

Improved Gaussian Approximation

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Abstract

In a recently developed approximation technique [1] for quantum field theory the standard one-loop result is used as a seed for a recursive formula that gives a sequence of improved Gaussian approximations for the generating functional. In this paper we work with the generic $\phi^3 + \phi^4$ model in $d = 0$ dimensions. We compare the first, and simplest, approximation in the above sequence with the one-loop and two-loop approximations, as well as the exact results (calculated numerically).

The central object in quantum field theory is the generating functional $Z[J]$. Functional derivatives of $Z[J]$ with respect to the external fields $J(x)$ give the Green's functions of the theory. The generating functional is determined from the (Euclidian) action $S[\phi]$ through the path integral

$$Z[J] = \int [d\phi] e^{-(S[\phi] - \int dx J(x)\phi(x))} . \quad (1)$$

The integration measure is, formally, simply $[d\phi] = \prod_{x \in \mathbb{R}^d} d\phi(x)$, where d is the dimension of space-time. In this paper we will work with models in $d = 0$ dimensions. In $d = 0$ functionals become functions, and the path integral reverts to a single definite integral over the whole real line

$$Z(J) = \int d\phi e^{-(S(\phi) - J\phi)} . \quad (2)$$

Two further important objects are $W(J)$ — the generator of connected diagrams (or free energy)

$$Z(J) = Z(0) e^{-W(J)} , \quad (3)$$

and the quantum average of the field $\varphi = \langle \phi \rangle = -\frac{\partial}{\partial J} W(J)$.

In the Gaussian approximation, we Taylor expand the action in the path integral around some reference point ϕ_{ref} , and keep terms that are at most quadratic in $\phi - \phi_{\text{ref}}$. The integral in (2) is now a Gaussian and we find

$$W_{\text{Gauss}}(J, \phi_{\text{ref}}) = S(\phi_{\text{ref}}) - J\phi_{\text{ref}} + \frac{1}{2} \ln S''(\phi_{\text{ref}}) - \frac{1}{2} \frac{(S'(\phi_{\text{ref}}) - J)^2}{S''(\phi_{\text{ref}})} . \quad (4)$$

For this approximation to make sense, the integral must get its dominant contribution from the vicinity of the reference point ϕ_{ref} . The standard Gaussian approximation (loop expansion) corresponds to the choice $\phi_{\text{ref}} = \phi_{\text{class}}(J)$, where ϕ_{class} is the solution of the classical equation of motion $S' = J$. The classical solution is the maximum of the integrand in (2).

In a previous paper [1] we expanded the integrand around the average field φ . As we have shown, although the classical solution gives the maximum of the integrand, expansion around φ gives a better approximation for the area under the curve. The Gaussian approximation around the average field φ is simply

$$W_{\text{Gauss}}(J, \varphi) = S(\varphi) - J\varphi + \frac{1}{2} \ln S''(\varphi) - \frac{1}{2} \frac{(S'(\varphi) - J)^2}{S''(\varphi)}. \quad (5)$$

To be able to calculate this *in closed form* we need to know $\varphi(J)$, which is tantamount to knowing how to do the theory exactly, since φ and its derivatives give all the connected Green's functions. The use of equation (5) comes about when one solves it iteratively. Using the definition of φ in terms of W , as well as equation (4) we obtain the following iterative process

$$\varphi_{n+1}(J) = -\frac{d}{dJ} W_{\text{Gauss}}(J, \varphi_n(J)). \quad (6)$$

For the seed of this iteration we chose the classical field, i.e. $\varphi_0 = \phi_{\text{class}}$. In this way one obtains a sequence of points $\varphi_0, \varphi_1, \varphi_2, \dots$ or equivalently of approximations to the connected generating functional W_1, W_2, W_3, \dots given by $W_{n+1}(J) = W_{\text{Gauss}}(J, \varphi_n(J))$. In [1] we have shown that this sequence gives better and better approximations and converges (though slowly) to the best Gaussian approximation φ_∞ . This is shown in Figure 1. Note

