Wick Rotation and Abelian Bosonization

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Abstract

We investigate the connection between Abelian bosonization in the Minkowski and Euclidean formalisms. The relation is best seen in the complex time formalism of S. A. Fulling and S. N. M. Ruijsenaars [11].

1 Introduction

It is well-known [1]-[3] that two-dimensional free massless Dirac theory is equivalent to free massless scalar field theory. In the operator formalism, this equivalence is established by giving the following explicit construction of fermionic fields in terms of bosonic fields:

$$\begin{aligned}\psi_1 &= \sqrt{\frac{c\mu}{2\pi}}\,\chi_1\ :e^{-i2\sqrt{\pi}\phi_1}: \\ \psi_2 &= \sqrt{\frac{c\mu}{2\pi}}\,\chi_2\ :e^{i2\sqrt{\pi}\phi_2}:\ .\end{aligned} \tag{1}$$

$\psi_{1/2}$ are two components of the spinor field ψ. On the equation of motion $\psi_1=\psi_1(x^+)$, $\psi_2=\psi_2(x^-)$. In Minkowski space $x^\pm$ are the light-cone coordinates, while in Euclidean space we are dealing with holomorphic and anti-holomorphic coordinates. Similary, on the equation of motion the scalar field $\phi(x)$ is a sum of a left-moving and right-moving part $\phi_1(x^+)$ and $\phi_2(x^-)$. In (1), μ is an infra-red cut off, necessary for dealing with massless scalar fields in two dimensions [6], while c is a conveniently chosen constant $\ln c=\Gamma'(0)$.

In the Minkowski formalism Mandelstam [1], [2] chose[1] $\chi_1=\chi_2=1$. On the other hands, in the Euclidean formalism [3]-[5], it is necessary to choose the χ prefactors to satisfy

$$\begin{aligned}\{\chi_\alpha,\chi_{\beta\neq\alpha}\} &= \{\chi_\alpha,\chi^\dagger_{\beta\neq\alpha}\}=0 \\ [\chi_\alpha,\chi_\alpha] &= [\chi_\alpha,\chi^\dagger_\alpha]=0\,.\end{aligned} \tag{2}$$

[1]With this choice, the bosonic correlator

$$\langle\phi(x)\phi(y)\rangle=-\frac{1}{4\pi}\,\ln c^2\mu^2\left[-(x-y)^2+i\varepsilon(x^0-y^0)\right]$$

automatically gives the correct fermi correlator and canonical anti-commutators.

Without this choice, one indeed gets the correct fermi correlator, but *not* the canonical anti-commutators. In most cases we are solely interested in correlators (for example in conformal field theory) and the choice of χ is not important. In this paper, however, we want to establish the precise connection between the Minkowski and Euclidean formalisms. For this reason it will be quite important to keep track of the χ's.

2 Minkowski Formalism

In his original paper, Mandelstam introduced a dual scalar field

$$\widetilde{\phi}(x) = \int_{-\infty}^{x^1} dy^1 \, \dot{\phi}(x^0, y^1) \,, \tag{3}$$

which satisfies $\partial_\mu \widetilde{\phi} = \varepsilon_\mu{}^\nu \partial_\nu \phi$. Both ϕ and $\widetilde{\phi}$ are solutions of the massless Klein-Gordon equation $\partial^2 \phi = \partial^2 \widetilde{\phi} = 0$. The corresponding two-point correlators are

$$\begin{aligned} \langle \phi(x)\phi(y) \rangle &= \Delta_+(x-y) \\ \langle \widetilde{\phi}(x)\widetilde{\phi}(y) \rangle &= \Delta_+(x-y) \\ \langle \phi(x)\widetilde{\phi}(y) \rangle &= \widetilde{\Delta}_+(x-y) + \frac{i}{4} \\ \langle \widetilde{\phi}(x)\phi(y) \rangle &= \widetilde{\Delta}_+(x-y) - \frac{i}{4} \,. \end{aligned} \tag{4}$$

As in [7], we have introduced the auxiliary functions:

$$\begin{aligned} \Delta_+(x) &\equiv -\frac{1}{4\pi} \ln c^2 \mu^2 (-x^2 + i\varepsilon x^0) \\ \widetilde{\Delta}_+(x) &\equiv -\frac{1}{4\pi} \ln \left| \frac{x^0 + x^1 - i\varepsilon}{x^0 - x^1 - i\varepsilon} \right| \,. \end{aligned} \tag{5}$$

