Eur. Phys. J. B **53**, 225–232 (2006)
DOI: 10.1140/epjb/e2006-00369-4

THE EUROPEAN PHYSICAL JOURNAL B

Linear kinetic equation: long-time behavior of one-particle distribution function

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Received 29 June 2006 / Received in final form 31 August 2006
Published online 13 October 2006 –

Abstract. We construct asymptotic (long-time) solution of the linear Boltzmann equation using the time-dependent perturbation theory generalized to non-Hermitian operators. We prove that for times much larger than the relaxation time τ_0, $t \gg \tau_0$, one-particle distribution function separates into spatio-temporal and velocity dependent parts, and provide the explicit expression for the long-time solution of the linear Boltzmann equation. Our analysis does not assume that relative density gradients $n^{-1}(\partial/\partial\vec{r})n$ are small. It relates the hydrodynamic form of the one-particle distribution function to spectral properties of operators involved in linear Boltzmann equation.

PACS. 51.10.+y Kinetic and transport theory of gases – 05.20.Dd Kinetic theory

1 Introduction

An important problem in non-equilibrium statistical mechanics is the quantitative description of transport properties of an impurity immersed in a bath of fluid particles and subjected to the action of one or more external fields. The transport of impurities is a common non-equilibrium problem and appears in various fields of physics. A typical example is the charged particle swarm in neutral gases. The kinetic theory of charged particle swarms (electrons, ions, positrons, muons, etc.) in the presence of electric and magnetic fields has been developed substantially over the last forty years. The reader is referred to the reviews of Kumar et al. [1,2] and White et al. [3].

In recent years, intensive theoretical and numerical studies of granular swarms have been carried out with the aim of clarifying the form and validity of a hydrodynamic description for granular gases [4–9]. Drift and diffusion of impurities in a dilute granular gas is the simplest example of non-equilibrium transport in a multicomponent granular gas. Understanding the transport theory of granular swarms is a necessary step in describing rapid flows in real granular mixtures.

The swarm particles may be subject to the gravitational force and/or, if they are ions, to electric and/or magnetic fields. We indicate with $\vec{F}^{\mathrm{ext}}$ the total external force acting on a swarm particle of mass m. Swarm particle concentrations are assumed sufficiently low that both the mutual interactions between the swarm particles and the influence of the swarm on the background fluid can be neglected. Under these conditions (tracer limit), the Boltzmann equation corresponding to the one-particle distribution function $f(\vec{r},\vec{v},t)$ of swarm particles can be written as

$$\left[\frac{\partial}{\partial t} + \vec{v}\cdot\frac{\partial}{\partial \vec{r}} + \vec{a}\cdot\frac{\partial}{\partial \vec{v}}\right] f(\vec{r},\vec{v},t) = J[f](\vec{r},\vec{v},t). \qquad (1.1)$$

Here vector $\vec{a} = \vec{F}^{\mathrm{ext}}/m$ is the acceleration on a particle produced by external field, which is assumed to be both space and time independent. The scalar operator J is local in space and in time and accounts for the rate of change of f due to various types of collisions between swarm particles and background fluid particles. By virtue of the low swarm particle concentrations it is linear operator which acts on f only through its $\vec{v}$ dependence. This property of linearity is shared by many systems studied in kinetic theory such as neutron or radiation transport and Lorentz gas model.

In swarm physics there are two timescales, a mean free time, and a macroscopic time which is usually drift time. Since energy is constantly being added by an external field, these systems do not approach equilibrium. However, after several collision times the velocity distribution relaxes to a 'local steady state', in which the energy gained by the swarm from external field is balanced by the energy loss due to collisions with the background fluid. This state is called the hydrodynamic regime (HDR) by analogy with the hydrodynamics of neutral gases.

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It is usually assumed that the space-time dependence of swarm one-particle distribution function is given by [1,2]:

$$f(\vec{r},\vec{v},t) = \sum_{p=0}^{\infty} \hat{f}^{(p)}(\vec{v}) \odot_p \left(-\frac{\partial}{\partial \vec{r}}\right)^p n(\vec{r},t). \qquad (1.2)$$

The coefficients $\hat{f}^{(p)}(\vec{v})$ are velocity-dependent tensors of rank p and $\odot_p$ indicates a p-fold scalar product. It should be noted that the space-time dependence of $f(\vec{r},\vec{v},t)$ in HDR is entirely carried by the number density. In contrast, the hydrodynamic regime of neutral gas is described by a coupled set of equations for five fields (n, $\vec{u}$ and T). The Chapman-Enskog method constructs these equations in successive approximations (Euler, Navier-Stokes etc.), where the linearized collision operator plays a distinctive role. Similarly, a coupled set of equations can be derived for $\hat{f}^{(p)}(\vec{v})$ with the operator $\mathcal{M} = -\vec{a}\cdot\partial/\partial\vec{v} + J$ playing a role similar to that of the linearized collision operator in neutral gas theory.

Density gradient expansion (1.2) is a priori assumed for one-particle distribution function $f(\vec{r},\vec{v},t)$ in the HDR. This form of functional relationship makes it possible to derive transport coefficients which are independent of time. Hydrodynamic regime does not pre-suppose small relative gradients $n^{-1}(\partial/\partial\vec{r})n$, yet equation (1.2) can be expected to hold only when density gradients are small. Our aim is to construct the hydrodynamic form (1.2) of one-particle distribution function from the more basic principles rather then to impose it. The question that concerns us in the present paper is to reveal which intrinsic properties of the operators present in the kinetic equation (1.1) are sufficient for the validity of the density gradient expansion (1.2). The starting point for this work comes from Kumar [10], who related the characteristic time of approach to HDR to the inverse of a gap in the spectrum of the aforementioned operator $\mathcal{M}$, i.e. to the distance between the lowest eigenvalue and the rest of the spectrum.

The analytical method used in the present paper builds on our previous works on the transport theory of swarm particles [11,6,12], where we have constructed transport theory of swarm particles as an initial value problem for a linear Boltzmann kinetic equation, using the time-dependent perturbation theory generalized to non-Hermitian operators. This general theory is used in this paper to clarify the concept of a hydrodynamic description of the swarms in its most general sense, and to establish the sufficient conditions for its validity. Finally, we want to mention that the problem of long-time behavior of one-particle distribution function has been studied for some special cases of linear kinetic equation [13]. However, our approach is applicable for arbitrary linear kinetic operator and requires usual assumptions on the spectral properties of operators involved in kinetic equation.

The plan of the paper is as follows. In Section 2 we briefly introduce the results from our earlier work, which is than used in Section 3 to construct the long-time solution of a linear Boltzmann equation. Finally, in Section 4 we discuss the obtained results and present conclusions.

2 Time-dependent perturbation treatment of linear Boltzmann equation

In this section we give a brief summary of the relevant results from our earlier work [6,12].

The starting point for our theory is the Fourier transform of equation (1.1)

$$\frac{\partial}{\partial t}\Phi_{\vec{q}}(\vec{v},t) = \mathcal{L}_{\vec{q}}\,\Phi_{\vec{q}}(\vec{v},t), \qquad (2.1)$$

where $\Phi_{\vec{q}}(\vec{v},t)$ is the spatial Fourier transform of the one-particle distribution function

$$\Phi_{\vec{q}}(\vec{v},t) = \int \mathrm{d}\vec{r}\, e^{-i\vec{q}\cdot\vec{r}} f(\vec{r},\vec{v},t). \qquad (2.2)$$

In equation (2.1) the operator $\mathcal{L}_{\vec{q}}$ is

$$\mathcal{L}_{\vec{q}} = \mathcal{M} + \mathcal{P}_{\vec{q}}, \qquad (2.3)$$

with

$$\mathcal{M}\Phi_{\vec{q}}(\vec{v},t) = -\vec{a}\cdot\frac{\partial}{\partial\vec{v}}\Phi_{\vec{q}}(\vec{v},t) + J\Phi_{\vec{q}}(\vec{v},t), \qquad (2.4)$$

$$\mathcal{P}_{\vec{q}}\,\Phi_{\vec{q}}(\vec{v},t) = -i\vec{q}\cdot\vec{v}\,\Phi_{\vec{q}}(\vec{v},t). \qquad (2.5)$$

It is convenient to interpret $\Phi_{\vec{q}}(\vec{v},t)$ as the velocity-space representation of the corresponding proper vector $|\Phi_{\vec{q}}(t)\rangle$ in an abstract Hilbert space $\mathcal{H}$, i.e., $\Phi_{\vec{q}}(\vec{v},t) = \langle\vec{v}|\Phi_{\vec{q}}(t)\rangle$. In Hilbert space $\mathcal{H}$, the scalar (inner) product between two arbitrary vectors $|\varphi\rangle$ and $|\psi\rangle$ is defined as

$$\langle\varphi|\psi\rangle = \int \mathrm{d}\vec{v}\ \frac{1}{\phi^0(\vec{v})}\varphi^*(\vec{v})\psi(\vec{v}), \qquad (2.6)$$

where φ^* denotes the complex conjugate of φ. Here $1/\phi^0(\vec{v})$ is a suitably chosen weight factor.