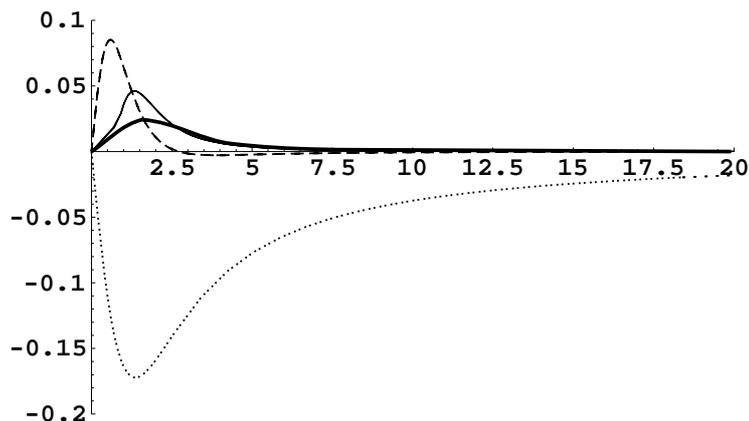


Figure 1: Plots of $\varphi - \varphi_0$ (dotted line), $\varphi - \varphi_1$ (dashed line), $\varphi - \varphi_2$ (thin line) and $\varphi - \varphi_\infty$ (thick line) as functions of J . The action is given in (8) with couplings $g_3 = 0$, $g_4 = 1$.

that φ_∞ is still not equal to the exact result φ . The reason for this is obvious: We used the

Gaussian approximation W_{Gauss} in defining our recursive relation, and there is no reason to expect that this converges to the exact result.

So far we have seen that we can improve on the usual loop expansion. However, the sequence of improved Gaussian approximations converges very slowly. For this reason it is interesting to look at the first approximations in this sequence and compare them to standard approximation schemes. As we have seen, $\varphi_0 = \phi_{\text{class}}$, so $W_1(J)$ is just the one-loop result. Our first new approximation is therefore $W_2(J) = W_{\text{Gauss}}(J, \varphi_1)$, where $\varphi_1 = -\frac{d}{dJ}W_1$. The nice thing about this approximation is that it is only a bit more complicated than the one-loop result. From now on we will study this approximation, and compare it to one-loop and two-loop results, as well as to the exact results that have been calculated numerically. In what follows we will designate W_2 as the improved Gaussian approximation.

For our comparison we have looked at the (full) n -particle Green's functions

$$G_n = \frac{\int d\phi \phi^n e^{-S(\phi)}}{\int d\phi e^{-S(\phi)}} . \quad (7)$$

The model we considered was

$$S(\phi) = \frac{1}{2} \phi^2 + \frac{1}{3!} g_3 \phi^3 + \frac{1}{4!} g_4 \phi^4 . \quad (8)$$

All our calculations have been done for $g_4 > \frac{3}{8} g_3^2$ where the above action has a unique minimum at $\phi = 0$. For the two-point function the one-loop approximation is given in Figure 2. In this plot we see the expected perturbative region in the vicinity of the

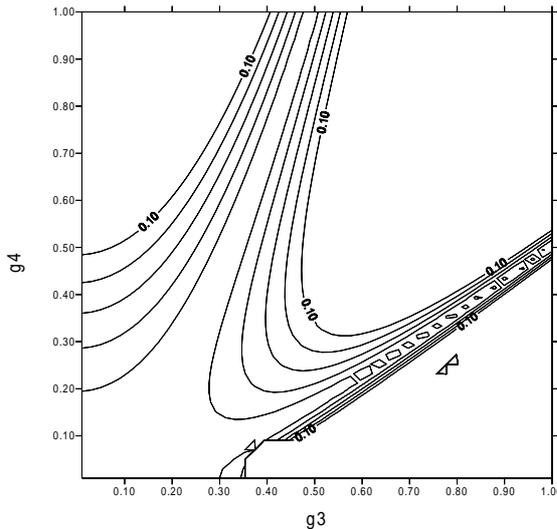


Figure 2: Absolute values of relative errors of the one-loop approximation to G_2 . Only the contours corresponding to $|\delta G_2^{1\text{-loop}}| < 0.1$ are shown.