The general solution of the equation of motion for the massless scalar field is of the form $\phi(x) = \phi_1(x^+) + \phi_2(x^-)$, where we have introduced light-cone coordinates $x^\pm = x^0 \pm x^1$. The left-moving and right-moving fields are determined up to a constant (corresponding to the zero mode). Mandelstam chose the particular decomposicion

$$\begin{aligned} \phi_1 &= \frac{1}{2}(\phi + \widetilde{\phi}) \\ \phi_2 &= \frac{1}{2}(\phi - \widetilde{\phi}) \,. \end{aligned} \tag{6}$$

For later convenience let us list the scalar correlators in terms of the above decomposition. We have

$$\begin{aligned} \langle \phi_1(x)\phi_1(y) \rangle &= \frac{1}{2}\left[\Delta_+(x-y) + \widetilde{\Delta}_+(x-y)\right] \\ \langle \phi_2(x)\phi_2(y) \rangle &= \frac{1}{2}\left[\Delta_+(x-y) - \widetilde{\Delta}_+(x-y)\right] \\ \langle \phi_1(x)\phi_2(y) \rangle &= -\frac{i}{8} \\ \langle \phi_2(x)\phi_1(y) \rangle &= \frac{i}{8} \,. \end{aligned} \tag{7}$$

From (6) we directly get the explicit bosonization formula

$$\begin{aligned}
\psi_1 &= \sqrt{\frac{c\mu}{2\pi}}\ :\exp\left[-i\sqrt{\pi}\phi(x) - i\sqrt{\pi}\int_{-\infty}^{x^1} dy^1\,\dot\phi(x^0,y^1)\right]: \\
\psi_2 &= \sqrt{\frac{c\mu}{2\pi}}\ :\exp\left[i\sqrt{\pi}\phi(x) - i\sqrt{\pi}\int_{-\infty}^{x^1} dy^1\,\dot\phi(x^0,y^1)\right]: .
\end{aligned} \tag{8}$$

A trivial check shows that (8) leads to the canonical anti-commutation relations for ψ.

For later convenience we will write the above results in a different way. Let us split the scalar field $\phi(x)$ into two pieces

$$\phi(x) = \varphi(x) + \rho(x)\,. \tag{9}$$

This split is chosen so that $\langle\varphi(x)\rho(y)\rangle = 0$. Our aim is to absorb all the ε-dependence of the two-point correlators into the ρ term. This is easily done, bearing in mind the identity $\ln(a \pm i\varepsilon) = \ln|a| \pm i\pi\theta(-a)$, where a is a real number. As a result the ρ correlators are

$$\begin{aligned}
\langle\rho_1(x)\rho_1(y)\rangle &= \frac{i}{8}\,H(x^+ - y^+) \\
\langle\rho_2(x)\rho_2(y)\rangle &= \frac{i}{8}\,H(x^- - y^-) \\
\langle\rho_1(x)\rho_2(y)\rangle &= -\frac{i}{8} \\
\langle\rho_2(x)\rho_1(y)\rangle &= \frac{i}{8}\,.
\end{aligned} \tag{10}$$

In these formulas $H(x)$ is the Heaviside function $H(x) \equiv \theta(x) - \theta(-x)$. Using (9), our basic bosonization formula becomes

$$\begin{aligned}
\psi_1 &= \sqrt{\frac{c\mu}{2\pi}}\ :e^{-i2\sqrt{\pi}\phi_1}: \ = \sqrt{\frac{c\mu}{2\pi}}\,\chi_1\ :e^{-i2\sqrt{\pi}\varphi_1}: \\
\psi_2 &= \sqrt{\frac{c\mu}{2\pi}}\ :e^{i2\sqrt{\pi}\phi_2}: \ = \sqrt{\frac{c\mu}{2\pi}}\,\chi_2\ :e^{i2\sqrt{\pi}\varphi_2}:,
\end{aligned} \tag{11}$$

where the above procedure gives us the following explicit form for the prefactors

$$\begin{aligned}
\chi_1 &\equiv\ :e^{-i2\sqrt{\pi}\rho_1}: \\
\chi_2 &\equiv\ :e^{i2\sqrt{\pi}\rho_2}: .
\end{aligned} \tag{12}$$