Using formal correspondence between operators $\mathcal{L}_{\vec{q}}$, $\mathcal{M}$ and $\mathcal{P}_{\vec{q}}$, and linear operators on Hilbert space $\mathcal{H}$,

$$\mathcal{M}\to\hat{H}_0, \quad \mathcal{P}_{\vec{q}}\to\hat{H}'_{\vec{q}}, \quad \mathcal{L}_{\vec{q}} = \mathcal{M}+\mathcal{P}_{\vec{q}}\to\hat{H}_{\vec{q}} = \hat{H}_0 + \hat{H}'_{\vec{q}}, \qquad (2.7)$$

we formulate the transport problem of swarm particles as an abstract initial value problem:

$$\frac{\partial}{\partial t}|\Phi_{\vec{q}}(t)\rangle = \hat{H}_{\vec{q}}\,|\Phi_{\vec{q}}(t)\rangle, \quad |\Phi_{\vec{q}}(t_0)\rangle = |\Phi^I_{\vec{q}}\rangle, \quad t\geqslant t_0. \qquad (2.8)$$

Vector $|\Phi^I_{\vec{q}}\rangle$ represents initial state of swarm at time $t=t_0$.

The formal solution of initial value problem (2.8) is [6,12]:

$$|\Phi_{\vec{q}}(t)\rangle = \sum_{p=0}^{\infty}(-i\vec{q})^p\odot_p\,\|\varkappa^{(p)}_{\vec{q}}(t)\rangle\!\rangle, \qquad t\geqslant t_0, \qquad (2.9)$$

where

$$\left\| \varkappa_{\vec{q}}^{(0)}(t) \right\rangle\!\!\Big\rangle = \hat{U}^{(0)}(t,t_0) \left| \Phi_{\vec{q}}^{I} \right\rangle, \quad t \geqslant t_0, \tag{2.10}$$

$$\left\| \varkappa_{\vec{q}}^{(p)}(t) \right\rangle\!\!\Big\rangle = \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \cdots \int_{t_0}^{t_{p-1}} \mathrm{d}t_p \, \hat{U}^{(0)}(t,t_1)\hat{\vec{v}} \times \hat{U}^{(0)}(t_1,t_2)\hat{\vec{v}} \cdots \hat{U}^{(0)}(t_{p-1},t_p)\hat{\vec{v}}\, \hat{U}^{(0)}(t_p,t_0)|\Phi_{\vec{q}}^{I}\rangle,$$
$$t \geqslant t_1 \geqslant t_2 \geqslant \cdots \geqslant t_{p-1} \geqslant t_0, \quad p \geqslant 1. \tag{2.11}$$

The quantities $(-i\vec{q})^p$ and $\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\!\rangle$ are tensors of rank p. The Cartesian components $\alpha_1,\ldots,\alpha_p = 1,2,3,\ p \geqslant 1$, of the tensor $(-i\vec{q})^p$ are $\mathbb{C}$-numbers given by

$$[(-i\vec{q})^p]_{\alpha_1\cdots\alpha_p} = (-i)^p q_{\alpha_1} q_{\alpha_2} \cdots q_{\alpha_p}, \tag{2.12}$$

while the components of tensors $\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\!\rangle$ are vectors of Hilbert space $\mathcal{H}$ given by

$$\left[\left\| \varkappa_{\vec{q}}^{(p)}(t) \right\rangle\!\!\Big\rangle\right]_{\alpha_1\cdots\alpha_p} = \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \cdots \int_{t_0}^{t_{p-1}} \mathrm{d}t_p \times \hat{U}^{(0)}(t,t_1)\hat{v}_{\alpha_1}\hat{U}^{(0)}(t_1,t_2)\hat{v}_{\alpha_2}\cdots\hat{U}^{(0)}(t_{p-1},t_p)\hat{v}_{\alpha_p} \times \hat{U}^{(0)}(t_p,t_0)|\Phi_{\vec{q}}^{I}\rangle \in \mathcal{H},$$
$$t \geqslant t_1 \geqslant t_2 \geqslant \cdots \geqslant t_{p-1} \geqslant t_0, \quad p \geqslant 1. \tag{2.13}$$

The symbol $\odot_p$ denotes the appropriate p-fold scalar product. Operator $\hat{U}^{(0)}(t,t_0)$ is the evolution operator corresponding to the unperturbed operator $\hat{H}_0$. It satisfies the following differential equation:

$$\frac{\partial}{\partial t}\hat{U}^{(0)}(t,t_0) = \hat{H}_0\hat{U}^{(0)}(t,t_0) = \hat{U}^{(0)}(t,t_0)\hat{H}_0,$$
$$\hat{U}^{(0)}(t_0,t_0) = \hat{I}, \quad t \geqslant t_0. \tag{2.14}$$

Tensors $\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\!\rangle$ obey the following hierarchy of coupled differential equations [6,12]:

$$\frac{\partial}{\partial t}\left\| \varkappa_{\vec{q}}^{(0)}(t) \right\rangle\!\!\Big\rangle = \hat{H}_0 \left\| \varkappa_{\vec{q}}^{(0)}(t) \right\rangle\!\!\Big\rangle, \quad \left\| \varkappa_{\vec{q}}^{(0)}(t_0) \right\rangle\!\!\Big\rangle = \left| \Phi_{\vec{q}}^{I} \right\rangle, \ t \geqslant t_0, \tag{2.15}$$

$$\frac{\partial}{\partial t}\left\| \varkappa_{\vec{q}}^{(p)}(t) \right\rangle\!\!\Big\rangle = \hat{H}_0 \left\| \varkappa_{\vec{q}}^{(p)}(t) \right\rangle\!\!\Big\rangle + \hat{\vec{v}} \left\| \varkappa_{\vec{q}}^{(p-1)}(t) \right\rangle\!\!\Big\rangle,$$
$$\left\| \varkappa_{\vec{q}}^{(p)}(t_0) \right\rangle\!\!\Big\rangle = 0, \quad t \geqslant t_0, \quad p \geqslant 1. \tag{2.16}$$

We define the transport coefficients by [6,12],

$$\frac{\partial}{\partial t}\hat{N}^{(p)}(\vec{q},t) = \hat{\omega}_{\vec{q}}^{(p)}(t)\hat{N}^{(0)}(\vec{q},t) + \sum_{r=0}^{p-1}\hat{\omega}_{\vec{q}}^{(r)}(t) \otimes \hat{N}^{(p-r)}(\vec{q},t), \quad p \geqslant 0, \tag{2.17}$$

where $\hat{\omega}_{\vec{q}}^{(r)}(t)$ denote tensor transport coefficients of rank r, and the quantities $\hat{N}^{(p)}(\vec{q},t)$ are analogous to the spatial moments of the number density $n(\vec{r},t)$ [2] defined as

$$\hat{N}^{(p)}(\vec{q},t) = \left\langle \phi^0 \right\| \varkappa_{\vec{q}}^{(p)}(t) \right\rangle\!\!\Big\rangle, \quad p \geqslant 0. \tag{2.18}$$

The symbol $\otimes$ denotes the standard symmetrized outer tensor product (see Eq. (3.7) in [12]).

