Constrained quantum dynamics and coarse-grained description of a quantum system of nonlinear oscillators

2012 Phys. Scr. 2012 014011

(http://iopscience.iop.org/1402-4896/2012/T149/014011)

IOP PUBLISHING PHYSICA SCRIPTA

Phys. Scr. **T149** (2012) 014011 (3pp) doi:10.1088/0031-8949/2012/T149/014011

Constrained quantum dynamics and coarse-grained description of a quantum system of nonlinear oscillators

Milan Radonjić, Slobodan Prvanović and Nikola Burić

Institute of Physics, University of Belgrade, Pregrevica 118, 11080 Belgrade, Serbia

E-mail: buric@ipb.ac.rs

Received 25 August 2011
Accepted for publication 2 November 2011
Published 27 April 2012
Online at stacks.iop.org/PhysScr/T149/014011

Abstract

Constrained Hamiltonian dynamics is exploited to provide the mathematical framework of a coarse-grained description of the quantum system of nonlinear interacting oscillators. The coarse graining is treated as an equivalence relation on the set of quantum states resulting in the emergence of classical phase space. The equivalence relation imposes constraints on the Hamiltonian dynamics of the quantum system. It is seen that the evolution of the coarse-grained system preserves constant and minimal quantum fluctuations of the fundamental observables. This leads to the emergence of the corresponding classical system on a sufficiently large scale.

PACS numbers: 03.65.Fd, 03.65.Sq

1. Introduction

The relation between quantum and classical mechanics (QC-relation) is very complex with many complementary aspects roughly belonging to two main groups. The first group is related to the problems of formal or mathematical relations between quantum and classical formalisms (for an excellent review, see [1]). Problems of the other group deal with the description of the physical reasons or processes that effectuate the quantum-to-classical transition [2–5]. Kibble [6, 7] pointed out that quantum evolution, determined by the linear Schrödinger equation, can be represented as a Hamiltonian dynamical system on an appropriate phase space. This approach developed into the full geometric Hamiltonian formulation of quantum mechanics [8–15] that provides a suitable framework for discussions of nonlinear constraints imposed on a quantum system [16–18].

In this paper, we consider a system of oscillators possibly nonlinear and interacting. Quantization of a classical system of oscillators is common knowledge from quantum theory [19]. We show that the quantum system of oscillators constrained with a specific type of constraints is equivalent to a finite-dimensional Hamiltonian system that preserves constant and minimal quantum fluctuations of the fundamental observables during the entire evolution. This Hamiltonian system approaches the classical one if some classical parameters are small. We give an interpretation of these results as the mathematical formulation of the emergence of classical systems from a coarse-grained description of quantum systems.

2. Constrained quantum dynamics of a system of nonlinear oscillators

A system of quantum nonlinear oscillators is given by the following Hamiltonian:

$$\hat{H} = \sum_{i=1}^{n} \frac{1}{2m_i}\hat{P}_i^2 + V(\hat{Q}_1, \hat{Q}_2, \ldots, \hat{Q}_n) = \sum_{i=1}^{n} \frac{1}{2m_i}\hat{P}_i^2 + \sum_{i=1}^{n} \frac{m_i\omega_i^2}{2}\hat{Q}_i^2 + \cdots, \qquad (1)$$

where V is some function of $(\hat{Q}_1, \hat{Q}_2, \ldots, \hat{Q}_n)$ having the properties $\partial^2 V/\partial Q_i^2|_{Q_i=0} = m_i\omega_i^2 (i = 1, 2, \ldots, n)$. In the general case when the Hamiltonian is not only quadratic in $\hat{Q}_i$, the dispersions $\Delta\hat{Q}_i$, $\Delta\hat{P}_i$ can have different arbitrary high values in the states along an orbit of $\hat{H}$. As a result, evolution starting from the coherent state manifold will not reside within it. However, we make a constrained system defined by the Hamiltonian (1) and the appropriate set of constraints so that the dispersions of all fundamental quantum observables are preserved during the evolution. The values of the dispersions, the minimal values that can be obtained simultaneously by the coordinates and momenta, are reached if and only if each oscillator is in a coherent state.

We focus on a single nonlinear oscillator case that is sufficient to point out the typical features of the general case. To formulate the constraints, we associate with each state ψ from the manifold of states $\mathcal{M}$ a state $\alpha(\psi)$ on the coherent state manifold Γ such that

$$\alpha(\psi) = (\langle\hat{Q}\rangle_\psi, \langle\hat{P}\rangle_\psi). \tag{2}$$

By definition, the operators $\hat{Q}$ and $\hat{P}$ have expectations in the coherent state $\alpha(\psi)$ the same as in the state ψ. The association of a single coherent state with the whole set of states establishes an equivalence relation, which plays a crucial role in the coarse-graining procedure. Equivalence classes of quantum states determine the corresponding quantum observables that can be seen as physically distinguishable. Thus, in the Hamiltonian system with constraints only functions defined on Γ are considered as physically distinguishable. If two functions on $\mathcal{M}$ correspond to two different operators but generate the same function on Γ, the two operators should be considered as physically indistinguishable. We see that imposing the constraint on the quantum system in fact provides the mathematical representation of a coarse-grained description of the quantum system.

