Introduction to Percolation Theory

Danica Stojiljkovic

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System in concern

- Discrete system in *d* dimensions
- Lattices
  - 1D: array
  - 2D: square, triangular, honeycomb
  - 3D: cubic, bcc, fcc, diamond
  - dD: hypercubic
  - Bethe lattice (Cayley tree)

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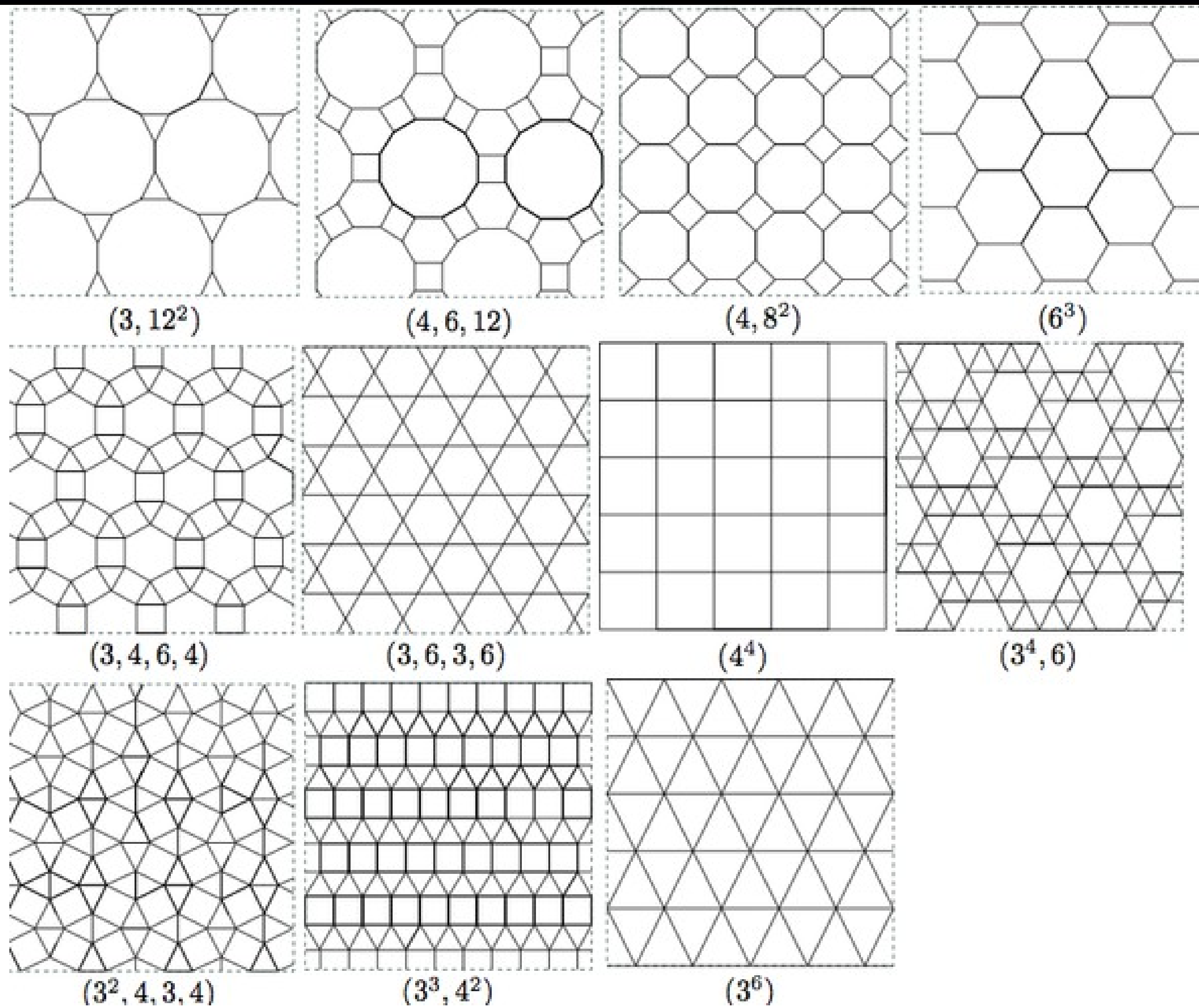
System in concern



System in concern

- Each lattice site is occupied randomly and independently with probability ***p***
- Nearest neighbors: sites with one side in common
- Cluster: a group of neighboring sites

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Site and Bond Percolation

- A "site" can be a field or a node of a lattice
- Bond percolation: bond is present with probability ***x***
- Site-bond percolation: continuous transition between site and bond percolation