Generalized diffusion equation (GDE) follows immediately from definitions (2.17) and (2.18) [6,12]:

$$\frac{\partial}{\partial t}n_{\vec{q}}(t) - \sum_{p=0}^{\infty}(-i\vec{q})^p \odot_p \hat{\omega}_{\vec{q}}^{(p)}(t) n_{\vec{q}}(t) = 0, \tag{2.19}$$

where $n_{\vec{q}}(t) = \langle\phi^0|\Phi_{\vec{q}}(t)\rangle$ is the spatial Fourier transform of the number density $n(\vec{r},t)$. It describes the temporal evolution of the $n_{\vec{q}}(t)$ in terms of an infinite set $\{\hat{\omega}_{\vec{q}}^{(p)} | p \geqslant 0\}$ of transport coefficients.

Combining the definition relations (2.17), and hierarchy of equations (2.15)–(2.16) we can write [6,12]:

$$\hat{\omega}_{\vec{q}}^{(0)}(t) = \frac{1}{\left\langle \phi^0 \right\| \varkappa_{\vec{q}}^{(0)}(t) \right\rangle\!\!\Big\rangle} \left\langle \phi^0 \right| \hat{H}_0 \left\| \varkappa_{\vec{q}}^{(0)}(t) \right\rangle\!\!\Big\rangle, \tag{2.20}$$

$$\hat{\omega}_{\vec{q}}^{(p)}(t) = \frac{1}{\langle\phi^0\|\varkappa_{\vec{q}}^{(0)}(t)\rangle\!\rangle}\Bigg[\langle\phi^0|\hat{\vec{v}}\,\|\varkappa_{\vec{q}}^{(p-1)}(t)\rangle\!\rangle + \langle\phi^0|\hat{H}_0\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\!\rangle - \sum_{r=0}^{p-1}\hat{\omega}_{\vec{q}}^{(r)}(t) \otimes \langle\phi^0\|\varkappa_{\vec{q}}^{(p-r)}(t)\rangle\!\rangle\Bigg], \quad p \geqslant 1. \tag{2.21}$$

These expressions give transport coefficients in terms of solutions of the kinetic equations (2.15)–(2.16).

3 Long-time behavior of solution

In this section we explore solution to the linear Boltzmann equation and characterize its hydrodynamic form. This is done through a formal analysis of eigenvalue problem for the associated linear operator.

Very little is known about the nature of the spectrum of $\hat{H}_0$ for collision operators corresponding to real interactions. Of the two operators of which it is a sum (Eq. (2.4)), operator $-\vec{a}\cdot\partial/\partial\vec{v}$ has a continuous spectrum and its eigenfunctions are not square integrable. The main features of the spectrum of collision operator J, but not all of its detailed properties are known for the hard sphere potential, and for r^{-s} repulsive potentials [14,15]. The r^{-4} potential (Maxwell molecules) is the only potential for which the spectrum is known completely [16]. Almost nothing is known for the potentials having an attractive component.

For simplicity we assume that the spectrum of $\hat{H}_0$ is entirely discrete. This assumption is not essential one, and is introduced to avoid cumbersome notation. The same formal results can be obtained without requiring the discreteness of the spectrum of $\hat{H}_0$. Consider than the unperturbed eigenvalue problem

$$\hat{H}_0 \left|\psi_{n\lambda}^{(0)}\right\rangle = \Lambda_n^{(0)} \left| \psi_{n\lambda}^{(0)}\right\rangle, \tag{3.1}$$

where $|\psi_{n\lambda}^{(0)}\rangle$ is an eigenvector and $\Lambda_n^{(0)}$ the corresponding eigenvalue. The index λ distinguishes between different eigenvectors belonging to some degenerate eigenvalue.

The operator $\hat{H}_0$ is not Hermitian, and as a consequence the eigenvectors $|\psi_{n\lambda}^{(0)}\rangle$ are not orthogonal. It is then useful to introduce the adjoint eigenvalue problem

$$\hat{H}_0^\dagger \left|\tilde{\psi}_{n\lambda}^{(0)}\right\rangle = \tilde{\Lambda}_n^{(0)} \left| \tilde{\psi}_{n\lambda}^{(0)}\right\rangle. \tag{3.2}$$

The sets $\{|\psi_{n\lambda}^{(0)}\rangle\}$ and $\{|\tilde{\psi}_{n\lambda}^{(0)}\rangle\}$ can always be made biorthonormal:

$$\left\langle \tilde{\psi}_{n\lambda}^{(0)} | \psi_{n'\lambda'}^{(0)} \right\rangle = \delta_{nn'}\delta_{\lambda\lambda'}. \tag{3.3}$$

In addition, we *assume* that these sets are complete, i.e.

$$\sum_{n\lambda} \left|\psi_{n\lambda}^{(0)}\right\rangle \left\langle \tilde{\psi}_{n\lambda}^{(0)}\right| = \hat{I}, \tag{3.4}$$

where $\hat{I}$ is the unit operator. The problem of justifying this simplifying assumption for actual operators $\hat{H}_0$ is not considered in this paper.

In our case, when $\hat{H}_0$ is time independent, the evolution operator $\hat{U}_0(t', t'')$ (Eq. (2.14)) becomes simply

$$\begin{aligned}\hat{U}_0(t', t'') &= e^{(t'-t'')\hat{H}_0} \\ &= \sum_{n,\lambda} e^{(t'-t'')\Lambda_n^{(0)}} \left|\psi_{n\lambda}^{(0)}\right\rangle\left\langle \tilde{\psi}_{n\lambda}^{(0)}\right|, \quad t' \geqslant t'' \geqslant t_0.\end{aligned} \tag{3.5}$$

Replacing the evolution operators in equations (2.10) and (2.11) with corresponding expansions in series of nonorthogonal projectors (Eq. (3.5)), one finds

$$\|\varkappa_{\vec{q}}^{(0)}(t)\rangle\rangle = \sum_{n,\lambda} c_{n\lambda}^{I(0)}(\vec{q})|\psi_{n\lambda}^{(0)}\rangle e^{(t-t_0)\Lambda_n^{(0)}}, \tag{3.6}$$

$$\begin{aligned}\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\rangle &= \sum_{n,\lambda} c_{n\lambda}^{I(0)}(\vec{q}) \sum_{n_1,\lambda_1}\sum_{n_2,\lambda_2} \cdots \\ &\times \sum_{n_p,\lambda_p} \langle \tilde{\psi}_{n_p\lambda_p}^{(0)}|\hat{\vec{v}}|\psi_{n_{p-1}\lambda_{p-1}}^{(0)}\rangle \cdots \\ &\cdots \langle \tilde{\psi}_{n_2\lambda_2}^{(0)}|\hat{\vec{v}}|\psi_{n_1\lambda_1}^{(0)}\rangle\langle \tilde{\psi}_{n_1\lambda_1}^{(0)}|\hat{\vec{v}}|\psi_{n\lambda}^{(0)}\rangle|\psi_{n_p\lambda_p}^{(0)}\rangle \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \cdots \\ &\cdots \int_{t_0}^{t_{p-1}} \mathrm{d}t_p e^{(t_p-t_0)\Lambda_n^{(0)}} e^{(t_{p-1}-t_p)\Lambda_{n_1}^{(0)}} \cdots \\ &\times e^{(t_1-t_2)\Lambda_{n_{p-1}}^{(0)}} e^{(t-t_1)\Lambda_{n_p}^{(0)}}, \\ & t \geqslant t_1 \geqslant t_2 \geqslant \cdots \geqslant t_{p-1} \geqslant t_0, \quad p \geqslant 1.\end{aligned} \tag{3.7}$$

Here

$$c_{n\lambda}^{I(0)}(\vec{q}) = \langle \tilde{\psi}_{n\lambda}^{(0)}|\Phi_{\vec{q}}^I\rangle. \tag{3.8}$$

Further discussion requires additional assumptions about the spectral properties of the operator $\hat{H}_0$.