Using the notation (2), we take the following two constraints:

$$\Phi_q = \langle V(\hat{Q})\rangle_\psi - \langle V(\hat{Q})\rangle_{\alpha(\psi)} = 0, \tag{3a}$$

$$\Phi_p = \langle \hat{P}^2\rangle_\psi - \langle \hat{P}^2\rangle_{\alpha(\psi)} = 0, \tag{3b}$$

to be imposed on the oscillator with arbitrary fixed potential $V(\hat{Q})$. The role of the constraints is to preserve during the evolution the association of the set of points $\psi(t)$ with the corresponding single coherent state $\alpha(\psi(t))$.

The total Hamiltonian takes the standard form [20, 21]: $H_{\mathrm{tot}} = \langle\hat{H}\rangle_\psi + \lambda_q\Phi_q + \lambda_p\Phi_p$, with the values of Lagrange multipliers $\lambda_q = -1$ and $\lambda_p = -1/(2m)$, yielding $H_{\mathrm{tot}} = \langle\hat{P}^2\rangle_{\alpha(\psi)}/(2m) + \langle V(\hat{Q})\rangle_{\alpha(\psi)} \equiv \langle\hat{H}\rangle_{\alpha(\psi)}$. Noting that $\langle\hat{P}^2\rangle_{\alpha(\psi)} = \langle\hat{P}\rangle^2_{\alpha(\psi)} + m\omega\hbar/2$ and dropping the irrelevant constant, we finally obtain the total constrained Hamiltonian

$$H_{\mathrm{tot}} = \langle\hat{P}\rangle^2_{\alpha(\psi)}/(2m) + \langle V(\hat{Q})\rangle_{\alpha(\psi)}, \tag{4}$$

which preserves the evolution on the manifold of the coherent states Γ. The total Hamiltonian (4) is, up to an additive constant, on the constrained manifold Γ equal to the initial Hamiltonian $H \equiv \langle\hat{H}\rangle_\psi$. However, while H_{tot} preserves constant and minimal quantum fluctuations of fundamental observables, the evolution with H can in general make them arbitrarily large.

An important fact is that the total Hamiltonian (4) depends only on the variables $q \equiv \langle\hat{Q}\rangle_{\alpha(\psi)}$ and $p \equiv \langle\hat{P}\rangle_{\alpha(\psi)}$, i.e. on the parameters of the coherent state manifold. Thus, the constrained evolution of the fields

$$\dot{\phi}(x) = \frac{\delta H_{\mathrm{tot}}(q,p)}{\delta\pi(x)}, \quad \dot{\pi}(x) = -\frac{\delta H_{\mathrm{tot}}(q,p)}{\delta\phi(x)}, \quad x \in \mathbf{R}^N \tag{5}$$

can be, up to the phase freedom of $\psi(x) = (\phi(x) + \mathrm{i}\,\pi(x))/\sqrt{2}$, inferred from the Hamiltonian evolution of the coherent state parameters

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{p}{m} = \frac{\partial H_{\mathrm{tot}}(q,p)}{\partial p}, \quad \frac{\mathrm{d}p}{\mathrm{d}t} = -\langle V'(\hat{Q})\rangle_{\alpha(\psi)} = -\frac{\partial H_{\mathrm{tot}}(q,p)}{\partial q}, \tag{6}$$

i.e. the dynamics of the constrained system is given by the Hamiltonian dynamics of the constrained manifold parameters.

An important consequence of the constraints is that the association (2) of the set of points $\psi(t)$ with the corresponding *single* coherent state $\alpha(\psi(t))$ is well defined during the entire evolution. In other words, relation (2) provides an equivalence relation on the manifold $\mathcal{M}$. Points from $\psi(t) \in \mathcal{M}$ which give the same expectations $\langle\hat{Q}\rangle$ and $\langle\hat{P}\rangle$ are identified with the single representative: the coherent state $\alpha(\psi)$ having the same expectations. We point out that the equivalence relation is not preserved by the unconstrained Schrödinger evolution of $\psi(t) \in \mathcal{M}$. On the other hand, the constrained evolution is precisely such that it preserves the equivalence of states over time because it can be expressed entirely in terms of the expectations $q = \langle\hat{Q}\rangle$ and $p = \langle\hat{P}\rangle$. In this sense the constrained evolution and the equivalence relation (2) imply each other.