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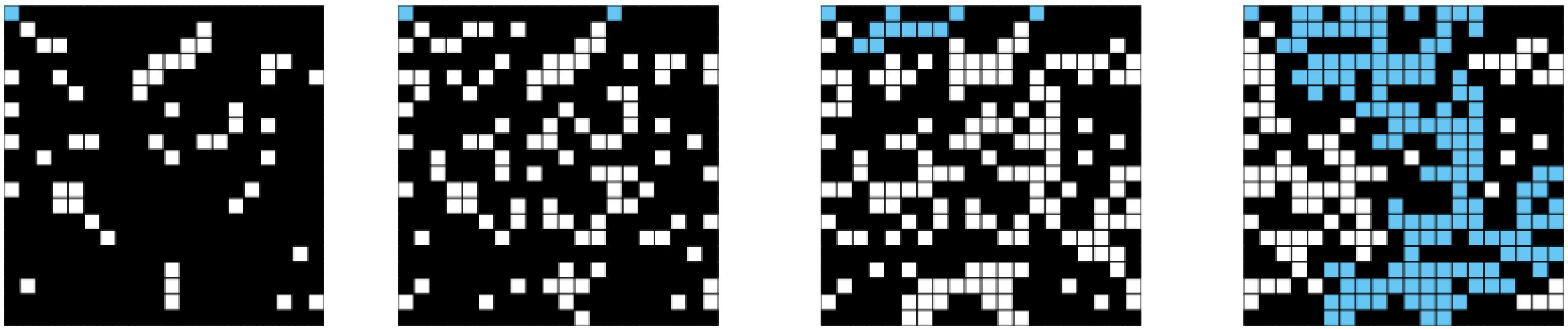
Percolation thresholds

20 x 20 site lattice



- At some occupation probability ***pc*** a spanning (infinite) cluster will appear for the first time
- Percolation thresholds are different for different systems

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Percolation thresholds

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Cluster numbers

- Number of cluster with ***s*** sites per lattice sites: ***ns***
- For 1D lattice:

$$n_s = p^s (1 - p)^2$$

- In general:

$$n_s = \sum_t g_{st} p^s (1 - p)^t$$

- t – perimeter
- gst –number of lattice animals with size s and perimeter t

Average cluster size

- Probability that random site belongs to a cluster of size s is $w_s = s n_s / \sum s n_s$
- Average size of a cluster that we hit if we point to and arbitrary occupied site that is not a part of an infinite cluster

$$S = \sum w_s s$$

$$= \sum s^2 n_s / \sum s n_s$$

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Average cluster size

- For 1D:

$$S = (1+p)/(1-p)$$
$$\sim |p_c - p|^{-1}$$

- For Bethe lattice:

$$S \sim |p_c - p|^{-1}$$

- In general

$$S \sim |p_c - p|^{-\gamma}$$

- compressibility, susceptibility

Cluster numbers at pc

- If ns at pc decays exponentially, than S would be finite, therefore:

$$ns(pc) \sim s^{-\tau}$$

- We calculate for Bethe

$$ns(p)/\,ns(pc) \sim exp\,(-Cs)$$

where

$$C \sim |\,pc - p\,|^{1/\sigma}$$

Scaling assumption

- Generalizing Bethe result:

$$n_s(p) \sim s^{-\tau} \exp(-|p-p_c|^{1/\sigma} s)$$

(1D result does not fit)

- We can derive all other critical exponents from τ and σ
- Scaling assumption

$$n_s(p) \sim s^{-\tau} F(|p-p_c|^{1/\sigma} s)$$

or $n_s(p) \sim s^{-\tau} \Phi((p-p_c)s^{\sigma})$

Universality

- $\Phi(z)$ depends only on dimensionality, not on lattice structure
- τ and σ, as well as function $\Phi(z)$ are universal
- Plotting $n_s(p)s^{-\tau}$ versus $(p-p_c)s^{\sigma}$ would fall on the same line

Strength of the infinite network

- P – probability of an arbitrary site belonging to the infinite network

$$P + \sum sns = p$$

- Applicable only at $p > pc$
- For Bethe lattice

$$P \sim (p - pc)$$

- In general

$$P \sim |p - pc|^{\beta}$$

- Spontaneous magnetization, difference between vapor and liquid density

Correlation function

- $g(r)$ -Probability that a site at distance r from an occupied site belongs to the same cluster
- In 1D array:

$$g(r) = pr = exp(-r/\xi)$$

$$\xi = -1/\ln(p) = 1/(pc-p)$$

- ξ - Correlation length

Radius of gyration

- Average of the square radii

$$R_s^2 = \sum_{i=1}^{s} \frac{\left| r_i - r_0 \right|^2}{s}$$

- ***r0*** – center of mass
- Relation of gyration radii and average site distance

$$2R_s^2 = \sum_{ij} \frac{\left| r_i - r_j \right|^2}{s^2}$$

Correlation length

- Average distance between two sites belonging to the same cluster:

$$\xi^2 = \frac{\sum_r r^2 g(r)}{\sum_r g(r)} = \frac{2\sum_s R_s^2 s^2 n_s}{\sum_r s^2 n_s}$$

- Near the percolation threshold

$$\xi \sim |p - p_c|^{-\nu}$$

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Fractal dimension

- At $p = pc$:

$$M \sim LD$$

where M is mass of the largest cluster in a box with sides L

- $D<d$ and it is called fractal dimension
- For gyration radii:

$$Rs \sim s1/D$$

Crossover phenomena

- For finite lattices whose linear dimension $L << \xi$, cluster behaves as fractals

$$M \sim L \; D$$

- ξ acts like a measuring stick, and in its absence all the relevant functions becomes power laws
- For $L > \xi$,

$$M \sim (L/\,\xi)d \bullet \xi D \sim L \; d$$

References

- **Introduction to Percolation Theory, 2nd revised edition,1993 by Dietrich Stauffer and Amnon Aharony**

To be continued...

- Exact solution for 1D and Bethe lattice
- Scaling assumption
- Deriving relations between critical coefficients
- Numerical methods
- Renormalization group