Let us emphasize that this is just Mandelstam's old result written in different way — in terms of the field φ whose correlators do not contain ε. For this reason, everything is consistent and we find that the prefactors satisfy

$$\begin{aligned}
\{\chi_\alpha(x), \chi_{\beta\neq\alpha}(y)\} &= \{\chi_\alpha(x), \chi^\dagger_{\beta\neq\alpha}(y)\} = 0 \\
[\chi_\alpha(x), \chi_\alpha(y)] &= [\chi_\alpha(x), \chi^\dagger_\alpha(y)] = 0\,.
\end{aligned} \tag{13}$$

3 Euclidean Formalism

Bosonization in Euclidean space proceeds in much the same way as in Minkowski space. The massless scalar field is $\varphi(x) = \varphi_1(z) + \varphi_2(\bar{z})$, where $z = x_0 + ix_1$ and $\bar{z} = x_0 - ix_1$. These fields satisfy

$$\begin{aligned}
\langle \varphi_1(z)\varphi_1(z')\rangle &= -\frac{1}{4\pi}\ln c\mu(z-z') \\
\langle \varphi_2(\bar{z})\varphi_2(\bar{z}')\rangle &= -\frac{1}{4\pi}\ln c\mu(\bar{z}-\bar{z}') \\
\langle \varphi_1(z)\varphi_2(\bar{z}')\rangle &= 0\,.
\end{aligned} \tag{14}$$

As we have stressed, in order to get the correct anti-commutation relation for the fermi fields, one needs to introduce non-trivial prefactors χ satisfying (2). From the previous section, we see that we have at our disposal an *explicit construction* of these prefactors. Turning that derivation on its head, we now introduce $\rho(x)$ satisfying

$$\begin{aligned}
\langle \rho_1(z)\rho_1(z')\rangle &= -\frac{1}{4\pi}\ln\left[1-\frac{i\varepsilon}{c\mu(z-z')}\right] \\
\langle \rho_2(\bar{z})\rho_2(\bar{z}')\rangle &= -\frac{1}{4\pi}\ln\left[1-\frac{i\varepsilon}{c\mu(\bar{z}-\bar{z}')}\right] \\
\langle \rho_1(z)\rho_2(\bar{z}')\rangle &= -\frac{i}{8} \\
\langle \rho_1(z)\rho_1(z')\rangle &= \frac{i}{8}\,.
\end{aligned} \tag{15}$$

This is just a straightforward extension to complex coordinates of the Minkowski result given in (10). Finally, we make a *new* scalar field $\phi(x) = \varphi(x) + \rho(x)$. The new field satisfies

$$\begin{aligned}
\langle \phi_1(z)\phi_1(z')\rangle &= -\frac{1}{4\pi}\ln c\mu(z-z'-i\varepsilon) \\
\langle \phi_2(\bar{z})\phi_2(\bar{z}')\rangle &= -\frac{1}{4\pi}\ln c\mu(\bar{z}-\bar{z}'-i\varepsilon) \\
\langle \phi_1(z)\phi_2(\bar{z}')\rangle &= -\frac{i}{8} \\
\langle \phi_2(\bar{z})\phi_1(z')\rangle &= \frac{i}{8}\,.
\end{aligned} \tag{16}$$

We seem to have come to a strange result: the $i\varepsilon$-regularization that is obviously necessory in Minkowski space saves the day in Euclidean space as well.

4 Complex Time Formalism

Our intuition concerning the $i\varepsilon$ prescription is based on the Feynman propagator. Based on that, $i\varepsilon$ wouldn't be necessary in the Euclidean theory — it doesn't do any damage, it is just *not necessary.* We shall, however, see that there exist objects for which $i\varepsilon$ is needed in both Euclidean and Minkowski formalisms. To do this, let us look at the following two-point functions

$$\begin{aligned} \Delta_+(x-y) &\equiv \langle \phi(x)\phi(y)\rangle \\ \Delta_-(x-y) &\equiv \langle \phi(y)\phi(x)\rangle = \Delta_+(x-y)^* \\ i\Delta_F(x-y) &\equiv \langle T\phi(x)\phi(y)\rangle = \theta(x^0-y^0)\Delta_+(x-y) + \theta(y^0-x^0)\Delta_-(x-y) \\ i\Delta(x-y) &\equiv [\phi(x),\phi(y)] = \Delta_+(x-y) - \Delta_-(x-y)\,. \end{aligned} \tag{17}$$