Assumption I: *There exists an isolated eigenvalue* $\Lambda_{\bar{n}}^{(0)}$ *such that*

$$\mathrm{Re}\ \Lambda_n^{(0)} < \mathrm{Re}\ \Lambda_{\bar{n}}^{(0)}, \quad \forall n \neq \bar{n}. \tag{3.9}$$

Let

$$\frac{1}{\tau_0} = d_0 = \inf_{n\neq\bar{n}} \left|\mathrm{Re}\ \Lambda_n^{(0)} - \mathrm{Re}\Lambda_{\bar{n}}^{(0)}\right|. \tag{3.10}$$

Assumption I was confirmed for the wide class of strictly repulsive potentials proportional to r^{-s} with $s > 2$, including the hard-sphere limit $s \to \infty$. Namely, Pao [14,15] has proved that for the class of power law potentials the spectrum of linearized collision operator is purely discrete and has no accumulation point. In kinetic theory of neutral gases, such an assumption is always implicit in any calculation of transport coefficients. This is a sufficient condition for the existence of hydrodynamics. Physically, it implies [17] a separation of the relaxation time scale $\tau_0 \propto (d_0)^{-1}$, and the hydrodynamic time scale $\tau_h \propto (q(k_BT)^{1/2})^{-1}$, where τ_h is the time a gas particle needs to travel the length of macroscopic gradients, and k_BT is the mean random energy of a gas particle.

Next we consider the long-time behavior of formal solution (2.9). We will examine the asymptotic limits of vector $|\Phi_{\vec{q}}(t)\rangle$ for times $t \gg \tau_0$ and *arbitrary* values of $\vec{q}$. This will be carried out using *Assumption I* (Eq. (3.9)). It will be shown that for the long-time limit, vector $|\Phi_{\vec{q}}(t)\rangle$ can be transformed into hydrodynamics form (1.2).

First let us find vectors $\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\rangle$ for times $t - t_0 \gg \tau_0$. For simplicity we put $t_0 = 0$. We present here the detailed calculation of $\|\varkappa_{\vec{q}}^{(0)}(t)\rangle\rangle$, $\|\varkappa_{\vec{q}}^{(1)}(t)\rangle\rangle$ and $\|\varkappa_{\vec{q}}^{(2)}(t)\rangle\rangle$ for long times $t \gg \tau_0$. From equations (3.6) and (3.7), we have

$$\|\varkappa_{\vec{q}}^{(0)}(t)\rangle\rangle = \sum_{n,\lambda} c_{n\lambda}^{I(0)}(\vec{q}) e^{t\Lambda_n^{(0)}}|\psi_{n\lambda}^{(0)}\rangle, \tag{3.11}$$

$$\begin{aligned}\|\varkappa_{\vec{q}}^{(1)}(t)\rangle\rangle &= \sum_{n,\lambda} c_{n\lambda}^{I(0)}(\vec{q}) t e^{t\Lambda_n^{(0)}} \langle \tilde{\psi}_{n\lambda}^{(0)}|\hat{\vec{v}}|\psi_{n\lambda}^{(0)}\rangle|\psi_{n\lambda}^{(0)}\rangle \\ &+ \sum_{n,\lambda} c_{n\lambda}^{I(0)}(\vec{q}) \sum_{n_1\neq n,\lambda_1} \frac{1}{\Lambda_n^{(0)} - \Lambda_{n_1}^{(0)}} \\ &\times \left(e^{t\Lambda_n^{(0)}} - e^{t\Lambda_{n_1}^{(0)}}\right) \langle \tilde{\psi}_{n_1\lambda_1}^{(0)}|\hat{\vec{v}}|\psi_{n\lambda}^{(0)}\rangle|\psi_{n_1\lambda_1}^{(0)}\rangle,\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}\|\varkappa_{\vec{q}}^{(2)}(t)\rangle\rangle &= \|K_1(\vec{q}, t)\rangle\rangle + \|K_2(\vec{q}, t)\rangle\rangle + \|K_3(\vec{q}, t)\rangle\rangle \\ &+ \|K_4(\vec{q}, t)\rangle\rangle + \|K_5(\vec{q}, t)\rangle\rangle + \|K_6(\vec{q}, t)\rangle\rangle,\end{aligned} \tag{3.13}$$

where

$$\|K_1(\vec{q}, t)\rangle\rangle = \sum_{n,\lambda} c_{n\lambda}^{I(0)}(\vec{q}) \frac{t^2}{2} e^{t\Lambda_n^{(0)}} \langle \tilde{\psi}_{n\lambda}^{(0)}|\hat{\vec{v}}|\psi_{n\lambda}^{(0)}\rangle\langle \tilde{\psi}_{n\lambda}^{(0)}|\hat{\vec{v}}|\psi_{n\lambda}^{(0)}\rangle|\psi_{n\lambda}^{(0)}\rangle, \tag{3.14}$$

$$\|K_2(\vec{q},t)\rangle\!\rangle = \sum_{n,\lambda} c_{n\lambda}^{I(0)}(\vec{q}) \sum_{n_2\neq n,\lambda_2} \left[\frac{1}{\Lambda_n^{(0)} - \Lambda_{n_2}^{(0)}} t e^{t\Lambda_n^{(0)}} - \frac{1}{\left(\Lambda_n^{(0)} - \Lambda_{n_2}^{(0)}\right)^2} e^{t\Lambda_n^{(0)}} + \frac{1}{\left(\Lambda_n^{(0)} - \Lambda_{n_2}^{(0)}\right)^2} e^{t\Lambda_{n_2}^{(0)}} \right] \times \langle \tilde{\psi}_{n_2\lambda_2}^{(0)} | \hat{\vec{v}} | \psi_{n\lambda}^{(0)} \rangle \langle \tilde{\psi}_{n\lambda}^{(0)} | \hat{\vec{v}} | \psi_{n\lambda}^{(0)} \rangle | \psi_{n_2\lambda_2}^{(0)} \rangle, \quad (3.15)$$

$$\|K_3(\vec{q},t)\rangle\!\rangle = \sum_{n,\lambda} c_{n\lambda}^{I(0)}(\vec{q}) \sum_{n_1\neq n,\lambda_1} \frac{1}{\Lambda_n^{(0)} - \Lambda_{n_1}^{(0)}} t e^{t\Lambda_n^{(0)}} \times \langle \tilde{\psi}_{n\lambda}^{(0)} | \hat{\vec{v}} | \psi_{n_1\lambda_1}^{(0)} \rangle \langle \tilde{\psi}_{n_1\lambda_1}^{(0)} | \hat{\vec{v}} | \psi_{n\lambda}^{(0)} \rangle | \psi_{n\lambda}^{(0)} \rangle, \quad (3.16)$$

$$\|K_4(\vec{q},t)\rangle\!\rangle = \sum_{n,\lambda} c_{n\lambda}^{I(0)}(\vec{q}) \sum_{n_1\neq n,\lambda_1} \sum_{n_2\neq n,\lambda_2} \left[\frac{1}{\Lambda_n^{(0)} - \Lambda_{n_1}^{(0)}} \frac{1}{\Lambda_n^{(0)} - \Lambda_{n_2}^{(0)}} e^{t\Lambda_n^{(0)}} - \frac{1}{\Lambda_n^{(0)} - \Lambda_{n_1}^{(0)}} \frac{1}{\Lambda_n^{(0)} - \Lambda_{n_2}^{(0)}} e^{t\Lambda_{n_2}^{(0)}} \right] \times \langle \tilde{\psi}_{n_2\lambda_2}^{(0)} | \hat{\vec{v}} | \psi_{n_1\lambda_1}^{(0)} \rangle \langle \tilde{\psi}_{n_1\lambda_1}^{(0)} | \hat{\vec{v}} | \psi_{n\lambda}^{(0)} \rangle | \psi_{n_2\lambda_2}^{(0)} \rangle, \quad (3.17)$$

