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Entanglement dynamics in systems of qubits with Markov environments

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Abstract
Entanglement dynamics in representative examples of Markov open quantum systems with qualitatively different dynamics are studied. Rings of qubits with thermal or dephasing local environment are used to study the qualitative properties of the entanglement dynamics depending on the interqubit interaction, type of environment and the initial state. It is demonstrated that the effect of the local environment is manifested as an exponential decrease of the entanglement superimposed on the entanglement dynamics in the isolated system.

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1. Introduction

Entanglement dynamics in solid-state systems are of great interest today. This is partly because of the potential use of solid-state devices for quantum information purposes, such as quantum computing and entanglement transport [1, 2], and partly because of the insights entanglement can give on the physics of such devices. The fact that quantum phase transitions can be recognized by a change in entanglement behaviour is an example of such an insight [3, 4].

Classically, the dynamics of a system can be either integrable or nonintegrable. In quantum mechanics, nonintegrability is not so easily defined since any bounded finite system must be quasi-periodic. Still, for spin systems, a system is often considered quantum integrable if it is completely solvable in the thermodynamic limit and nonintegrable otherwise [5]. It is just as difficult to define quantum chaos. A working definition that is consistent with the observed data uses the nearest-neighbour level spacing (NNLS) of the energy spectra of systems. For integrable systems, a Poissonian NNLS is observed, whereas systems considered to be quantum chaotic have a Wigner NNLS. The Wigner NNLS is also observed in Hamiltonians constructed from random matrices [6].

There are indications that a change in entanglement behaviour could also serve as a general indicator of quantum chaos similar to the classical Lyaponov exponents [7, 8]. For example, for some spin chains, the von Neumann entropy of half the chain increases linearly for quantum chaotic parameters and logarithmically for integrable parameters with the chain length [9, 10]. The entanglement in a quantum chaotic spin chain is expected to be predominantly multipartite, at the cost of pairwise entanglement [9–11]. However, in [10], it was questioned whether this behaviour is just the result of mixing of eigenfunctions rather than quantum chaos.

Since no realistic quantum system is really isolated, it makes sense to ask: what is the behaviour of entanglement in integrable and nonintegrable spin chains exposed to an outer environment? The influence of different temperatures of a thermal environment has been studied for the Heisenberg and transverse Ising models [12] and a realistic model of Josephson junctions [13]. Interestingly, measured by the logarithmic negativity, some parameter values were found to have a steady-state entanglement different from 0 even in the presence of an environment [13].

We have used the model of [5, 9] to closely study the decay of entanglement under different dynamics and environments. As a measure of pairwise entanglement, we have used entanglement of formation [14], $E(|\Psi\rangle)$. We have found an exponential decrease in entanglement, resulting in the vanishing of all entanglement. Our results will be presented for one of the states examined.

2. Methods

Instead of solving the Lindblad equation, we have used quantum state diffusion (QSD) formalism. In the QSD picture, a quantum state diffuses through state space not unlike the diffusive motion of a Brownian particle in water. Itô calculus, which has been developed for stochastic differentials, serves as the mathematical backbone of the theory. In complex Itô calculus, there are two kinds of differentials, ordinary dt as well as complex stochastic differentials dξ [15].

2.1. The QSD equation

QSD is just one way to get a consistent stochastic Schrödinger equation. It is based on the Itô calculus diffusion equation [15]

$$|\mathrm{d}\Psi\rangle = \sum_i (|v_i\rangle \mathrm{d}t + |a_i\rangle \mathrm{d}\xi_i). \tag{1}$$

To this is added that the stochastic differentials dξ_i are time independent, that the state vector remains normalized and that the ensemble mean of all state vector tractories must generate a density matrix obeying the Lindblad equation,

$$\rho(t) = M|\Psi(t)\rangle\langle\Psi(t)|. \tag{2}$$

These four assumptions give the following unique form to the QSD equation [15]:

$$|\mathrm{d}\Psi(t)\rangle = -\mathrm{i}H_\Delta|\Psi(t)\rangle\mathrm{d}t - \sum_k \tfrac{1}{2}L^2_{\Delta k}|\Psi(t)\rangle\,\mathrm{d}t + \sum_k L_{\Delta k}|\Psi(t)\rangle\,\mathrm{d}\xi_k. \tag{3}$$

The operators H_Δ and $L_{\Delta k}$ are shifted versions of the system Hamiltonian H and the environment Lindblad operators L_k [15]:

$$\begin{aligned} H_\Delta &= H - \langle H\rangle, \\ L_{\Delta k} &= L_k - \langle L_k\rangle. \end{aligned} \tag{4}$$

This equation has been numerically solved using enough trajectories not to see any difference depending on the random number series used.

