The Use of Path Integral Ideals: Deriving the Euler Summation Formula for Path Integrals

Aleksandar Bogojević*, Antun Balaž* and Aleksandar Belić*

*Institute of Physics, P. O. Box 57, 11001 Belgrade, SERBIA AND MONTENEGRO
emails: alex@phy.bg.ac.yu , antun@phy.bg.ac.yu , abelic@phy.bg.ac.yu

Abstract. We present and comment on a new quantity that we have recently introduced: the path integral ideal. The new quantity governs the flow of a discrete quantum theory to its continuum limit. Path integral ideals satisfy a unique integral equation – the distinction between different quantum theories being in the boundary conditions. An asymptotic expansion of this equation has led to the derivation of a generalization of Euler's summation formula for path integrals. The new analytical method has brought about a systematic improvement of the convergence of path integrals. Applied to numerical procedures, the new analytical input has resulted in the speedup of numerical simulations by many orders of magnitude. On the analytical side, the integral equation for ideals may turn out to be a useful setting for extending the obtained results to a wider setting – e.g. to p-adic valued theories and theories on non-commuting space-times.

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1. INTRODUCTION

Path integrals present a rich and flexible formalism for dealing with quantum and statistical theories [1, 2] that has proven extremely useful for handling symmetries, deriving non-perturbative results, establishing connections between different theories [3, 4], and extending the quantization procedure to ever more complicated systems. They have served as catalysts for the exchange of key ideas between different areas of physics, most notably high energy and condensed matter physics [5, 6]. Today, analytical and numerical approaches to path integrals [7, 8, 9, 10] play important roles not only in physics but also in chemistry and materials science, and are acquiring a prominent role in mathematics and modern finance [11].

Further development of the path integral method is constrained by the small number of solvable models, as well as by our rather limited knowledge of their precise mathematical properties. In fact, most of our knowledge is negative, e.g. we know which trajectories do not contribute to the path integral rather than which do. One of the few positive statements concerning path integrals is that relevant trajectories exhibit stochastic self-similarity [1]. As a result they have non-trivial fractal dimension and jaggedness [12, 13]. Researchers working on numerical approaches to path integrals have successfully utilized these kinematic consequences of self-similarity to produce efficient path-generating algorithms [9, 10].

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In a recent series of papers [14, 15] we have investigated the dynamical implications of stochastic self-similarity by studying the relation between discretizations of path integrals with different coarseness. This has resulted in a systematic analytical procedure that may be used to reduce path integral error to $O(\varepsilon_N^p)$ for arbitrary $p \in \mathbb{N}$, where ε_N is the discrete time step. Note that $\varepsilon_N = T/N$, T being the time of propagation and N the discretization coarseness. This reduction of error brings about a substantial increase in the speed of numerical algorithms. Additional information can be found on our web site [16]. Self-similarity played a crucial role in this procedure in that it allowed us to derive an integral equation relating discretized theories viewed at different coarseness and to solve it in terms of an asymptotic series. The asymptotic expansion, however, implies that the obtained method is directly applicable only for $\varepsilon_N < 1$.

The fact that we can arbitrarily decrease the error points to the possibility that one can extend the formalism and obtain exact information (i.e not given as a power series in ε_N, and so valid even for large values of ε_N) about the continuum theory. Large ε_N corresponds to long times of propagation, precisely what interests us in quantum field theory (or in modern finance). Equivalently, in condensed matter and materials science this corresponds to the physically most interesting region of small temperatures. Large ε_N behavior is also central for determining the energy spectrum of a given model, and as such is applicable in many areas of physics [11] (e.g. atomic and molecular physics, quantum dots).

The central quantity we will work with in this paper is the path integral ideal which governs the flow of a generic discrete theory to the continuum. The ideal was first introduced in [17]. In that paper we showed that the flow to the continuum is classified according to the degree of divergence of the potential at spatial infinity. In addition we derived certain asymptotic properties of ideals. We will here show how the formalism of ideals may be used to derive a generalization of the Euler summation formula to path integrals [18].

