\frac{1}{(2\pi)^2}\left\{\mu\,\Lambda^2\,2\pi + \left(1 + \ln\frac{4\Lambda^2}{\Delta_s^2}\right) \times \pi\mu\mathcal{M}\right\}, \tag{B3}$$

where

$$\mathcal{M} = \Delta_s^2 + \tfrac{1}{2}\Delta_s\beta_2\mu + \tfrac{1}{4}\left(\alpha_1^2 + \alpha_2^2 + 3\beta_1^2 + 3\beta_2^2\right)\mu^2, \tag{B4}$$

with g_1 and g_2 as positive coupling constants which drive the instabilities in the respective channels. The last contribution of the quadratic order in the anisotropic parameters, proportional to $\sum_i(\alpha_i^2 + 3\beta_i^2)$, was derived assuming $\Lambda \ll \frac{\beta_2\mu}{\Delta_s}\mu$.

To find this result for the free-energy density we applied the same set of approximations as in Appendix A. We derived Eq. (B3) with Eq. (B4) by expanding the value of the square root for large μ and then performing the integral over k. Before the final angular integration, we further simplified the result of the integration over k by assuming the stated ordering of scales.

In the BCS weak-coupling limit, by minimizing the free energy, i.e., the total ground state energy, assuming $\Delta_s \gg \omega\mu$, where ω can be α_1, α_2, β_1, or β_2, we obtain

$$\Delta_s \approx 2\Lambda \exp\left\{-\frac{4\pi g_1}{\mu}\right\}, \quad \beta_2 \approx \frac{\mu^2}{32\pi}\frac{1}{g_2}\Delta_s\left(1 + \frac{8\pi g_1}{\mu}\right), \quad \alpha_1 = \alpha_2 = \beta_1 = 0. \tag{B5}$$

Thus we can conclude that for cutoff Λ, $\mu\beta_2 \ll \Delta_s \ll \Lambda \ll \mu$, and in the presence of the isotropic instability Δ_s we can expect the presence of the anisotropic pairing with the order parameter $\sim i\beta_2 k_y$. This is due to the cross term in $\mathcal{F}$ with Δ_s and β_2—see Eqs. (B4) and (B3).

In the presence of the mass r the dispersion of the Bogoliubons is modified as

$$E^2 = \mu^2 + r^2 + \Delta_s^2 + k^2\left(1 + f_1^2 + f_2^2\right) - 2\sqrt{\mu^2k^2 + \Delta_s^2 f_2^2 k^2 + k_x^2\left(f_1^2 + f_2^2\right)k^2 + \mathcal{R}},$$

where

$$\mathcal{R} = r^2\left(\Delta_s^2 + \mu^2\right) + 2\Delta_s k(-k_y f_2\mu + k_x f_1 r). \tag{B6}$$

We can notice that, besides the cross term $\sim\Delta_s f_2 k_y$ under the square root in the above equation, we have, in the presence of the mass r, the term $\sim\Delta_s f_1 k_x$. By performing a mean-field analysis similar to that performed previously, we find that this term will lead to the development of the real component proportional to k_x in the anisotropic pairing, $\alpha k_x + \beta k_y = \alpha_1 k_x + i\beta_2 k_y$, with $\alpha_1/\beta_2 \sim r/\mu$ for $r \ll \mu$. Eventually, for $r \lesssim \mu$, we expect that $\Delta_s = 0$, and we expect the presence of the rotationally symmetric p wave, $\alpha k_x + \beta k_y \sim (k_x \pm i k_y)$, of one-component quadratically dispersing HLR composite fermions. Indeed the assumption that $\Delta_s = 0$ and the presence of the p wave are compatible with $r < \mu$, and there is no closing of the gap.

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