2.2. Entanglement of formation

In the asymptotic limit the entanglement of formation $E(|\mathrm{Psi}\rangle)$ of a two-qubit state $|\mathrm{Psi}\rangle$ is a unique measure of pairwise entanglement [16]. Even if entanglement of formation loses its unique status for finite numbers of pairs $|\Psi\rangle$, it is still a useful entanglement measure, since it can be calculated exactly for the mixed state density matrix $\rho_{(ij)}$ of two qubits, i and j. This is done in terms of the quantity called concurrence, $C(\rho_{(ij)})$ [17, 18],

$$E(\rho) = H\left(\frac{1+\sqrt{1-C(\rho)^2}}{2}\right), \tag{5}$$

with $H(x)$ being the binary entropy

$$H(x) = -x\log_2 x - (1-x)\log_2(1-x). \tag{6}$$

Concurrence itself can be calculated from the eigenvalues of the matrix $\rho\tilde{\rho}$ with $\tilde{\rho} = (\sigma_2\otimes\sigma_2)\rho^*(\sigma_2\otimes\sigma_2)$. If $\lambda_1\cdots\lambda_4$ are the eigenvalues in descending order, the concurrence is [18]

$$C(\rho) = \max\left(0, \sqrt{\lambda_1}-\sqrt{\lambda_2}-\sqrt{\lambda_3}-\sqrt{\lambda_4}\right). \tag{7}$$

In order to obtain the relevant subsystem density matrix for the qubits i and j, the expectation values $\langle\sigma^i_k\sigma^j_l\rangle$ have been extracted from the QSD simulations. The indices $k, l = 1, \ldots, 3$ label the qubit Pauli matrices and $k, l = 0$ is the identity matrix. Since the 16 $\sigma^i_k\otimes\sigma^j_l$ matrices are a basis for all 4×4 matrices, the two-qubit density matrices can be reconstructed.

To better see the loss of entanglement, we have also calculated the quantity

$$E_{\mathrm{avr}}(\rho_{(ij)}(t)) = \frac{1}{t}\int_0^t E(\rho_{(ij)}(s))\,\mathrm{d}s. \tag{8}$$

Since E_{avr} takes a long time to go to zero when compared with the open system entanglement evolution we have also calculated

$$\begin{aligned} E_{\mathrm{avr}}(\rho_{(ij)}(t), \Delta T) &= \frac{1}{\Delta T'}\int_{t-\Delta T}^t E(\rho_{(ij)}(s))\,\mathrm{d}s, \\ \Delta T' &= \begin{cases} t, & t < \Delta T, \\ \Delta T, & \Delta T \leqslant t. \end{cases} \end{aligned} \tag{9}$$

3. Model

Our model is the same as that studied in [9] and also described in [5]. A kicked version of it has been studied in [11]. We have used a ring of six qubits with the Hamiltonian

$$H = \sum_{i=1}^{6}(h_x\sigma^i_x + h_z\sigma^i_z) + J\sum_{i=1}^{6}\sigma^i_x\sigma^{i+1}_x. \tag{10}$$

Since the qubits are in a ring, we have the boundary condition $\sigma^7_x = \sigma^1_x$. When $h_z = 0$ or $h_x = 0$ the model is quantum integrable. The case $J = 1$, $h_x = 1.4$ and $h_z = 0$ will henceforth be called the symmetric case and $J = 1$, $h_x = 0$ and $h_z = 1.4$ will be called the nonsymmetric case. The nonsymmetric case is the same as the transverse Ising model and thus only solvable by the Jordan–Wigner transformation [19]. As an example of a nonintegrable Hamiltonian, we have used $J = 1$, $h_x = 1.0$ and $h_z = 1.0$.

The Lindblad operators of a thermal environment are [20, 21]

$$L_i = \frac{\Gamma(\bar{n}+1)\sigma^i_-}{2} + \frac{\Gamma\bar{n}\sigma^i_+}{2} \tag{11}$$

and those for the dephasing environment are [20]

$$L_i = \Gamma\sigma^i_+\sigma^i_-. \tag{12}$$

We have treated the parameters Γ and $\bar{n}$ as constants and used the values $\Gamma = 0.03$ and $\bar{n} = 0.5$ in our calculations. Temperature dependence has been studied in [12, 13].