2. EULER SUMMATION FORMULA FOR ORDINARY INTEGRALS

The current status of the development of the path integral formalism is quite similar to that of ordinary integrals before the setting up of integration theory by Riemann. In those days integrals were calculated directly from the defining formula, i.e. one looked at a specific discretization of the integral (Darboux sum), attempted to do the sum explicitly, and finally tried to calculate the continuum limit. For example,

$$I[f] \equiv \int_0^T f(t)dt = \lim_{N\to\infty} I_N[f] \text{ , where } \quad I_N[f] = \sum_{n=1}^{N} f(t_n)\varepsilon_N \text{ ,} \tag{1}$$

$\varepsilon_N = T/N$ and $t_n = n\varepsilon_N$. It goes without saying that done this way, even the simplest ordinary integrals presented a challenge. The mathematicians of the 18th century did not have computers at their disposal or the development of integration theory might have

come much later, i.e. they might have succumbed to doing brute force numerical calculations of integrals of all but the simplest functions. The problem with these hypothetical numerical calculations would have been two fold: they would have been inefficient (the discretized sums converge slowly to the continuum value), and they would have worked (thus quite probably slowing down the further development of integration theory). Luckily, this early numerical road was not open. The last great step in the development of integration before Riemann was made by Euler.

Discretization is not unique. This makes it possible to change $f(t)$ to some other function (adding terms proportional to ε_N, ε_N^2, etc.) without changing the integral. Let us assume that $f^*(t)$ is such an equivalent function with the added property that the sums $I_N[f^*]$ do not depend on N. In fact we shall present a way of explicitly constructing $f^*(t)$ for any given $f(t)$. We first look at the simple case of $f(t) = 1$. Now

$$I_N[1] = \sum_{n=1}^{N} \varepsilon_N = T \ , \tag{2}$$

which is already N-independent. Hence, in this case, all the additional terms vanish. Note that $f^*(t)$ is completely determined by the original function $f(t)$ (and by ε_N), so that the additional terms necessarily depend only on the derivatives f', f'', etc.

The second step is to take $f(t) = t$. In this case we get

$$I_N[t] = \sum_{n=1}^{N} t_n \varepsilon_N = \frac{N(N+1)}{2} \frac{T^2}{N^2} = \frac{T^2}{2} + \frac{T^2}{2N} \ . \tag{3}$$

From this it follows that $I_N[t - \frac{\varepsilon_N}{2}] = \frac{T^2}{2}$. Therefore, up to f'' and higher derivatives of f that all vanish for linear $f(t)$, we have $f^*(t) = f(t) - \frac{\varepsilon_N}{2} f'(t)$.

We continue this procedure by looking at $f(t) = t^2$. In this case we find

$$I_N[t^2] = \sum_{n=1}^{N} t_n^2 \varepsilon_N = \frac{N(N+1)(2N+1)}{6} \frac{T^3}{N^3} = \frac{T^3}{3} + \frac{T^3}{2N} + \frac{T^3}{6N^2} \ . \tag{4}$$

It follows that $I_N[t^2 - \varepsilon_N t_n - \frac{2}{3}\varepsilon_N^2] = \frac{T^3}{3}$. In terms of f^* this gives $f^*(t) = f(t) - \frac{\varepsilon_N}{2} f'(t) - \frac{2\varepsilon_N^2}{3} f''(t) + \ldots$. The additional terms now depend on higher powers of ε_N as well as on higher derivatives and are determined by considering $I_N[t^3]$, and so on. In this way we have constructed a procedure for finding $f^*(t)$ for any given $f(t)$. Remembering that $I_N[f^*]$ does not depend on N we find

$$\int_0^T f(t)dt = \sum_{n=1}^{N} f(t_n)\varepsilon_N - \frac{\varepsilon_N}{2} \sum_{n=1}^{N} f'(t_n)\varepsilon_N - \frac{2\varepsilon_N^2}{3} \sum_{n=1}^{N} f''(t_n)\varepsilon_N + \ldots \ . \tag{5}$$

This is the well-known Euler summation formula. We may also write it more compactly as

$$I[f] = I_N[f^{(p)}] + O(\varepsilon_N^p) \ , \tag{6}$$

where $f^{(p)}$ is the truncation of f^* to the first p terms. The Euler formula gives the analytical relation between integrals and their discretized sums. Looked at numerically, this formula allows us to increase the speed of convergence of discretized expressions to the continuum limit. For example, in the defining relation the discretized expressions differ from the continuum by a term of order $O(1/N)$. By using the Euler sum formula with p terms we can reduce that error to $O(1/N^p)$. All that is needed to do this is that the integrand is differentiable $p-1$ times. the following sections we will generalize the above approach to path integrals.