$$\|K_5(\vec{q},t)\rangle\!\rangle = -\sum_{n,\lambda} c_{n\lambda}^{I(0)}(\vec{q}) \sum_{n_1\neq n,\lambda_1} \frac{1}{\Lambda_n^{(0)} - \Lambda_{n_1}^{(0)}} t e^{t\Lambda_{n_1}^{(0)}} \times \langle \tilde{\psi}_{n_1\lambda_1}^{(0)} | \hat{\vec{v}} | \psi_{n_1\lambda_1}^{(0)} \rangle \langle \tilde{\psi}_{n_1\lambda_1}^{(0)} | \hat{\vec{v}} | \psi_{n\lambda}^{(0)} \rangle | \psi_{n_1\lambda_1}^{(0)} \rangle, \quad (3.18)$$

$$\|K_6(\vec{q},t)\rangle\!\rangle = -\sum_{n,\lambda} c_{n\lambda}^{I(0)}(\vec{q}) \sum_{n_1\neq n,\lambda_1} \sum_{n_2\neq n_1,\lambda_2} \left[\frac{1}{\Lambda_n^{(0)} - \Lambda_{n_1}^{(0)}} \frac{1}{\Lambda_{n_1}^{(0)} - \Lambda_{n_2}^{(0)}} e^{t\Lambda_{n_1}^{(0)}} - \frac{1}{\Lambda_n^{(0)} - \Lambda_{n_1}^{(0)}} \frac{1}{\Lambda_{n_1}^{(0)} - \Lambda_{n_2}^{(0)}} e^{t\Lambda_{n_2}^{(0)}} \right] \times \langle \tilde{\psi}_{n_2\lambda_2}^{(0)} | \hat{\vec{v}} | \psi_{n_1\lambda_1}^{(0)} \rangle \langle \tilde{\psi}_{n_1\lambda_1}^{(0)} | \hat{\vec{v}} | \psi_{n\lambda}^{(0)} \rangle | \psi_{n_2\lambda_2}^{(0)} \rangle. \quad (3.19)$$

For $t \gg \tau_0$, using the *Assumption I* (see Eq. (3.9)), we obtain from equations (3.11)–(3.13),

$$\|\varkappa_{\vec{q}}^{(0)}(t)\rangle\!\rangle \simeq c_{\bar{n}}^{I(0)}(\vec{q}) e^{t\Lambda_{\bar{n}}^{(0)}} |\psi_{\bar{n}}^{(0)}\rangle, \quad t \gg \tau_0, \quad (3.20)$$

$$\|\varkappa_{\vec{q}}^{(1)}(t)\rangle\!\rangle \simeq c_{\bar{n}}^{I(0)}(\vec{q}) e^{t\Lambda_{\bar{n}}^{(0)}} \langle \tilde{\psi}_{\bar{n}}^{(0)} | \hat{\vec{v}} | \psi_{\bar{n}}^{(0)} \rangle |\psi_{\bar{n}}^{(0)}\rangle + c_{\bar{n}}^{I(0)}(\vec{q}) e^{t\Lambda_{\bar{n}}^{(0)}} \times \sum_{n_1\neq\bar{n},\lambda_1} \frac{1}{\Lambda_{\bar{n}}^{(0)} - \Lambda_{n_1}^{(0)}} \langle \tilde{\psi}_{n_1\lambda_1}^{(0)} | \hat{\vec{v}} | \psi_{\bar{n}}^{(0)} \rangle |\psi_{n_1\lambda_1}^{(0)}\rangle, \quad t \gg \tau_0, \quad (3.21)$$

and

$$\|\varkappa_{\vec{q}}^{(2)}(t)\rangle\!\rangle \simeq \|L_1(\vec{q},t)\rangle\!\rangle + \|L_2(\vec{q},t)\rangle\!\rangle + \|L_3(\vec{q},t)\rangle\!\rangle + \|L_4(\vec{q},t)\rangle\!\rangle, \quad t \gg \tau_0, \quad (3.22)$$

where

$$\|L_1(\vec{q},t)\rangle\!\rangle = c_{\bar{n}}^{I(0)}(\vec{q}) \frac{t^2}{2} e^{t\Lambda_{\bar{n}}^{(0)}} \langle \tilde{\psi}_{\bar{n}}^{(0)} | \hat{\vec{v}} | \psi_{\bar{n}}^{(0)} \rangle \langle \tilde{\psi}_{\bar{n}}^{(0)} | \hat{\vec{v}} | \psi_{\bar{n}}^{(0)} \rangle |\psi_{\bar{n}}^{(0)}\rangle, \quad (3.23)$$

$$\|L_2(\vec{q},t)\rangle\!\rangle = c_{\bar{n}}^{I(0)}(\vec{q}) t e^{t\Lambda_{\bar{n}}^{(0)}} \langle \tilde{\psi}_{\bar{n}}^{(0)} | \hat{\vec{v}} | \psi_{\bar{n}}^{(0)} \rangle \sum_{n_2\neq\bar{n},\lambda_2} \frac{1}{\Lambda_{\bar{n}}^{(0)} - \Lambda_{n_2}^{(0)}} \times \langle \tilde{\psi}_{n_2\lambda_2}^{(0)} | \hat{\vec{v}} | \psi_{\bar{n}}^{(0)} \rangle |\psi_{n_2\lambda_2}^{(0)}\rangle, \quad (3.24)$$

$$\|L_3(\vec{q},t)\rangle\!\rangle = c_{\bar{n}}^{I(0)}(\vec{q}) t e^{t\Lambda_{\bar{n}}^{(0)}} \sum_{n_1\neq\bar{n},\lambda_1} \frac{1}{\Lambda_{\bar{n}}^{(0)} - \Lambda_{n_1}^{(0)}} \times \langle \tilde{\psi}_{\bar{n}}^{(0)} | \hat{\vec{v}} | \psi_{n_1\lambda_1}^{(0)} \rangle \langle \tilde{\psi}_{n_1\lambda_1}^{(0)} | \hat{\vec{v}} | \psi_{\bar{n}}^{(0)} \rangle |\psi_{\bar{n}}^{(0)}\rangle, \quad (3.25)$$

$$\|L_4(\vec{q},t)\rangle\!\rangle = c_{\bar{n}}^{I(0)}(\vec{q}) e^{t\Lambda_{\bar{n}}^{(0)}} \left[\sum_{n_1\neq\bar{n},\lambda_1} \sum_{n_2\neq\bar{n},\lambda_2} \frac{1}{\Lambda_{\bar{n}}^{(0)} - \Lambda_{n_1}^{(0)}} \frac{1}{\Lambda_{\bar{n}}^{(0)} - \Lambda_{n_2}^{(0)}} \times \left\langle \tilde{\psi}_{n_2\lambda_2}^{(0)} | \hat{\vec{v}} | \psi_{n_1\lambda_1}^{(0)} \right\rangle \left\langle \tilde{\psi}_{n_1\lambda_1}^{(0)} | \hat{\vec{v}} | \psi_{\bar{n}}^{(0)} \right\rangle \left| \psi_{n_2\lambda_2}^{(0)} \right\rangle - \left\langle \tilde{\psi}_{\bar{n}}^{(0)} | \hat{\vec{v}} | \psi_{\bar{n}}^{(0)} \right\rangle \sum_{n_2\neq\bar{n},\lambda_2} \frac{1}{\left(\Lambda_{\bar{n}}^{(0)} - \Lambda_{n_2}^{(0)}\right)^2} \times \left\langle \tilde{\psi}_{n_2\lambda_2}^{(0)} | \hat{\vec{v}} | \psi_{\bar{n}}^{(0)} \right\rangle \left| \psi_{n_2\lambda_2}^{(0)} \right\rangle \right]. \quad (3.26)$$

