The Measure from Schwinger-Dyson Equations

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Abstract

We review a new prescription for calculating the Lagrangian path integral measure directly from the Hamiltonian Schwinger-Dyson equations. The method agrees with the usual way of deriving the measure in which one has to perform the path integration over momenta.

Linearity (of amplitudes) lies at the heart of any quantum theory. In the formalism of quantum field theory this linearity is encoded in the Schwinger-Dyson equations. In terms of the generating functional $Z[J]$ we have

$$\left(\frac{\delta S}{\delta q}\Big|_{q=\frac{\hbar}{i}\frac{\delta}{\delta J}}+J\right)Z[J]=0\ , \tag{1}$$

where $S[q]$ generates the Feynman rules: S'' is the inverse of the Feynman propagator, $S^{(n)}$ are the n-point vertices. Equation (1) is a linear (functional) differential equation for $Z[J]$. A formal functional Fourier transform of this is just the Feynman path integral

$$Z[J]=\int[dq]\,e^{\frac{i}{\hbar}\left(S[q]+\int dt\,Jq\right)}\ . \tag{2}$$

The $\hbar\to 0$ limit of $Z[J]$ is dominated by configurations near to the solutions of $\frac{\delta S}{\delta q}+J=0$. On the other hand, $\hbar\to 0$ corresponds to classical physics given by $\frac{\delta I}{\delta q}+J=0$, where I is the action. From this we see that

$$S[q]=I[q]+\frac{\hbar}{i}\,M[q]\ . \tag{3}$$

$M[q]$ is the measure term. This is as far as the usual functional formalism takes us — namely, there is no way to determine the measure term[1]. The only way to do this has

[1]It is common to write the integrand of (2) in terms of the action. The path integral measure is then $d\mu=[dq]\,\exp M[q]$. Lack of knowledge about M translates into lack of knowledge about $d\mu$.

been to make connection with the operator formalism. From it we find an expression for the generating functional in terms of a Hamiltonian path integral

$$Z[J] = \int [dp\, dq] \exp \frac{i}{\hbar} \int dt \,(p\dot{q} - H(q,p) + Jq) \ . \tag{4}$$

Here the measure is trivial. The Lagrangian expressions, including the corresponding measure, are obtained by doing the momentum path integral (see for example [1]).

In a previous paper [2] we have developed an alternate way for calculating the measure *inside* the functional formalism. To do this we start from the Hamiltonian form of the Schwinger-Dyson equations. These may be written as

$$\left(\dot{P} + \frac{\partial H(Q,P)}{\partial Q} - J\right) Z[J,K] \;\; = \;\; 0 \tag{5}$$

$$\left(\dot{Q} - \frac{\partial H(Q,P)}{\partial P} + K\right) Z[J,K] \;\; = \;\; 0 \ , \tag{6}$$

where K is an additional source term that couples to momenta[2], $P = \frac{\hbar}{i}\frac{\delta}{\delta K}$, and $Q = \frac{\hbar}{i}\frac{\delta}{\delta J}$. With this nomenclature the above Schwinger-Dyson equations look just like the classical Hamiltonian equations of motion. The only difference is that we have the following non-zero commutators

$$[P, K] = [Q, J] = \frac{\hbar}{i} \ . \tag{7}$$

Note that in this formalism P and Q commute. By using the above equations we can derive the Lagrangian path integral measure. As an example let us look at a model whose Hamiltonian is simply

$$H(q,p) = \frac{1}{2}\, p^2 + V(q) \ . \tag{8}$$

In this case the Schwinger-Dyson equations read

$$\left(\dot{P} + V'(Q) - J\right) Z[J,K] \;\; = \;\; 0 \tag{9}$$

$$\left(\dot{Q} - P + K\right) Z[J,K] \;\; = \;\; 0 \ . \tag{10}$$

Differentiating the second of these equations with respect to time, and then adding it to the first, we get an equation for Q alone

$$\left(\ddot{Q} - V'(Q) - J\right) Z[J] = 0 \ , \tag{11}$$

[2]The corresponding Hamiltonian path integral is

$$Z[J,K] = \int [dp\, dq] \, \exp \frac{i}{\hbar} \int dt \ (p\dot{q} - H(q,p) + Jq + Kp) \ ,$$

where the $[dp\, dq]$ measure is trivial.

where we have now turned off the source for momenta. The action for this model is $I[q] = \int dt\, (\frac{1}{2}\,\dot{q}^2 - V(q))$. In terms of it the Schwinger-Dyson equations are simply

$$\left(\frac{\delta I}{\delta Q} + J\right) Z[J] = 0 \ . \tag{12}$$

Fourier transforming this we obtain the usual Lagrangian path integral

$$Z[J] = \int [dq]\, \exp \frac{i}{\hbar} \int dt\, \left(\frac{1}{2}\,\dot{q}^2 - V(q) + Jq\right) \ . \tag{13}$$

We have just derived the well known result that the path integral measure is trivial for models whose Hamiltonian is of the simple form given in (8).