origin. What is not immediately obvious is the meaning of the two regions sprouting off to large values of the coupling constants. Still, it is rather easy to give a simple hand-waving argument: The one-loop result for G_n is an $(n - 1)^{\text{st}}$ order polynomial in the couplings, while the exact result is a relatively slowly varying, monotonous function (on the range of interest). As a consequence, $\delta G_n^{1\text{-loop}}$ can vanish on (at most) n curves in the g_3, g_4 plane. The sprouting regions in Figure 2 flank these two curves. A similar plot of $\delta G_1^{1\text{-loop}}$ has one sprouting region. What is important to note is that the sprouting regions corresponding to different Green's function have nothing to do with each other. In fact, they are just a manifestation of the old saw that a stopped clock gives the correct time of day twice during each day. The same plot for the two-loop and improved Gaussian approximations is shown in Figure 3. As before, we again have central (perturbative)

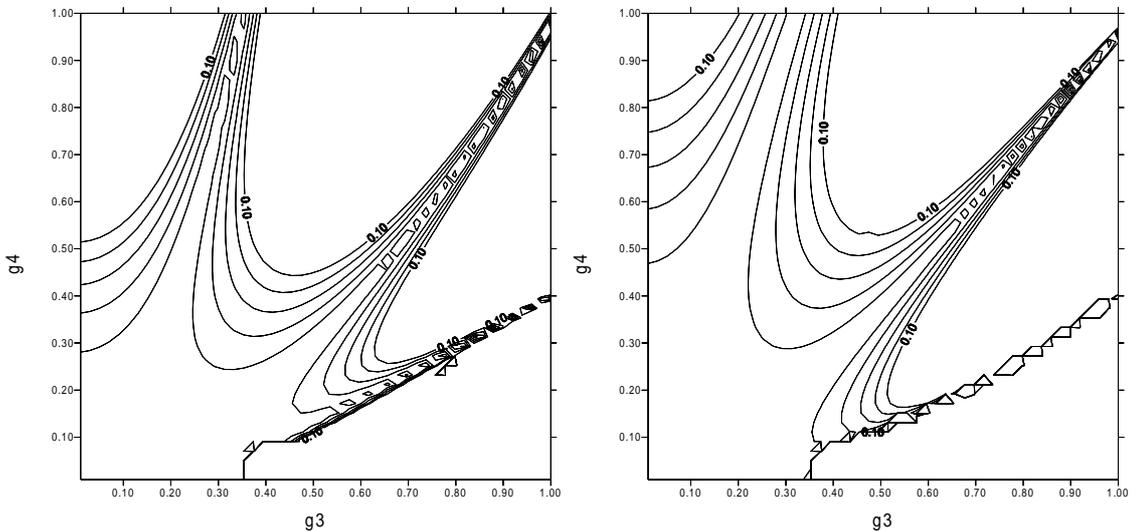


Figure 3: Contour plot of $|\delta G_2^{2\text{-loop}}| < 0.1$ (left), and $|\delta G_2^{\text{improved}}| < 0.1$ (right)

regions and a certain number of sprouting regions. These plots clearly show that the improved Gaussian approximation outperforms both the one-loop and two-loop results. Figure 4 gives the same comparison of one-loop, two-loop and improved results for the one-point Green's function.

We have looked at all the Green's function from G_1 to G_6 . In all cases the improved Gaussian approximation gives the best result, however the advantage becomes less marked when one looks at higher Green's function.

As a conclusion, we have shown in this paper that even the simplest improved Gaussian approximation gives better agreement with exact results than both one-loop and two-loop approximations. At the same time, the computational cost of the improved Gaussian approximation is negligibly greater than that of the one-loop result, and significantly smaller than of the two-loop result. In our previous paper we looked at the benefits of the

