Local Bifurcations of equilibria in delayed coupled type II excitable systems

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Introduction

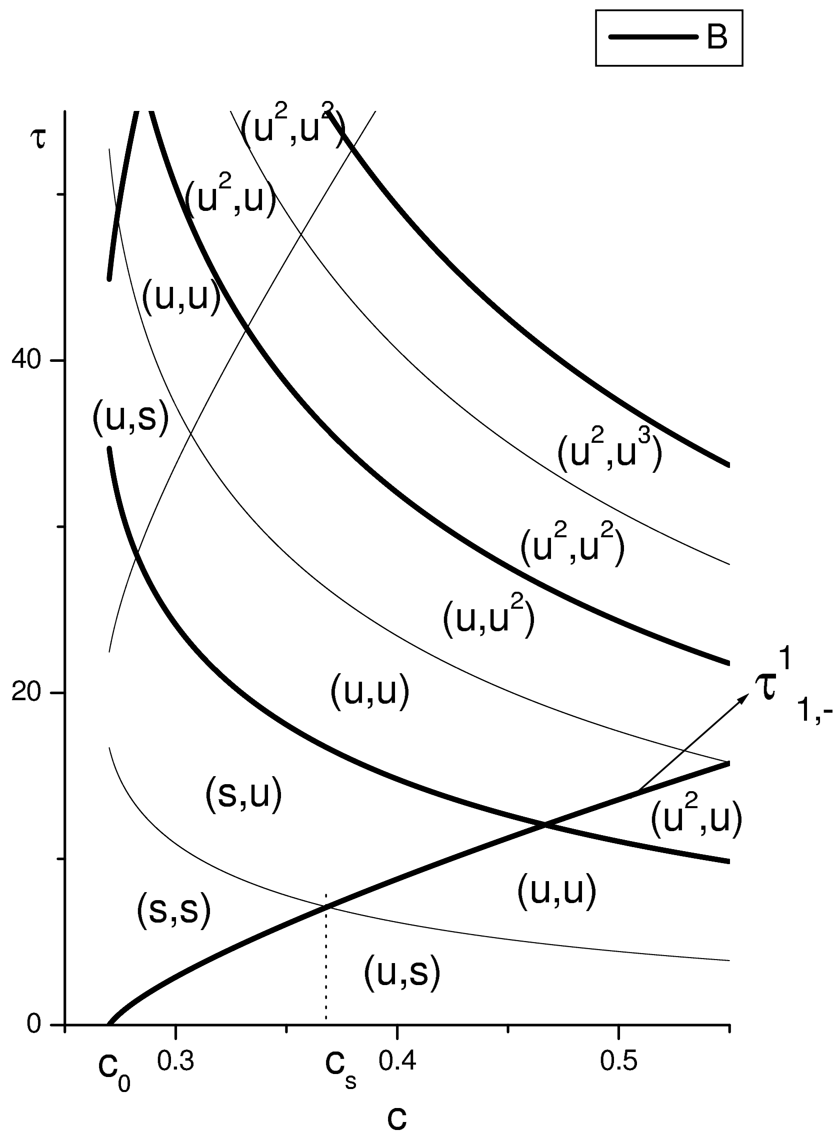
- excitable dynamical systems
- type II excitable systems
- prime example of type II excitable systems

Two coupled FHN excitable systems with delayed coupling:

$$
\begin{aligned}
\dot{x}_1 &= -x_1^3 + (a+1)x_1^2 - ax_1 - y_1 + c\tan^{-1}(x_2^\tau), \\
\dot{y}_1 &= bx_1 - \gamma y_1, \\
\dot{x}_2 &= -x_2^3 + (a+1)x_2^2 - ax_2 - y_2 + c\tan^{-1}(x_1^\tau), \\
\dot{y}_2 &= bx_2 - \gamma y_2,
\end{aligned}
$$

where $x^\tau(t) = x(t-\tau)$

Bifurcation curves in (c, τ) plane



Numerical calculations — two different types of excitable behavior

- single stationary solution as the only attractor
- two coexisting attractors, the stable stationary solution and a stable limit cycle.

The domain in (c, τ) plain — oscillator death.
Both types of excitable behavior occur for relatively small time–lag.
Approximation: $f(x(t-\tau)) \approx f(x - \tau\dot{x})$

The approximate system of ODE's is then given by:

$$
\begin{aligned}
\dot{x}_1 &= -x_1^3 + (a+1)x_1^2 - ax_1 - y_1 + \\
&\quad + c\tan^{-1}(x_2 - \tau(-x_2^3 + (a+1)x_2^2 - \\
&\quad - ax_2 - y_2 + c\tan^{-1}(x_1))), \\
\dot{y}_1 &= bx_1 - \gamma y_1, \\
\dot{x}_2 &= -x_2^3 + (a+1)x_2^2 - ax_2 - y_2 + \\
&\quad + c\tan^{-1}(x_1 - \tau(-x_1^3 + (a+1)x_1^2 - \\
&\quad - ax_1 - y_1 + c\tan^{-1}(x_2))), \\
\dot{y}_2 &= bx_2 - \gamma y_2,
\end{aligned}
$$

Main result

- The organizing center for the dynamics — codimension 2 generalized Hopf (Bautin) bifurcation.
- quite a small time–lag — bifurcation curves of the exact and approximate system almost coincide and the dynamics is qualitatively the same.

Exact system: $\dot{X} = F(X, X^{\tau})$.