3. GENERAL PROPERTIES OF PATH INTEGRALS

In the functional formalism the quantum mechanical amplitude $A(a,b;T) = \langle b|e^{-T\hat{H}}|a\rangle$ is given in terms of a path integral which is simply the $N \to \infty$ limit of the $(N-1)$– fold integral expression

$$A_N(a,b;T) = \left(\frac{1}{2\pi\varepsilon_N}\right)^{\frac{N}{2}} \int dq_1 \cdots dq_{N-1}\, e^{-S_N} \ . \tag{7}$$

The Euclidean time interval $[0,T]$ has been subdivided into N equal time steps of length $\varepsilon_N = T/N$, with $q_0 = a$ and $q_N = b$. S_N is the naively discretized action of the theory. We focus on actions of the form

$$S = \int_0^T dt\, \left(\frac{1}{2}\dot{q}^2 + V(q)\right) \ , \tag{8}$$

whose naive discretization is simply

$$S_N = \sum_{n=0}^{N-1} \left(\frac{\delta_n^2}{2\varepsilon_N} + \varepsilon_N V_n\right) \ , \tag{9}$$

where $\delta_n = q_{n+1} - q_n$, $V_n = V(\bar{q}_n)$, and $\bar{q}_n = \frac{1}{2}(q_{n+1} + q_n)$. We use units in which $\hbar$ and particle mass equal 1.

As was the case with ordinary integrals the definition of the path integrals also makes it necessary to make the transition from the continuum to the discretized theory, a process that is far from unique. For theories described by eq. (8) we have the freedom to choose any point in $[q_n, q_{n+1}]$ in which to evaluate the potential without changing physics – the discretized amplitudes do differ, but they tend to the same continuum limit. The calculations we present turn out to be simplest in the mid-point prescription where the potential V is evaluated at $\bar{q}_n$. A more important freedom related to our choice of discretized action has to do with the possibility of introducing additional terms that explicitly vanish in the continuum limit. Actions with such additional terms will be called effective. For example, the term $\sum_{n=0}^{N-1} \varepsilon_N\, \delta_n^2\, g(\bar{q}_n)$, where g is regular when $\varepsilon_N \to 0$, does not change the continuum physics since it goes over into $\varepsilon_N^2 \int_0^T dt\, \dot{q}^2\, g(q)$,

i.e. it vanishes as ε_N^2. Such terms do not change the physics, but they do affect the speed of convergence. A systematic study of the relation between different discretizations of the same path integral will allow us to explicitly construct a series of effective actions with progressively faster convergence to the continuum. Before we do this we will parallel the derivation in the previous section and derive some general properties of the best effective action.

The amplitude $A(a,b;T)$ of some theory with action S satisfies

$$A(a,b;T) = \int dq_1 \cdots dq_{n-1} A(b,q_{n-1};\varepsilon_N) \cdots A(q_1,a;\varepsilon_N) \; , \tag{10}$$

for all N. This general relation is a direct consequence of the linearity of states in a quantum theory. In analogy with ordinary integrals let us now suppose that there exists an effective action S^* that is equivalent to S (i.e that leads to the same continuum limit for all path integrals) with the additional property that its N-fold discretized amplitude $A_N^*(a,b;T)$ does not depend on N, i.e. that satisfies

$$A_N^*(a,b;T) = A(a,b;T) \; . \tag{11}$$

As was the case in the previous section we will in fact construct a general procedure for evaluating this effective action. For actions of the form given in eq. (8) we may write the amplitude as

$$A(q_{n+1},q_n;\varepsilon_N) = \left(\frac{1}{2\pi\varepsilon_N}\right)^{\frac{1}{2}} \exp\left(-\frac{\delta_n^2}{2\varepsilon_N}\right) \mathscr{A}(q_{n+1},q_n;\varepsilon_N) \; , \tag{12}$$

where the reduced amplitude $\mathscr{A} \to 1$ as $\varepsilon_N \to 0$. Writing S_N^* as

$$S_N^* = \sum_{n=o}^{N-1} \left(\frac{\delta_n^2}{2\varepsilon_N} + \varepsilon_N W_n^*\right) \; , \tag{13}$$

and using eq. (7), (10) and (11) we find

$$\exp\left(-\varepsilon_N W_n^*\right) = \mathscr{A}(q_{n+1},q_n;\varepsilon_N) \; . \tag{14}$$