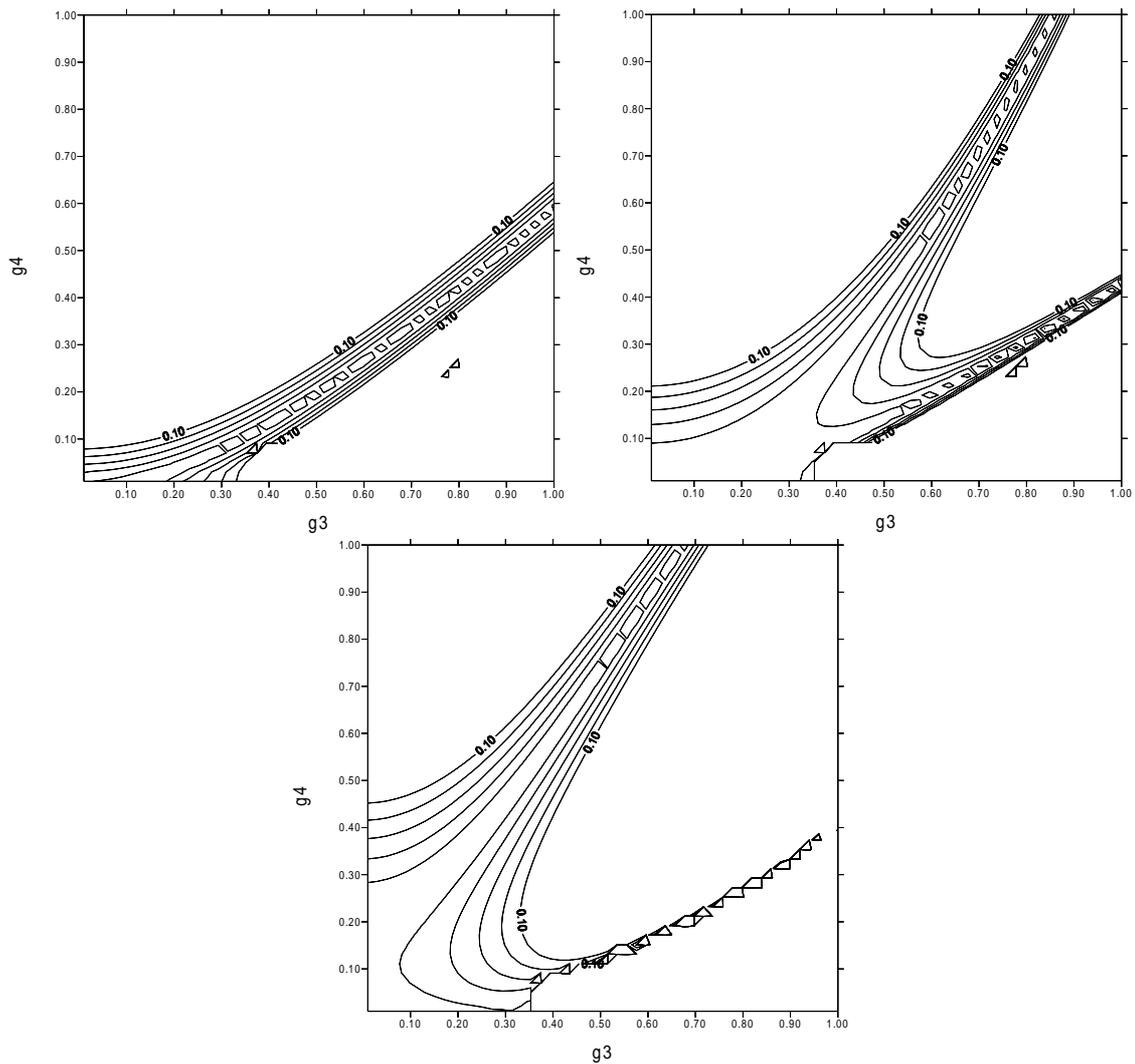


Figure 4: $|\delta G_1^{1\text{-loop}}| < 0.1$ (top left), $|\delta G_1^{2\text{-loop}}| < 0.1$ (top right), $|\delta G_1^{\text{improved}}| < 0.1$ (bottom)

improved Gaussian approximation both from the analytic and numerical (Monte Carlo) sides. We are currently working [2] on extending those results to interesting models in $d \geq 1$.

References

- [1] A. Balaž, A. Belić and A. Bogojević, Preprint IP-HET-98/11, April 1998.
[hep-th/9804207](#)
- [2] A. Balaž, A. Belić and A. Bogojević, Work in progress