Introduction to Percolation Theory

Danica Stojiljkovic
System in concern

- Discrete system in $d$ dimensions
- Lattices
  - 1D: array
  - 2D: square, triangular, honeycomb
  - 3D: cubic, bcc, fcc, diamond
  - dD: hypercubic
  - Bethe lattice (Cayley tree)
System in concern
System in concern

• Each lattice site is occupied randomly and independently with probability $p$
• Nearest neighbors: sites with one side in common
• Cluster: a group of neighboring sites
Site and Bond Percolation

• A “site” can be a field or a node of a lattice

• Bond percolation: bond is present with probability $x$

• Site-bond percolation: continuous transition between site and bond percolation
Percolation thresholds

20 x 20 site lattice

- At some occupation probability $p_c$ a spanning (infinite) cluster will appear for the first time
- Percolation thresholds are different for different systems
Cluster numbers

- Number of cluster with $s$ sites per lattice sites: $ns$
- For 1D lattice:
  \[ n_s = p^s (1 - p)^2 \]
- In general:
  \[ n_s = \sum g_{st} p^s (1 - p)^t \]
- $t$ – perimeter
- $g_{st}$ – number of lattice animals with size $s$ and perimeter $t$
Average cluster size

• Probability that random site belongs to a cluster of size $s$ is
  \[ w_s = \frac{s n_s}{\sum s n_s} \]

• Average size of a cluster that we hit if we point to an arbitrary occupied site that is not a part of an infinite cluster
  \[ S = \sum w_s s \]
  \[ = \frac{\sum s^2 n_s}{\sum s n_s} \]
Average cluster size

• For 1D:
  \[ S = \frac{(1+p)}{(1-p)} \sim |pc-p|^{-1} \]

• For Bethe lattice:
  \[ S \sim |pc-p|^{-1} \]

• In general
  \[ S \sim |pc-p|^{-\gamma} \]

- compressibility, susceptibility
Cluster numbers at $pc$

- If $ns$ at $pc$ decays exponentially, then $S$ would be finite, therefore:
  
  $$ns(pc) \sim s^{-\tau}$$

- We calculate for Bethe
  
  $$\frac{ns(p)}{ns(pc)} \sim \exp(-Cs)$$

where

$$C \sim |pc - p|^{-1/\sigma}$$
Scaling assumption

• Generalizing Bethe result:
  \[ ns(p) \sim s^{-\tau} \exp(-|p-p_c|^{1/\sigma} s) \]
  \( (1D \text{ result does not fit}) \)

• We can derive all other critical exponents from \( \tau \) and \( \sigma \)

• Scaling assumption
  \[ ns(p) \sim s^{-\tau} F(|p-p_c|^{1/\sigma} s) \]

or
  \[ ns(p) \sim s^{-\tau} \Phi((p-p_c)s \ \sigma) \]
Universality

• $\Phi(z)$ depends only on dimensionality, not on lattice structure
• $\tau$ and $\sigma$, as well as function $\Phi(z)$ are universal
• Plotting $ns(p)s-\tau$ versus $(p-pc)s\sigma$ would fall on the same line
Strength of the infinite network

- $P$ – probability of an arbitrary site belonging to the infinite network
  \[ P + \sum_s n_s = p \]
- Applicable only at $p > p_c$
- For Bethe lattice
  \[ P \sim (p - p_c) \]
- In general
  \[ P \sim |p - p_c|^\beta \]
- Spontaneous magnetization, difference between vapor and liquid density
Correlation function

• $g(r)$ - Probability that a site at distance $r$ from an occupied site belongs to the same cluster

• In 1D array:
  
  $$g(r) = pr = \exp\left(-\frac{r}{\xi}\right)$$
  
  $$\xi = -\frac{1}{\ln(p)} = \frac{1}{pc-p}$$

• $\xi$ - Correlation length
Radius of gyration

• Average of the square radii

\[ R_s^2 = \sum_{i=1}^{s} \left| r_i - r_0 \right|^2 \]

• \( r_0 \) – center of mass

• Relation of gyration radii and average site distance

\[ 2R_s^2 = \sum_{ij} \frac{\left| r_i - r_j \right|^2}{s^2} \]
Correlation length

• Average distance between two sites belonging to the same cluster:

\[ \xi = \sum \frac{r^2 g(r)}{\sum g(r)} \]

\[ = \frac{2 \sum R_s^2 s^2 n_s}{\sum s^2 n_s} \]

• Near the percolation threshold

\[ \xi \sim |p - p_c|^{-\nu} \]
At $p = p_c$:

$$M \sim L^D$$

where $M$ is mass of the largest cluster in a box with sides $L$

$D < d$ and it is called fractal dimension

For gyration radii:

$$R_s \sim s^{1/D}$$
Crossover phenomena

• For finite lattices whose linear dimension $L \ll \xi$, cluster behaves as fractals
  \[ M \sim L \, D \]

• $\xi$ acts like a measuring stick, and in its absence all the relevant functions becomes power laws

• For $L > \xi$,
  \[ M \sim (L/\xi)^d \, \xi D \sim L \, d \]
• Introduction to Percolation Theory, 2nd revised edition, 1993 by Dietrich Stauffer and Amnon Aharony
To be continued...

- Exact solution for 1D and Bethe lattice
- Scaling assumption
- Deriving relations between critical coefficients
- Numerical methods
- Renormalization group