Note that W_n^* is reminiscent of some effective potential, so it should depend on $\bar{q}_n$, however, from the above relation we see that it must also depend on δ_n. In addition, W^* also has an explicit dependence on the discrete time step ε_N, hence

$$W_n^* = W^*(\delta_n,\bar{q}_n;\varepsilon_N) \; . \tag{15}$$

As we have seen, the above functional form is a direct consequence of the linearity of quantum theory. The equivalence of S and S^* implies that $W^* \to V(\bar{q})$ when ε_N and δ go to zero. The final general property of W^* follows from the reality of amplitudes in the Euclidean formalism. Using the hermiticity of the Hamiltonian we find $A(a,b;T) = A(a,b;T)^\dagger = \langle b|e^{-T\hat{H}}|a\rangle^\dagger = \langle a|e^{-T\hat{H}}|b\rangle = A(b,a;T)$. In terms of W^* this gives us

$$W^*(\delta_n,\bar{q}_n;\varepsilon_N) = W^*(-\delta_n,\bar{q}_n;\varepsilon_N) \; , \tag{16}$$

or, said another way, only even powers of δ_n are present in the expansion of W^*:

$$W^*(\delta_n, \bar{q}_n; \varepsilon_N) = g_0(\bar{q}_n; \varepsilon_N) + \delta_n^2\, g_1(\bar{q}_n; \varepsilon_N) + \delta_n^4\, g_2(\bar{q}_n; \varepsilon_N) + \dots \ . \tag{17}$$

All the functions g_k are regular in the $\varepsilon_N \to 0$ limit. The link to the starting theory is now simply $g_0(\bar{q}_n; \varepsilon_N) \to V(\bar{q}_n)$ as ε_N goes to zero. This concludes the general properties of W^*. The remaining properties will be analyzed in the following section by studying the relation of discretizations of different coarseness.

We next derive an equation for path integral ideals by studying the relation between the $2N$-fold and N-fold discretizations of the same theory. From eq. (7) we see that we can write the $2N$-fold amplitude as an N-fold amplitude given in terms of a new action $\widetilde{S}_N$ determined by

$$e^{-\widetilde{S}_N} = \left(\frac{2}{\pi\varepsilon_N}\right)^{\frac{N}{2}} \int dx_1 \cdots dx_N\, e^{-S_{2N}} \ , \tag{18}$$

where S_{2N} is the $2N$-fold discretization of the starting action. We have written the $2N$-fold discretized coordinates $Q_0, Q_1, \dots, Q_{2N}$ in terms of q's and x's in the following way: $Q_{2k} = q_k$ and $Q_{2k-1} = x_k$. Note that we have $q_0 = a$, $q_N = b$, while the $N-1$ remaining q's play the role of the dynamical coordinates in the N-fold discretized theory. The x's are the N remaining intermediate points that we integrate over in eq. (18). It is not difficult to see that if we use the naively discretized action S_N one obtains for $\widetilde{S}_N$ an expression that is not of the same form as S_N. Having in mind the results derived at the beginning of this section it is best to use the effective action

$$S_N^* = \sum_{n=o}^{N-1} \left(\frac{\delta_n^2}{2\varepsilon_N} + \varepsilon_N W^*(\delta_n, \bar{q}_n; \varepsilon_N)\right) \ , \tag{19}$$

which gives the same result for both the $2N$-fold and N-fold discretizations. Therefore, in this case we get

$$e^{-S_N^*} = \left(\frac{2}{\pi\varepsilon_N}\right)^{\frac{N}{2}} \int dx_1 \cdots dx_N\, e^{-S_{2N}^*} \ . \tag{20}$$

From this it is straightforward to show that path integral ideals satisfy

$$\begin{aligned} \exp\left(-\varepsilon_N W^*(\delta, q; \varepsilon_N)\right) &= \sqrt{\frac{2}{\pi\varepsilon_N}} \int_{-\infty}^{+\infty} dy\, e^{-2y^2/\varepsilon_N} \\ &\times \exp\left(-\frac{\varepsilon_N}{2} W^*\left(\frac{\delta}{2} - y, q + \frac{\delta}{4} + \frac{y}{2}; \frac{\varepsilon_N}{2}\right) - \frac{\varepsilon_N}{2} W^*\left(\frac{\delta}{2} + y, q - \frac{\delta}{4} + \frac{y}{2}; \frac{\varepsilon_N}{2}\right)\right) . \end{aligned} \tag{21}$$