Now we introduce the tensor quantities

$$\hat{\omega}_*^{(0)} = \langle \tilde{\psi}_{\bar{n}}^{(0)} | \hat{H}_0 \| \chi_{\bar{n}}^{(0)} \rangle\!\rangle, \quad \hat{\omega}_*^{(p)} = \langle \tilde{\psi}_{\bar{n}}^{(0)} | \hat{\vec{v}} \, \| \chi_{\bar{n}}^{(p-1)} \rangle\!\rangle, \quad p \geqslant 1, \quad (3.27)$$

where

$$\|\chi_{\bar{n}}^{(0)}\rangle\!\rangle = |\psi_{\bar{n}}^{(0)}\rangle, \quad \|\chi_{\bar{n}}^{(1)}\rangle\!\rangle = [\hat{\omega}_*^{(0)} - \hat{H}_0]^{-1} \hat{Q} \, \hat{\vec{v}} \, \|\chi_{\bar{n}}^{(0)}\rangle\!\rangle,$$
$$\|\chi_{\bar{n}}^{(p)}\rangle\!\rangle = [\hat{\omega}_*^{(0)} - \hat{H}_0]^{-1} \hat{Q} \times \left[\hat{\vec{v}} \, \|\chi_{\bar{n}}^{(p-1)}\rangle\!\rangle - \sum_{r=1}^{p-1} \hat{\omega}_*^{(r)} \otimes \|\chi_{\bar{n}}^{(p-r)}\rangle\!\rangle \right], \quad p \geqslant 2. \quad (3.28)$$

Operator

$$\hat{Q} = \hat{I} - \left| \psi_{\bar{n}}^{(0)} \rangle \langle \tilde{\psi}_{\bar{n}}^{(0)} \right|, \quad (3.29)$$

is projector (but not orthogonal) onto subspace complementary to the subspace spanned by basic eigenvector $|\psi_{\bar{n}}^{(0)}\rangle$. Since operator $\hat{\omega}_*^{(0)} - \hat{H}_0$ is singular, it is clear that the inverse operator $[\hat{\omega}_*^{(0)} - \hat{H}_0]^{-1}$ cannot be defined on the whole Hilbert space $\mathcal{H}$. In equations (3.28) operator $[\hat{\omega}_*^{(0)} - \hat{H}_0]^{-1}$ is well defined because the range of $\hat{Q}$ does not contain the vectors of the kernel of $\hat{\omega}_*^{(0)} - \hat{H}_0$, $\mathrm{im}\hat{Q} \cap \ker(\hat{\omega}_*^{(0)} - \hat{H}_0) = \emptyset$. Furthermore, from the closure relation (3.4) and definition (3.29), it is obvious that the operator $[\hat{\omega}_*^{(0)} - \hat{H}_0]^{-1}\hat{Q}$ can be written in the form of a series of elementary, non-orthogonal projectors:

$$\left[\hat{\omega}_*^{(0)} - \hat{H}_0\right]^{-1} \hat{Q} = \sum_{n\neq\bar{n}} \sum_{\lambda} \frac{1}{\hat{\omega}_*^{(0)} - \Lambda_n^{(0)}} \left| \psi_{n\lambda}^{(0)} \rangle \langle \tilde{\psi}_{n\lambda}^{(0)} \right|. \quad (3.30)$$

Explicit expressions for the tensors $\hat{\omega}_*^{(p)}$ and vectors $\|\chi_{\bar{n}}^{(p)}\rangle\rangle$ (Eqs. (3.27) and (3.28)) in terms of solutions of the eigenvalue problems (3.1) and (3.2) can be derived by using the spectral decomposition of operator $[\hat{\omega}_*^{(0)} - \hat{H}_0]^{-1}\hat{Q}$ (Eq. (3.30)). Inserting decomposition (3.30) into definition equations (3.27) and (3.28) we find that the vectors $\|\chi_{\bar{n}}^{(p)}\rangle\rangle$, $p = 1, 2$ can be expressed as

$$\left\|\chi_{\bar{n}}^{(1)}\right\rangle\!\!\left.\right\rangle = \sum_{n\neq\bar{n},\lambda} \frac{1}{\hat{\omega}_*^{(0)} - \Lambda_n^{(0)}} \left\langle \tilde{\psi}_{n\lambda}^{(0)} \middle| \hat{\vec{v}} \middle| \psi_{\bar{n}}^{(0)} \right\rangle \left|\psi_{n\lambda}^{(0)}\right\rangle, \quad (3.31)$$

$$\begin{aligned}\left\|\chi_{\bar{n}}^{(2)}\right\rangle\!\!\left.\right\rangle = &\sum_{n\neq\bar{n},\lambda} \sum_{n_1\neq\bar{n},\lambda_1} \frac{1}{\hat{\omega}_*^{(0)} - \Lambda_n^{(0)}} \frac{1}{\hat{\omega}_*^{(0)} - \Lambda_{n_1}^{(0)}} \\ &\times \left\langle \tilde{\psi}_{n\lambda}^{(0)} \middle| \hat{\vec{v}} \middle| \psi_{n_1\lambda_1}^{(0)} \right\rangle \left\langle \tilde{\psi}_{n_1\lambda_1}^{(0)} \middle| \hat{\vec{v}} \middle| \psi_{\bar{n}}^{(0)} \right\rangle \left|\psi_{n\lambda}^{(0)}\right\rangle \\ &- \left\langle \tilde{\psi}_{\bar{n}}^{(0)} \middle| \hat{\vec{v}} \middle| \psi_{\bar{n}}^{(0)} \right\rangle \sum_{n\neq\bar{n},\lambda} \frac{1}{\left(\hat{\omega}_*^{(0)} - \Lambda_n^{(0)}\right)^2} \left\langle \tilde{\psi}_{n\lambda}^{(0)} \middle| \hat{\vec{v}} \middle| \psi_{\bar{n}}^{(0)} \right\rangle \left|\psi_{n\lambda}^{(0)}\right\rangle, \end{aligned} \quad (3.32)$$

and that tensors $\hat{\omega}_*^{(p)}$, $p = 1, 2, 3$ have the form

$$\hat{\omega}_*^{(1)} = \left\langle \tilde{\psi}_{\bar{n}}^{(0)} \middle| \hat{\vec{v}} \middle| \psi_{\bar{n}}^{(0)} \right\rangle, \quad (3.33)$$

$$\hat{\omega}_*^{(2)} = \sum_{n\neq\bar{n}} \sum_{\lambda} \frac{1}{\hat{\omega}_*^{(0)} - \Lambda_n^{(0)}} \left\langle \tilde{\psi}_{\bar{n}}^{(0)} \middle| \hat{\vec{v}} \middle| \psi_{n\lambda}^{(0)} \right\rangle \left\langle \tilde{\psi}_{n\lambda}^{(0)} \middle| \hat{\vec{v}} \middle| \psi_{\bar{n}}^{(0)} \right\rangle, \quad (3.34)$$

$$\begin{aligned}\hat{\omega}_*^{(3)} = &\sum_{n\neq\bar{n}} \sum_{\lambda} \sum_{n'\neq\bar{n}} \sum_{\lambda'} \frac{1}{\hat{\omega}_*^{(0)} - \Lambda_n^{(0)}} \frac{1}{\hat{\omega}_*^{(0)} - \Lambda_{n'}^{(0)}} \\ &\times \left\langle \tilde{\psi}_{\bar{n}}^{(0)} \middle| \hat{\vec{v}} \middle| \psi_{n\lambda}^{(0)} \right\rangle \left\langle \tilde{\psi}_{n\lambda}^{(0)} \middle| \hat{\vec{v}} \middle| \psi_{n'\lambda'}^{(0)} \right\rangle \left\langle \tilde{\psi}_{n'\lambda'}^{(0)} \middle| \hat{\vec{v}} \middle| \psi_{\bar{n}}^{(0)} \right\rangle \\ &- \left\langle \tilde{\psi}_{\bar{n}}^{(0)} \middle| \hat{\vec{v}} \middle| \psi_{\bar{n}}^{(0)} \right\rangle \sum_{n\neq\bar{n}} \sum_{\lambda} \frac{1}{\left(\hat{\omega}_*^{(0)} - \Lambda_n^{(0)}\right)^2} \\ &\times \left\langle \tilde{\psi}_{\bar{n}}^{(0)} \middle| \hat{\vec{v}} \middle| \psi_{n\lambda}^{(0)} \right\rangle \left\langle \tilde{\psi}_{n\lambda}^{(0)} \middle| \hat{\vec{v}} \middle| \psi_{\bar{n}}^{(0)} \right\rangle. \end{aligned} \quad (3.35)$$

The case of higher order tensors can be treated in an analogous way.