Now let us look at a bit more complicated example. We consider a model with Hamiltonian given by

$$H(q,p) = \frac{1}{2}\, g^{-1}(q) p^2 + V(q) \ . \tag{14}$$

The Hamiltonian Schwinger-Dyson equations are now

$$\begin{aligned} \left(\dot{P} - \frac{1}{2} g^{-2}(Q) g'(Q) P^2 + V'(Q) - J\right) Z[J,K] &= 0 && (15)\\ \left(\dot{Q} - g^{-1}(Q) P + K\right) Z[J,K] &= 0 \ . && (16) \end{aligned}$$

We may write the second equation as $PZ = g\,(\dot{Q} + K) Z$ and use this to get rid of the P terms in the first equation. Therefore

$$\begin{aligned} P^2 Z &= P(g\dot{Q} + gK)Z = \\ &= (g\dot{Q} + gK)PZ + [P,K]gZ = \left((g\dot{Q} + gK)^2 + \frac{\hbar}{i}\, g\right) Z \ , \end{aligned} \tag{17}$$

as well as

$$\dot{P}Z = (g'\dot{Q}(\dot{Q} + K) + g(\ddot{Q} + \dot{K}))Z \ . \tag{18}$$

Setting $K = 0$ we find

$$(g\ddot{Q} + \frac{1}{2} g'\dot{Q}^2 + V' - \frac{1}{2}\,\frac{\hbar}{i}\,(\ln g)' - J) Z[J] = 0 \ . \tag{19}$$

This equation can again be written as $\left(\frac{\delta S}{\delta Q} + J\right) Z[J] = 0$, where we have $S = I + \frac{\hbar}{i} M$. The first term is just the action $I[q] = \int dt\, (\,\frac{1}{2} g(q)\dot{q}^2 - V(q))$. The measure term equals $M = \int dt\, \ln\sqrt{g}$. Fourier transforming the last equation we find

$$Z[J] = \int \prod_t \left(dq(t)\sqrt{g(q)}\right) \exp \frac{i}{\hbar}(I + \int dt\, Jq) \ . \tag{20}$$

This agrees with the standard derivation of the Lagrangian path integral in which one performs the Gaussian momentum integration in the Hamiltonian path integral.

The generalization of the previous example to more variables gives us the σ-model

$$L = \frac{1}{2}\, g_{\alpha\beta}(q)\dot{q}^\alpha \dot{q}^\beta \ . \tag{21}$$

The Hamiltonian is given in terms of the inverse metric $g^{\alpha\beta}$, and equals $H = \frac{1}{2}\, g^{\alpha\beta} p_\alpha p_\beta$. The Schwinger-Dyson equations become

$$\begin{aligned} \left(\dot{P}_\alpha + \frac{1}{2} g^{\gamma\delta}_{\ \ ,\alpha} P_\gamma P_\delta - J_\alpha\right) Z[J,K] &= 0 & (22)\\ \left(\dot{Q}^\alpha - g^{\alpha\beta} P_\beta + K^\alpha\right) Z[J,K] &= 0 \ . & (23) \end{aligned}$$

Just as in the previous example it is a simple exercise to get rid of the P terms and derive the Lagrangian Schwinger-Dyson equation. It may be compactly written as

$$\left(\frac{\delta I}{\delta Q^\alpha} - i\hbar\, \frac{1}{\sqrt{g}}\, \partial_\alpha \sqrt{g} + J_\alpha\right) Z[J] = 0 \ , \tag{24}$$

where $g = \det\, g_{\alpha\beta}$. The corresponding path integral has the familiar form

$$Z[J] = \int \prod_t \left(dq(t)\sqrt{g(q)}\right) \exp\left(\frac{i}{\hbar}(I + \int dt\, J_\alpha q^\alpha)\right) \ . \tag{25}$$

From these examples it is obvious that the generalization from 1-dimensional field theory, i.e. quantum mechanics, to d-dimensional field theory is trivial. The d-dimensional expressions just contain more dummy labels. What is not trivial, when one tackles full-fledged field theory, is how to deal with gauge symmetries. Therefore, it would be very interesting to extend this work to the treatment of gauge theories, and re-derive the measures obtained by Faddev-Popov and Batalin-Vilkovisky. Another interesting avenue of research is to try to use the above method to derive an explicit differential equation satisfied by the measure term. Doing this would enable us to complete what Dirac and Feynman started: To define a complete quantum theory in terms of the Lagrangian.

References

[1] C. Itzykson and J.-B. Zuber, “Quantum Field Theory”,
McGraw-Hill, New York, 1980

[2] A. Bogojević and D. S. Popović, Preprint IP-HET-98/9, April 1998.