We end this section by briefly commenting on some general properties of the above integral equation. All quantum theories have been reduced to a single integral equation. We must first solve this equation and only then impose boundary conditions ($\varepsilon_N \to 0$ limit) that link us to a specific theory. The integral equation is easily solved for quadratic ideals, in which case we recover the usual free particle and harmonic oscillator results. Eq. (21) is an good starting point for developing various approximation schemes and for

analyzing non-perturbative properties of ideals [17]. We also believe that thus integral equation will have natural extensions to fermions, higher dimensions, as well as to more complex settings such as p-adic theories and quantum theories on non-commuting space-times. The general solution of the above integral equation in the form of an asymptotic expansion will be given in the following section.

4. EULER SUMMATION FORMULA FOR PATH INTEGRALS

In order to solve eq. (21) in the form of an asymptotic expansion we write it as

$$e^{-\varepsilon_N W^*(\delta_n,\bar{q}_n;\varepsilon_N)} = \left(\frac{2}{\pi\varepsilon_N}\right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} dy\, \exp\left(-\frac{2}{\varepsilon_N}y^2\right)\, F\left(\bar{q}_n+y;\frac{\varepsilon_N}{2}\right)\,, \tag{22}$$

where

$$\begin{aligned} -\frac{2}{\varepsilon_N}\ln F(x;\varepsilon_N) &= g_0\Big(\frac{q_{n+1}+x}{2};\varepsilon_N\Big) + g_0\Big(\frac{x+q_n}{2};\varepsilon_N\Big) \\ &+ (q_{n+1}-x)^2\, g_1\Big(\frac{q_{n+1}+x}{2};\varepsilon_N\Big) + (x-q_n)^2\, g_1\Big(\frac{x+q_n}{2};\varepsilon_N\Big) + \ldots \end{aligned} \tag{23}$$

Note the integral in eq. (22) is in a form that is ideal for an asymptotic expansion [19]. The time step ε_N is playing the role of small parameter (in complete parallel to the role $\hbar$ plays in standard semi-classical, or loop, expansion). As is usual, the above asymptotic expansion is carried through by first Taylor expanding $F\left(\bar{q}_n+y;\frac{\varepsilon_N}{2}\right)$ around $\bar{q}_n$ and then by doing the remaining Gaussian integrals. Assuming that $\varepsilon_N < 1$ (i.e. $N > T$) we have

$$\begin{aligned} g_0(\bar{q}_n;\varepsilon_N) + \delta_n^2\, g_1(\bar{q}_n;\varepsilon_N) + \delta_n^4\, g_2(\bar{q}_n;\varepsilon_N) + \ldots = \\ = -\frac{1}{\varepsilon_N}\ln\left[\sum_{m=0}^{\infty}\frac{F^{(2m)}\left(\bar{q}_n;\frac{\varepsilon_N}{2}\right)}{(2m)!!}\left(\frac{\varepsilon_N}{4}\right)^m\right]\,. \end{aligned} \tag{24}$$

Note that $F^{(2m)}(x;\varepsilon_N)$ denotes the corresponding derivative with respect to x. All that remains is to calculate these expressions using eq. (23) and to expand all the g_k's around the mid-point $\bar{q}_n$. This is a straight forward though tedious calculation. In this paper we will illustrate the general procedure for calculating S^* by explicitly giving its expansion to order ε_N^3:

$$\begin{aligned} g_0(\bar{q}_n;\varepsilon_N) &= g_0\left(\bar{q}_n;\frac{\varepsilon_N}{2}\right) + \varepsilon_N\left[\frac{1}{4}\,g_1\left(\bar{q}_n;\frac{\varepsilon_N}{2}\right) + \frac{1}{32}\,g_0''\left(\bar{q}_n;\frac{\varepsilon_N}{2}\right)\right] \\ &+ \varepsilon_N^2\left[\frac{3}{16}\,g_2\left(\bar{q}_n;\frac{\varepsilon_N}{2}\right) - \frac{1}{32}\,{g_0'}^2\left(\bar{q}_n;\frac{\varepsilon_N}{2}\right) + \frac{1}{2048}\,g_0^{(4)}\left(\bar{q}_n;\frac{\varepsilon_N}{2}\right)\right. \\ &\left. + \frac{3}{128}\,g_1''\left(\bar{q}_n;\frac{\varepsilon_N}{2}\right)\right] \\ g_1(\bar{q}_n;\varepsilon_N) &= \frac{1}{4}\,g_1\left(\bar{q}_n;\frac{\varepsilon_N}{2}\right) + \frac{1}{32}\,g_0''\left(\bar{q}_n;\frac{\varepsilon_N}{2}\right) \end{aligned} \tag{25}$$

$$+\,\varepsilon_N\left[\frac{3}{8}\,g_2\left(\bar{q}_n;\frac{\varepsilon_N}{2}\right)+\frac{1}{1024}\,g_0^{(4)}\left(\bar{q}_n;\frac{\varepsilon_N}{2}\right)-\frac{1}{64}\,g_1''\left(\bar{q}_n;\frac{\varepsilon_N}{2}\right)\right]$$

$$g_2(\bar{q}_n;\varepsilon_N) \;=\; \frac{1}{16}\,g_2\left(\bar{q}_n;\frac{\varepsilon_N}{2}\right)+\frac{1}{6144}\,g_0^{(4)}\left(\bar{q}_n;\frac{\varepsilon_N}{2}\right)+\frac{1}{128}\,g_1''\left(\bar{q}_n;\frac{\varepsilon_N}{2}\right)\;.$$

In the above relations we expanded g_0 up to ε_N^2, g_1 up to ε_N, etc. We also disregarded all the higher g_k's. The reason for this is that the short time propagation of any theory satisfies $\delta_n^2 \propto \varepsilon_N$ while the g_k term enters the action multiplied by δ_n^{2k}. In general, if we wish to expand the effective action to ε_N^p we need to evaluate only g_0 (up to ε_N^{p-1}) and the remaining $p-1$ functions g_k (up to ε_N^{p-1-k}). The task of calculating the effective action to large powers of ε_N is time-consuming and is best done with the help of a standard package for algebraic calculations such as Mathematica. Using Mathematica we determined the corresponding expressions for $p \leq 9$.

Although the above system of recursive relations is non-linear, it is in fact quite easy to solve if we remember that the system itself was derived via an expansion in ε_N. Having this in mind we first write all the functions as expansions in powers of ε_N that are appropriate to the level p we are working at. For $p=3$, we have

$$\begin{aligned} g_0(\bar{q}_n;\varepsilon_N) &= V(\bar{q}_n)+\varepsilon_N R_1(\bar{q}_n)+\varepsilon_N^2 R_2(\bar{q}_n) \\ g_1(\bar{q}_n;\varepsilon_N) &= R_3(\bar{q}_n)+\varepsilon_N R_4(\bar{q}_n) \\ g_2(\bar{q}_n;\varepsilon_N) &= R_5(\bar{q}_n)\;. \end{aligned} \tag{26}$$

Putting this into the Eq. (25) we determine the functions R_1 to R_5 in terms of V. The $p=3$ level solution equals

$$\begin{aligned} g_0 &= V+\varepsilon_N\frac{V''}{12}+\varepsilon_N^2\left[-\frac{V'^2}{24}+\frac{V^{(4)}}{240}\right] \\ g_1 &= \frac{V''}{24}+\varepsilon_N\frac{V^{(4)}}{480} \\ g_2 &= \frac{V^{(4)}}{1920}\;. \end{aligned} \tag{27}$$

Note that W^* depends only on the initial potential V and its derivatives (as well as on ε_N). One can similarly calculate the effective action S^* to any desired level p. We denote the p level truncation of the effective action as $S^{(p)}$. $S^{(p)}$ has the property that its N-fold amplitudes deviate from the continuum expressions as $O(\varepsilon_N^p)$

$$A(a,b;T) = A_N^{(p)}(a,b;T)+O(\varepsilon_N^p)\;. \tag{28}$$

Comparing this to eq. (6) we see that we have just derived the generalization of the Euler summation formula to path integrals. Just as with the ordinary Euler formula it gives the relation between path integrals and their discretizations to any given precision.