Finally, from equations (3.20)–(3.22) and using equations (3.31)–(3.35), we obtain the following asymptotic formulae for the first three vectors:

$$\left\|\varkappa_{\vec{q}}^{(0)}(t)\right\rangle\!\!\left.\right\rangle \simeq c_{\bar{n}}^{I(0)}(\vec{q}) e^{t\hat{\omega}_*^{(0)}} \left\|\chi_{\bar{n}}^{(0)}\right\rangle\!\!\left.\right\rangle, t \gg \tau_0, \quad (3.36)$$

$$\left\|\varkappa_{\vec{q}}^{(1)}(t)\right\rangle\!\!\left.\right\rangle \simeq c_{\bar{n}}^{I(0)}(\vec{q}) e^{t\hat{\omega}_*^{(0)}} \left[t\hat{\omega}_*^{(1)} \left\|\chi_{\bar{n}}^{(0)}\right\rangle\!\!\left.\right\rangle + \left\|\chi_{\bar{n}}^{(1)}\right\rangle\!\!\left.\right\rangle\right], t \gg \tau_0, \quad (3.37)$$

$$\begin{aligned}\left\|\varkappa_{\vec{q}}^{(2)}(t)\right\rangle\!\!\left.\right\rangle \simeq &c_{\bar{n}}^{I(0)}(\vec{q}) e^{t\hat{\omega}_*^{(0)}} \left[\frac{t^2}{2}\hat{\omega}_*^{(1)} \otimes \hat{\omega}_*^{(1)} \left\|\chi_{\bar{n}}^{(0)}\right\rangle\!\!\left.\right\rangle \right. \\ &\left. + t\hat{\omega}_*^{(1)} \otimes \left\|\chi_{\bar{n}}^{(1)}\right\rangle\!\!\left.\right\rangle + t\hat{\omega}_*^{(2)} \|\chi_{\bar{n}}^{(0)}\rangle\!\!\left.\right\rangle + \left\|\chi_{\bar{n}}^{(2)}\right\rangle\!\!\left.\right\rangle\right], t \gg \tau_0. \end{aligned} \quad (3.38)$$

The calculation of asymptotic expressions for the vectors of the order higher than two is cumbersome. We present the final result only:

$$\begin{aligned}\left\|\varkappa_{\vec{q}}^{(3)}(t)\right\rangle\!\!\left.\right\rangle \simeq &c_{\bar{n}}^{I(0)}(\vec{q}) e^{t\hat{\omega}_*^{(0)}} \left[\frac{t^3}{6}\hat{\omega}_*^{(1)} \otimes \hat{\omega}_*^{(1)} \otimes \hat{\omega}_*^{(1)} \left\|\chi_{\bar{n}}^{(0)}\right\rangle\!\!\left.\right\rangle \right. \\ &+ \frac{t^2}{2}\hat{\omega}_*^{(1)} \otimes \hat{\omega}_*^{(1)} \otimes \left\|\chi_{\bar{n}}^{(1)}\right\rangle\!\!\left.\right\rangle + \frac{t^2}{2}\hat{\omega}_*^{(1)} \otimes \hat{\omega}_*^{(2)} \left\|\chi_{\bar{n}}^{(0)}\right\rangle\!\!\left.\right\rangle \\ &+ t\hat{\omega}_*^{(1)} \otimes \left\|\chi_{\bar{n}}^{(2)}\right\rangle\!\!\left.\right\rangle + \frac{t^2}{2}\hat{\omega}_*^{(2)} \otimes \hat{\omega}_*^{(1)} \left\|\chi_{\bar{n}}^{(0)}\right\rangle\!\!\left.\right\rangle \\ &\left. + t\hat{\omega}_*^{(2)} \otimes \left\|\chi_{\bar{n}}^{(1)}\right\rangle\!\!\left.\right\rangle + t\hat{\omega}_*^{(3)} \left\|\chi_{\bar{n}}^{(0)}\right\rangle\!\!\left.\right\rangle + \left\|\chi_{\bar{n}}^{(3)}\right\rangle\!\!\left.\right\rangle\right], \quad t \gg \tau_0. \end{aligned} \quad (3.39)$$

Substituting the above asymptotic expressions for $\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\rangle$ into equation (2.9), after a suitable rearrangement of the terms, gives the following form of $|\Phi_{\vec{q}}(t)\rangle$ for $t \gg \tau_0$:

$$|\Phi_{\vec{q}}(t)\rangle \simeq \sum_{p=0}^{\infty} \|\chi_{\bar{n}}^{(p)}\rangle\rangle \odot_p (-i\vec{q})^p c_{\bar{n}}^{I(0)}(\vec{q}) e^{t\hat{\omega}_*^{(0)}} \Omega_{\vec{q}}(t), \quad (3.40)$$

where

$$\begin{aligned}\Omega_{\vec{q}}(t) = &1 + (-i\vec{q}) \odot_1 \hat{\omega}_*^{(1)} t \\ &+ (-i\vec{q})^2 \odot_2 \hat{\omega}_*^{(1)} \otimes \hat{\omega}_*^{(1)} \frac{t^2}{2} + (-i\vec{q})^2 \odot_2 \hat{\omega}_*^{(2)} t \\ &+ (-i\vec{q})^3 \odot_3 \hat{\omega}_*^{(1)} \otimes \hat{\omega}_*^{(1)} \otimes \hat{\omega}_*^{(1)} \frac{t^3}{6} \\ &+ (-i\vec{q})^3 \odot_3 \hat{\omega}_*^{(1)} \otimes \hat{\omega}_*^{(2)} \frac{t^2}{2} \\ &+ (-i\vec{q})^3 \odot_3 \hat{\omega}_*^{(2)} \otimes \hat{\omega}_*^{(1)} \frac{t^2}{2} + (-i\vec{q})^3 \odot_3 \hat{\omega}_*^{(3)} t + \cdots \end{aligned} \quad (3.41)$$

Using the multiplication formula for infinite series $\left(\sum_{n=0}^{\infty} a_n\right) \left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k}$, we readily obtain

$$\begin{aligned}\Omega_{\vec{q}}(t) = &\left[1 + (-i\vec{q}) \odot_1 \hat{\omega}_*^{(1)} t + \frac{1}{2}(-i\vec{q})^2 \odot_2 \hat{\omega}_*^{(1)} \otimes \hat{\omega}_*^{(1)} t^2 \right. \\ &\left. + \frac{1}{6}(-i\vec{q})^3 \odot_3 \hat{\omega}_*^{(1)} \otimes \hat{\omega}_*^{(1)} \otimes \hat{\omega}_*^{(1)} t^3 + \cdots\right] \\ &\times \left[1 + (-i\vec{q})^2 \odot_2 \hat{\omega}_*^{(2)} t + \frac{1}{2}(-i\vec{q})^4 \odot_4 \hat{\omega}_*^{(2)} \otimes \hat{\omega}_*^{(2)} t^2 + \cdots\right] \\ &\times \left[1 + (-i\vec{q})^3 \odot_3 \hat{\omega}_*^{(3)} t + \cdots\right] [1 + \cdots] \cdots \end{aligned} \quad (3.42)$$

By induction, it is easy to verify that

$$\begin{aligned}\Omega_{\vec{q}}(t) &= \prod_{s=1}^{\infty} \left\{1 + \sum_{r=1}^{\infty} \frac{1}{r!} \left[(-i\vec{q})^s \odot_s \hat{\omega}_*^{(s)}\right]^r t^r\right\} \\ &= \prod_{s=1}^{\infty} e^{(-i\vec{q})^s \odot_s \hat{\omega}_*^{(s)} t}. \end{aligned} \quad (3.43)$$

Substituting the expression (3.43) into equation (3.40) leads to

$$|\Phi_{\vec{q}}(t)\rangle \simeq \sum_{p=0}^{\infty} \|\chi_{\bar{n}}^{(p)}\rangle\rangle \odot_p (-i\vec{q})^p \bar{n}_{\vec{q}}^{(0)}(t), \quad t \gg \tau_0, \quad (3.44)$$

where

$$\bar{n}^{(0)}_{\vec{q}}(t) = c^{I(0)}_{\bar{n}}(\vec{q}) e^{\sum_{s=0}^{\infty}(-i\vec{q})^s \odot_s \hat{\omega}^{(s)}_* t}. \tag{3.45}$$

Thus we have found the explicit expression for arbitrary Fourier mode $|\Phi_{\vec{q}}(t)\rangle$ of one particle distribution function $|f(\vec{r},t)\rangle$ in the limit of long times, $t \gg \tau_0$.