3.1. Initial states

A common initial state for simulating entanglement dynamics in spin chains is

$$|\Psi_{\max}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\otimes|\downarrow\downarrow\downarrow\downarrow\rangle. \tag{13}$$

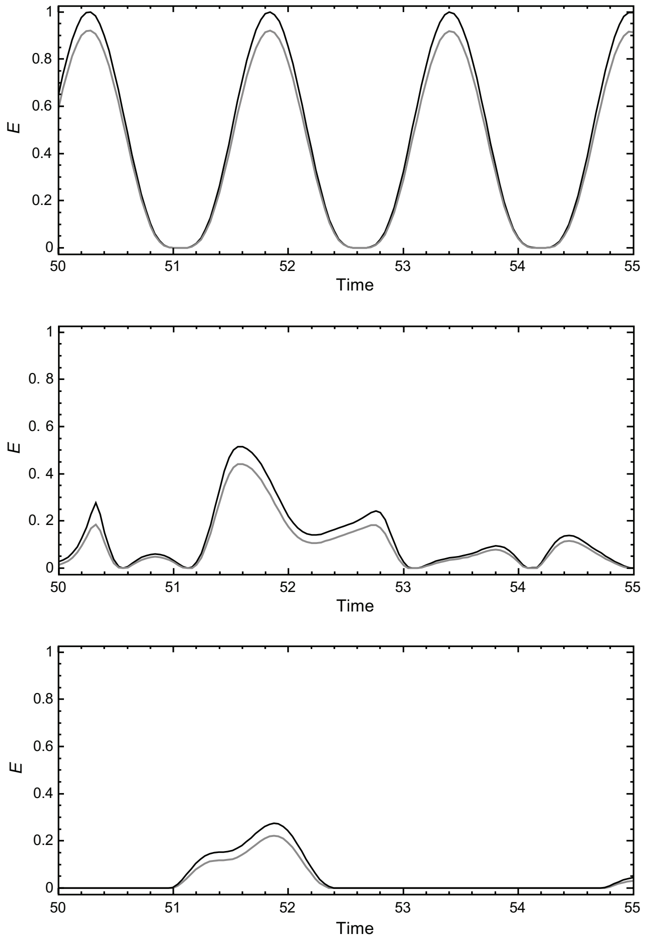

Figure 1. Entanglement dynamics for the qubits of the Bell pair of $|\Psi_{\max}\rangle$. Lines in black (grey) show entanglement for the isolated (open) system. Top: symmetric Hamiltonian. Middle: nonsymmetric Hamiltonian. Bottom: nonintegrable Hamiltonian.

This state, for example, allows for the demonstration of entanglement transport. We have studied this state as well as separable states such as $|\mathrm{Psi}_{\mathrm{sep}}\rangle = |\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\rangle$ and the W state [22]. We have also studied these states with all spins in the opposite direction.

4. Results

All the states studied have given qualitatively the same results and therefore we will present only those for $|\Psi_{\max}\rangle$. Both dephasing and thermal environments were found to give an exponential dampening of the entanglement dynamics. Only the rate of dampening differed. Time is given in dimensionless units $t \cdot J$.

The dynamics of the two qubits of the Bell pair of $|\Psi_{\max}\rangle$ are shown for the three cases in figure 1. The dynamics between the three cases are radically different. For the symmetric case, the other qubit pairs are uninfluenced by the Bell pair. However, in the nonsymmetric case, the Bell state entanglement is periodically transported between the original Bell state and the qubit pair on the other side of the ring. For the nonintegrable dynamics, the pairwise entanglement disappears quickly, which is as expected [9–11].

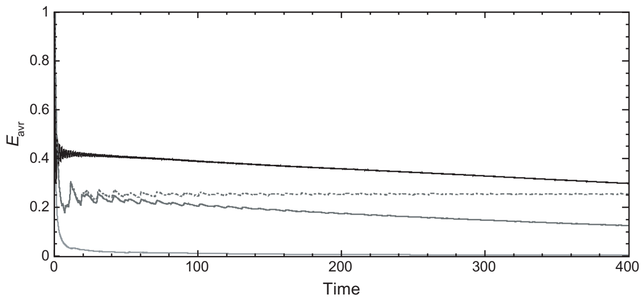

Figure 2. E_{avr} is plotted for $|\Psi_{\max}\rangle$ in an open system. Black: symmetric Hamiltonian. Dark grey upper line (dashed): isolated E_{avr} for the nonsymmetric Hamiltonian. Dark grey lower line: nonsymmetric Hamiltonian. Light grey line: nonintegrable Hamiltonian.

$E_{\mathrm{avr}}(\rho_{12}(t))$ for $|\Psi_{\max}\rangle$ is shown in figure 2. The averaged entanglement $E_{\mathrm{avr}}(\rho_{ij}(t))$ converges to an exponential decay. This is also true of $E_{\mathrm{avr}}(\rho_{(ij)}(t), \Delta T)$ large enough to average out the oscillations of entanglement.

5. Summary

We have examined the behaviour of three distinct entanglement dynamics in an open system for various initial states. The amount of entanglement is certainly highly dependent on the state and the Hamiltonian. The qualitative complexity of entanglement dynamics mainly depended on the integrability as well as symmetry of the Hamiltonian. Both the thermal and dephasing environments were seen to cause an exponential decrease in entanglement, resulting in the disappearance of all two-qubit entanglements.

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