It is important to note that one solves for the effective action at level p but once for all theories, i.e. the solution that is found holds for all initial potentials. The only

requirement for the level p solution is that the starting potential is differentiable $2p-2$ times. Solutions for larger values of p are a bit more cumbersome, however, they are just as easy to use in simulations. We have found that the growth in complexity of the effective actions with increasing p has little effect on computation time for $p \leq 4$, while simulations with $p = 9$ are roughly ten times slower due to this. However, this is an extremely small price to pay for a gain of eight orders of magnitude in the speed of convergence. Expressions for effective actions up to $p = 9$ can be found on our web site [16]. The analytical derivations presented work equally well in both the Euclidean

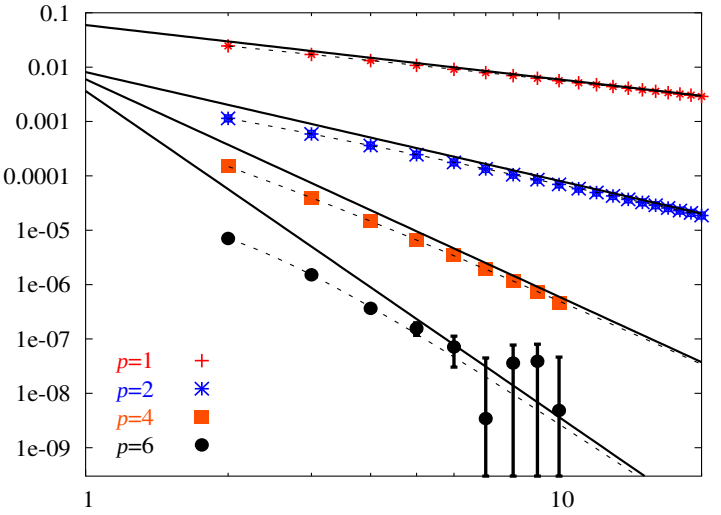

FIGURE 1. Deviations from the continuum limit $|A_N^{(p)} - A|$ as functions of N for $p = 1, 2, 4$ and 6 for an anharmonic oscillator with quartic coupling $\lambda = 10$, time of propagation $T = 1$ from $a = 0$ to $b = 1$. N_{MC} was $9.2 \cdot 10^9$ for $p = 1, 2$, $9.2 \cdot 10^{10}$ for $p = 4$, and $3.68 \cdot 10^{11}$ for $p = 6$. Dashed lines correspond to appropriate $1/N$ polynomial fits to the data. Solid lines give the leading $1/N$ behavior. The level p curve has a $1/N^p$ leading behavior.

and Minkowski formalism (with appropriate $i\varepsilon_N$ regularization), i.e. they are directly applicable to quantum systems as well as to statistical ones. However, the Monte Carlo simulations used to numerically document our analytical results necessarily needed to be done in the Euclidean formalism. We analyzed in detail several models: the anharmonic oscillator with quartic coupling $V = \frac{1}{2}q^2 + \frac{\lambda}{4!}q^4$ and a particle moving in a modified Pöschl-Teller potential over a wide range of parameters. In all cases we found agreement with eq. (28). Fig. 1 illustrates this behavior in the case of an anharmonic oscillator. We see that the p level data indeed differs from the continuum amplitudes as a polynomial starting with $1/N^p$. The deviations from the continuum limit $|A_N^{(p)} - A|$ become exceedingly small for larger values of p making it necessary to use ever larger values of N_{MC} so that the MC statistical error does not mask these extremely small deviations. For $p = 6$ we see that although we used an extremely large number of MC samples ($N_{MC} = 3.68 \cdot 10^{11}$) the statistical errors become of the same order as the

deviations already at $N \gtrsim 8$. For $p = 9$ this is the case even for $N = 2$, i.e. we already get the continuum limit within a MC error of around 10^{-8}.

5. CONCLUSION

A general quantum theory may be written in terms of a quantity which we designate the path integral ideal. We have determined the integral equation satisfied by ideals and have solved it in terms of an asymptotic series in discretized time step. This solution represents a generalization of the well known Euler summation formula to path integrals and leads to the speedup of numerical simulations of path integrals of a general theory by many orders of magnitude. We have also briefly commented on the use of ideals as a natural starting point for extensions of quantization to more complex settings such as p-adic theories.

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