Fourier inversion of the equation (3.44) gives

$$|f(\vec{r},t)\rangle \simeq \sum_{p=0}^{\infty} \left\| \chi^{(p)}_{\bar{n}} \right\rangle\!\!\right\rangle \odot_p \left(-\frac{\partial}{\partial \vec{r}}\right)^p e^{\hat{\omega}^{(0)}_* t} \times \int \mathrm{d}\vec{q}\, c^{I(0)}_{\bar{n}}(\vec{q})\, e^{+i\vec{q}\cdot\vec{r}} \prod_{s=1}^{\infty} e^{(-i\vec{q})^s \odot_s \hat{\omega}^{(s)}_* t}, \quad t \gg \tau_0. \tag{3.46}$$

Coefficients $c^{I(0)}_{\bar{n}}(\vec{q}) = \langle \tilde{\psi}^{(0)}_{\bar{n}} | \Phi^I_{\vec{q}} \rangle$ depend on $\vec{q}$ through their dependence on the initial state of the swarm $|\Phi^I_{\vec{q}}\rangle$. Our approach strongly suggests that solution (3.46) of linear Boltzmann equation (1.1), obtained for arbitrary initial condition, properly describes long-time behavior of a swarm of charged particles with arbitrary varying inhomogeneities.

Finally, let us consider the long-time behavior of the transport coefficients $\hat{\omega}^{(p)}_{\vec{q}}(t)$, $p \geqslant 0$. We are interested in times $t \gg \tau_0$. Introducing asymptotic expressions for vectors $\| \varkappa^{(p)}_{\vec{q}}(t) \rangle\!\rangle$ (see Eqs. (3.36)–(3.39)) into equations (2.20) and (2.21), we obtain that

$$\hat{\omega}^{(p)}_{\vec{q}}(t) \simeq \hat{\omega}^{(p)}_*, \quad t \gg \tau_0, \quad p \geqslant 0. \tag{3.47}$$

The present derivation, valid for any $\vec{q}$, shows that all time-dependent transport coefficients $\hat{\omega}^{(p)}_{\vec{q}}(t)$ achieve their hydrodynamic values $\hat{\omega}^{(p)}_*$ (Eq. (3.27)) in the same characteristic time.

The gradient expansions (1.2) and (3.46) are still not identical, but are equivalent in a sense of separation of velocity and space-time dependence of the one-particle distribution function. In order to bring them into an identical form we use the expression (3.47) to solve GDE (2.19) in the asymptotic regime $t \gg \tau_0$, and obtain

$$n_{\vec{q}}(t) \simeq n_{\vec{q}}(0)\, e^{\sum_{s=0}^{\infty}(-i\vec{q})^s \odot_s \hat{\omega}^{(s)}_* t}, \quad t \gg \tau_0. \tag{3.48}$$

Inserting equation (3.48) into (3.45) we get

$$\bar{n}^{(0)}_{\vec{q}}(t) = \frac{c^{I(0)}_{\bar{n}}(\vec{q})}{n_{\vec{q}}(0)} = \frac{\left\langle \tilde{\psi}^{(0)}_{\bar{n}} \middle| \Phi^I_{\vec{q}} \right\rangle}{\left\langle \phi^0 \middle| \Phi^I_{\vec{q}} \right\rangle} n_{\vec{q}}(t). \tag{3.49}$$

This expression is valid for any arbitrary initial condition. However, in the physically interesting case where initial state $|\Phi^I_{\vec{q}}\rangle = |f_0\rangle\, n_{\vec{q}}(0)$ separates velocity and space-time dependence, we find that $\bar{n}^{(0)}_{\vec{q}}(t)$ and $n_{\vec{q}}(t)$ are proportional:

$$\bar{n}^{(0)}_{\vec{q}}(t) = \frac{\left\langle \tilde{\psi}^{(0)}_{\bar{n}} \middle| f_0 \right\rangle}{\left\langle \phi^0 \middle| f_0 \right\rangle} n_{\vec{q}}(t). \tag{3.50}$$

The constant of proportionality can be absorbed into the definition of coefficients $\hat{f}^{(p)}(\vec{v})$, and, for a class of initial conditions $|\Phi^I_{\vec{q}}\rangle = |f_0\rangle\, n_{\vec{q}}(0)$, the asymptotic solution (3.46) reduces to the hydrodynamic form (1.2).

4 Concluding remarks

In the present paper we have demonstrated that the *Assumption I* is sufficient to obtain the expression equation (3.46) for the one-particle distribution function in a long-time regime $t \gg \tau_0$. In addition, we have shown that in this limit ($t \gg \tau_0$) *all* transport coefficients $\hat{\omega}^{(p)}_{\vec{q}}(t)$ become time and $\vec{q}$ independent in the *same* characteristic time and reach their hydrodynamics values $\hat{\omega}^{(p)}_*$. Transport coefficients $\hat{\omega}^{(p)}_*$ are explicitely given by the microscopic expressions (3.33)–(3.35) (see also Eq. (3.27)) in terms of solutions of the eigenvalue problems (3.1) and (3.2). All these results are not restricted to small gradients in the density of swarm particles.

Analogous results can be obtained for small values of wave vectors $\vec{q}$ and for long times $t \gg \tau_0$ by using the Résibois method of derivation of linear transport coefficients [11,16]. Résibois treated the eigenvalue problem associated to the linear generalized Boltzmann equation [18] by stationary perturbation method in powers of the uniformity parameter q. In the long-wavelength limit $\vec{q} \to 0$, the usual expressions for the transport coefficients in neutral gases come out as q^p-coefficients of the eigenvalues. Résibois method requires assumption of upper semicontinuity of spectrum [16,19]:

Assumption II *The small perturbation $\hat{H}'_{\vec{q}}$ shifts slightly the eigenvalues of $\hat{H}_0$, but assumption* (3.9) *remains valid for eigenvalues $\Lambda_n(\vec{q})$ of $\hat{H}_{\vec{q}}$ in the long-wavelength limit $\vec{q} \to 0$:*

$$\mathrm{Re}\ \Lambda_n(\vec{q}) < \mathrm{Re}\ \Lambda_{\bar{n}}(\vec{q}), \quad \forall n \neq \bar{n}; \quad \vec{q} \to 0. \tag{4.1}$$

This assumption states that perturbation $\hat{H}'_{\vec{q}}$, for small values of the wave vector $\vec{q}$, makes a small effect on the spectrum of the operator $\hat{H}_0$. Hence, spectrum of the perturbed operator $\hat{H}_{\vec{q}}$ (at least for sufficiently small $\vec{q}$) also has an isolated eigenvalue separated from the rest of the spectrum along the real axis. Assumption of upper semicontinuity of the spectrum is not explicitly used in our analysis. However, this assumption is an essential one in order to ensure convergence of perturbative expansion (2.9).

In conclusion, in this work we have not limited our analysis to weakly inhomogeneous swarms (i.e. small relative density gradients $n^{-1}(\partial/\partial\vec{r})n$). Also, we have demonstrated the separation of one particle distribution function into the spatio-temporal and velocity part for arbitrary initial conditions. Presented results depend only on intrinsic properties of the operators occurring in the linear kinetic equation, rather than the detailed form of collision operator. Consequently, it is expected that the implications of our analysis are applicable to wide class of collision operators with appropriate spectral properties.

The work presented in this paper was supported in part by the Ministry of Science and Environmental Protection of the Republic of Serbia under grant ON 141035.

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