Let us next make an analytical continuation of these two-point functions to complex time. Following Fulling and Ruijsenaars [11], [12], we take

$$t \longrightarrow s = t + i\tau\,, \tag{18}$$

and define the single function

$$\mathcal{D}(s) \equiv \begin{cases} \Delta_-(s)\,, & \mathrm{Im}\, s > 0 \\ \Delta_+(s)\,, & \mathrm{Im}\, s < 0\,. \end{cases} \tag{19}$$

This function is analytic in the s-plane except for a branch cut along the t-axis[2] for $|t| > |x|$. From this and (5), it follows that we have

$$\mathcal{D}(s) = -\frac{1}{4\pi}\, \ln c^2\mu^2(-s^2+x^2)\,. \tag{20}$$

In terms of $\mathcal{D}(s)$, the standard Minkowski two-point functions $\Delta_\pm$, Δ_F correspond to the contours $C_\pm$ and C_F in Figure 1. For example $\Delta_\pm(t) = \mathcal{D}\left(t \mp \frac{i\varepsilon}{2}\right)$.

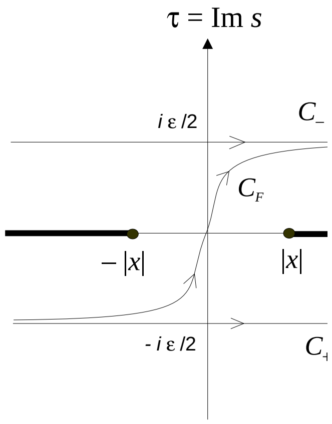

Figure 1: Contours corresponding to two-point functions $\Delta_\mp$ and Δ_F.

Let us now look at the Euclidean theory. The Euclidean propagator satisfies

$$\left(\partial_\tau^2 + \partial_x^2\right)\Delta_E(\tau,x) = -\delta(\tau)\delta(x)\,. \tag{21}$$

It can also be given in terms of $\mathcal{D}(s)$. This time the contour is the imaginary s-axis. Wick rotation of the propagator corresponds to taking C_F into C_E, as in Figure 2. Further, there are no obstructions, so C_E can be deformed to the imaginary s-axis (hence there is no reason for $i\varepsilon$).

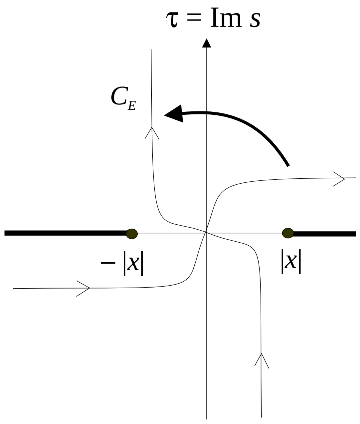

Figure 2: Wick rotation of the propagator: $C_F \to C_E$.

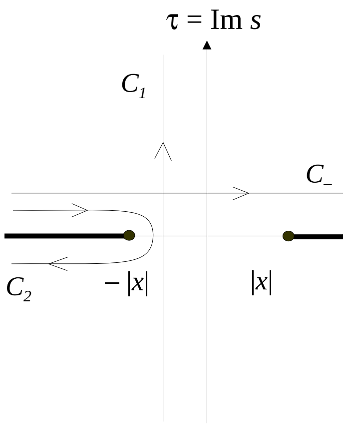

Figure 3: Wick rotation of Δ_- function: $C_- \to C_1 \cup C_2$.

Let us next Wick rotate the Δ_- function. The contour C_- rotates into $C_1 \cup C_2$, as is shown in Figure 3. As before, C_1 can be deformed to the imaginary s-axis (it is not sensitive to $i\varepsilon$). However, this time we have an additional piece corresponding to C_2. From equations (17) and (19), it follows that this is proportional to the canonical commutator. A naive Wick rotation of Δ_- would not have seen this contribution. The $i\varepsilon$ prescription *is* important in the whole complex time plane.