For more than one oscillator, which might be nonlinear and interacting, the condition that $\Delta\hat{Q}_i$ and $\Delta\hat{P}_i$ are simultaneously minimal implies that each of the oscillators is always in some pure coherent state $|\alpha_i(t)\rangle$. Thus, the total state $|\psi(t)\rangle$ is always given by the tensor product of the single oscillator's pure coherent states $|\psi(t)\rangle = \otimes_i|\alpha_i(t)\rangle$, implying for example $\langle\psi(t)|\hat{Q}_1 \otimes \hat{Q}_2|\psi(t)\rangle = \langle\hat{Q}_1\rangle_{\alpha_1(t)} \times \langle\hat{Q}_2\rangle_{\alpha_2(t)} = q_1(t) \times q_2(t)$. Suppression of quantum fluctuations for each oscillator's degree of freedom implies that the degrees of freedom of different oscillators do not get entangled during the evolution. This is enough to generalize the results of the single oscillator analysis to the general case of an arbitrary number of interacting oscillators with constraints.

We now compare the total Hamiltonian (4) on the constrained manifold Γ of the coherent states with $h_{\mathrm{cl}} = p^2/(2m) + V(q)$ representing the Hamilton function of a classical nonlinear oscillator with potential $V(q)$. It can be proven [22] that the total Hamiltonian at a point $\alpha \equiv (q, p)$ on the constrained manifold is

$$H_{\mathrm{tot}} = \frac{p^2}{2m} + V(q) + \sum_{k=1}^{\infty}\frac{1}{2^k k!}\frac{\hbar^k V^{(2k)}(q)}{(2m\omega)^k} \equiv h_{\mathrm{cl}} + \sum_{k=1}^{\infty}\frac{1}{2^k k!}\frac{\hbar^k V^{(2k)}(q)}{(2m\omega)^k}. \tag{7}$$

In the classical limit $\hbar \to 0$ [23], the terms in the sum in (7) tend to zero, yielding $H_{\mathrm{tot}} \to h_{\mathrm{cl}}, \hbar \to 0$.

3. Summary

We have formulated a consistent set of dynamical equations for a system of quantum nonlinear oscillators that maintain the evolution on the coherent state manifold. Because such an evolution preserves minimal fluctuations of fundamental observables, the total Hamiltonian (including the constraints) on coherent state manifold differs from the Hamilton function of a classical nonlinear oscillator with the same interaction potential by terms that are small in the classical limit. This yields the corresponding classical behavior on a sufficiently large scale.

References

[1] Landsman N P 1998 *Mathematical Topics Between Classical and Quantum Mechanics* (New York: Springer)
[2] Zurek W H 2003 *Rev. Mod. Phys* **73** 715
[3] Schlosshauer M 2007 *Decoherence and the Quantum-to-Classical Transition* (Berlin: Springer)
[4] Percival I C 1999 *Quantum State Diffusion* (Cambridge: Cambridge University Press)
[5] Kofler J and Brukner Č 2007 *Phys. Rev. Lett.* **99** 180403
[6] Kibble T W B 1978 *Commun. Math. Phys.* **64** 73
[7] Kibble T W B 1979 *Commun. Math. Phys.* **65** 189
[8] Heslot A 1985 *Phys. Rev.* D **31** 1341
[9] Weinberg S 1989 *Phys. Rev. Lett.* **62** 485
[10] Weinberg S 1989 *Phys. Rev. Lett.* **63** 1115
[11] Weinberg S 1989 *Ann. Phys.* **194** 336
[12] Brody D C and Hughston L P 1996 *Phys. Rev. Lett.* **77** 2851
[13] Brody D C and Hughston L P 1997 *Phys. Lett.* A **236** 257
[14] Gibbons G W 1992 *J. Geom. Phys.* **8** 147
[15] Anandan J and Aharonov Y 1990 *Phys. Rev. Lett.* **65** 1697
[16] Burić N 2008 *Ann. Phys.* **233** 17
[17] Brody D C, Gustavsson A C T and Hughston L 2008 *J. Phys. A: Math. Theor. Phys.* **41** 475301
[18] Brody D C, Gustavsson A C T and Hughston L 2009 *J. Phys. A: Math. Theor. Phys.* **42** 295303
[19] Mackey G W 1963 *Mathematical Foundations of Quantum Mechanics* (New York: Benjamin)
[20] Dirac P A M 1950 *Can. J. Math.* **2** 129
[21] Klauder J R 2010 *A Modern Approach to Functional Integration* (New York: Birkhäuser)
[22] Radonjić M, Prvanović S and Burić N 2011 *Phys. Rev.* A **84** 022103
[23] Kobayashi T 1995 *Nuovo Cimento* B **110** 61