Let us emphasize this point once again. From the last relation in (17), we see that the canonical commutator may be written in terms of $\Delta_\pm$ as

$$[\phi(x), \phi(y)] = \frac{1}{i}\,[\Delta_+(x-y) - \Delta_-(x-y)]\ . \tag{22}$$

Naive Wick rotation of the right hand side gives zero. On the other hand, doing things carefully, *i.e.* keeping track of the branch cut (or equivalently of $i\varepsilon$), the right hand side of (22) goes over into the correct canonical commutator of the Euclidean theory. This is precisely the same thing that we saw in the previous section, where we looked at bosonization in Euclidean space. The naive treatment gets wrong results for fermi commutators — they vanish. The correct fermi commutators follow from incorporating the $i\varepsilon$ prescription.

[2]The fact that the cut is absent for $|t| < |x|$ is a consequence of the causality condition which states that $[\phi, \phi]$ vanishes outside the light-cone.

5 A Generalization

At the end, let us look at a slight generalization of the standard bosonization sheme along the lines of Nakanishi [7]. In his paper, Nakanishi looks at two scalar fields ϕ and $\widetilde{\phi}$, connected by the duality relation

$$\partial_\mu \widetilde{\phi} = \varepsilon_\mu{}^\nu \partial_\nu \phi \,. \tag{23}$$

Treating both fields as fundamental, we impose

$$\begin{aligned} [\phi(x),\, \phi(y)] &= i\Delta(x-y) \\ [\widetilde{\phi}(x),\, \widetilde{\phi}(y)] &= i\Delta(x-y)\,. \end{aligned} \tag{24}$$

The mixed correlator is assumed to also give a c-number function. From this and (24), we find

$$[\phi(x), \widetilde{\phi}(y)] = i\widetilde{\Delta}(x-y) + i\alpha\,, \tag{25}$$

where $i\widetilde{\Delta} \equiv \widetilde{\Delta}_+ - \widetilde{\Delta}_-$ and $\widetilde{\Delta}_- \equiv \widetilde{\Delta}_+^*$. The parameter α is a new undetermined real constant. Mandelstam's bosonization sheme corresponds to the choice $\alpha = 1$. Different authors have considered the cases $\alpha = \frac{1}{2}$ [7], and $\alpha = 0$ [5], [8]-[10]. Essentially, different choices for α correspond to different boundary conditions. At the level of correlators, the general α case gives

$$\begin{aligned} \langle \phi(x)\phi(y) \rangle &= \Delta_+(x-y) \\ \langle \widetilde{\phi}(x)\widetilde{\phi}(y) \rangle &= \Delta_+(x-y) \\ \langle \phi(x)\widetilde{\phi}(y) \rangle &= \widetilde{\Delta}_+(x-y) + \frac{i}{4}\,\alpha \\ \langle \widetilde{\phi}(x)\phi(y) \rangle &= \widetilde{\Delta}_+(x-y) - \frac{i}{4}\,\alpha\,, \end{aligned} \tag{26}$$

or, equivalently, using (6) we find

$$\begin{aligned} \langle \phi_1(x)\phi_1(y) \rangle &= \frac{1}{2}\left[\Delta_+(x-y) + \widetilde{\Delta}_+(x-y)\right] \\ \langle \phi_2(x)\phi_2(y) \rangle &= \frac{1}{2}\left[\Delta_+(x-y) - \widetilde{\Delta}_+(x-y)\right] \\ \langle \phi_1(x)\phi_2(y) \rangle &= -\frac{i}{8}\,\alpha \\ \langle \phi_2(x)\phi_1(y) \rangle &= \frac{i}{8}\,\alpha\,. \end{aligned} \tag{27}$$

The above authors do not look at fermi commutators. A simple calculation, as in the first section, shows that one gets the correct results for $\{\psi_1, \psi_1\}$ and $\{\psi_2, \psi_2\}$ for all values of α. On the other hand, the mixed commutator $\{\psi_1, \psi_2\}$ is found to be proportional to $1 + (-1)^\alpha$. Therefore, by insisting on canonical fermi commutators, we impose the condition $\alpha = 2n + 1$, where n is an integer. Note that Mandelstam's bosonization scheme satisfies this condition, while the others do not.

The issue of boundary conditions in bosonization is a very important one. It is still an open problem. We are currently looking into that on the example of the bosonization of the massive Thirring model.

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