Approximate system: $\dot{X} = F_{app}(X)$,
$X \in R^4$, $X = (x_1, y_1, x_2, y_2)$.

Bifurcations of the stationary solution

Parameters a, b, γ, c satisfy conditions

$$4\frac{b}{\gamma} < (a-1)^2, \qquad c < c_1 \equiv a + \frac{b}{\gamma}$$

For $\tau = 0$, F_{app} has only one stationary solution.
Each unit is excitable when decoupled.

Theorem 1 The system $\mathcal{F}_{app}$ has a pitchfork bifurcation for any $(c,\tau) \in \mathcal{B}_{E_0;p} \equiv \{(c,\tau)|\tau = 1/c, c < c_1\}$.

The standard calculations of the normal form prove the theorem.

Theorem 2 For the parameter values

$$(c,\tau) \in \mathcal{B}_{E_0;H} \equiv \left\{ (c,\tau) \middle| \tau = \frac{c-a-\gamma}{c(c-a)}, c \in (a, c_1) \right\}$$

the system $\mathcal{F}_{app}$ has either the supercritical Hopf or the subcritical Hopf or the generalized Hopf bifurcation. Furthermore, there are such values of a, b and γ that the value c_B for which the system has the generalized Hopf bifurcation satisfies $c_B \in (c_0, c_1)$.

Proof: For the parameter values in $\mathcal{B}_{E_0;H}$ linear part of the approximate system has a pair of purely imaginary eigenvalues $\lambda_{1,2} = \pm iv$, $v > 0$, and no other nonhyperbolic eigenvalues.

$$d \equiv \frac{dRe\lambda_{1,2}}{d\tau}|_{\mathcal{B}_{E_0;H}} = \frac{1}{2}\frac{d(-\gamma + F + D)}{d\tau}|_{\mathcal{B}_{E_0;H}} = c(a-c)/2 < 0$$

$\Rightarrow (c, \tau) \in \mathcal{B}_{E_0;H}$ corresponds to the Hopf bifurcation.
The approximate system expanded up to the terms of the third order:

$$\dot{X} = AX + \frac{1}{2}\mathcal{F}_{app,2}(X, X) + \frac{1}{6}\mathcal{F}_{app,3}(X, X, X),$$

where

$$A = \begin{bmatrix} F & -1 & D & E \\ b & -\gamma & 0 & 0 \\ D & E & F & -1 \\ 0 & 0 & b & -\gamma \end{bmatrix}, \ F = -a-c^2\tau, \ D = c+ca\tau, \ E = c\tau$$

$$\mathcal{F}_{app,2}(X,X) = \begin{bmatrix} (a+1)x_1^2 - c(a+1)\tau x_2^2 \\ 0 \\ (a+1)x_2^2 - c(a+1)\tau x_1^2 \\ 0 \end{bmatrix},$$

and $\mathcal{F}_{app,3}(X,X,X)$ with terms of the third order.

First we introduce a complex eigenvector $Q \in R^4$ of A ie.
$AQ = ivQ$ and corresponding eigenvector P of A^T,
$A^T P = -ivP$, normalized to $< P, Q >= \overline{P}^T Q = 1$.
$\forall R \in E^C \; R = \alpha Q + \overline{\alpha}\overline{Q}$, where $\alpha =< P, R >$.

The relation between the original system $\dot{X} = F_{app}(X)$ and the complex normal form of the system on the center manifold $X = H(\alpha, \overline{\alpha})$ of the following form:

$$\dot{\alpha} = iv\alpha + l_1\alpha|\alpha|^2 + l_2\alpha|\alpha|^4 + O(|\alpha|^6)$$

is contained in the corresponding homological equation:

$$\frac{\partial H}{\partial \alpha}\dot{\alpha} + \frac{\partial H}{\partial \bar{\alpha}}\dot{\bar{\alpha}} = \mathcal{F}_{app}(H(\alpha, \bar{\alpha})).$$

$$H(\alpha, \overline{\alpha}) = \alpha Q + \overline{\alpha}\overline{Q} + \sum_{1<j+k\leq 5} \frac{1}{j!k!} h_{jk}\alpha^j\overline{\alpha}^k + O(|\alpha|^6)$$

$$
\begin{aligned}
l_1 &= \frac{1}{2}\mathbf{Re} < P, \mathcal{F}_{app,3}(Q, Q, \bar{Q}) + \\
&\quad + \mathcal{F}_{app,2}(\bar{Q}, (2ivI_4 - A)^{-1}\mathcal{F}_{app,2}(Q, Q)) \\
&\quad - 2\mathcal{F}_{app,2}(Q, A^{-1}\mathcal{F}_{app,2}(Q, \bar{Q})) > \\
&= \frac{\gamma c^3 + B(a,b,\gamma)c^2 + C(a,b,\gamma)c + D(a,b,\gamma)}{2(a-c)(b + a\gamma - c\gamma)},
\end{aligned}
$$

$$
\begin{aligned}
B(a,b,\gamma) &= -b - 3a\gamma - 2\gamma^2 \\
C(a,b,\gamma) &= 2ab + 3a^2\gamma + 2b\gamma - 3\gamma^2 + 3a\gamma^2 \\
D(a,b,\gamma) &= -a^2 b - a^3\gamma + 3b\gamma - ab\gamma - 2\gamma^2 - \\
&- \; a\gamma^2 - 3a^2\gamma^2.
\end{aligned}
$$

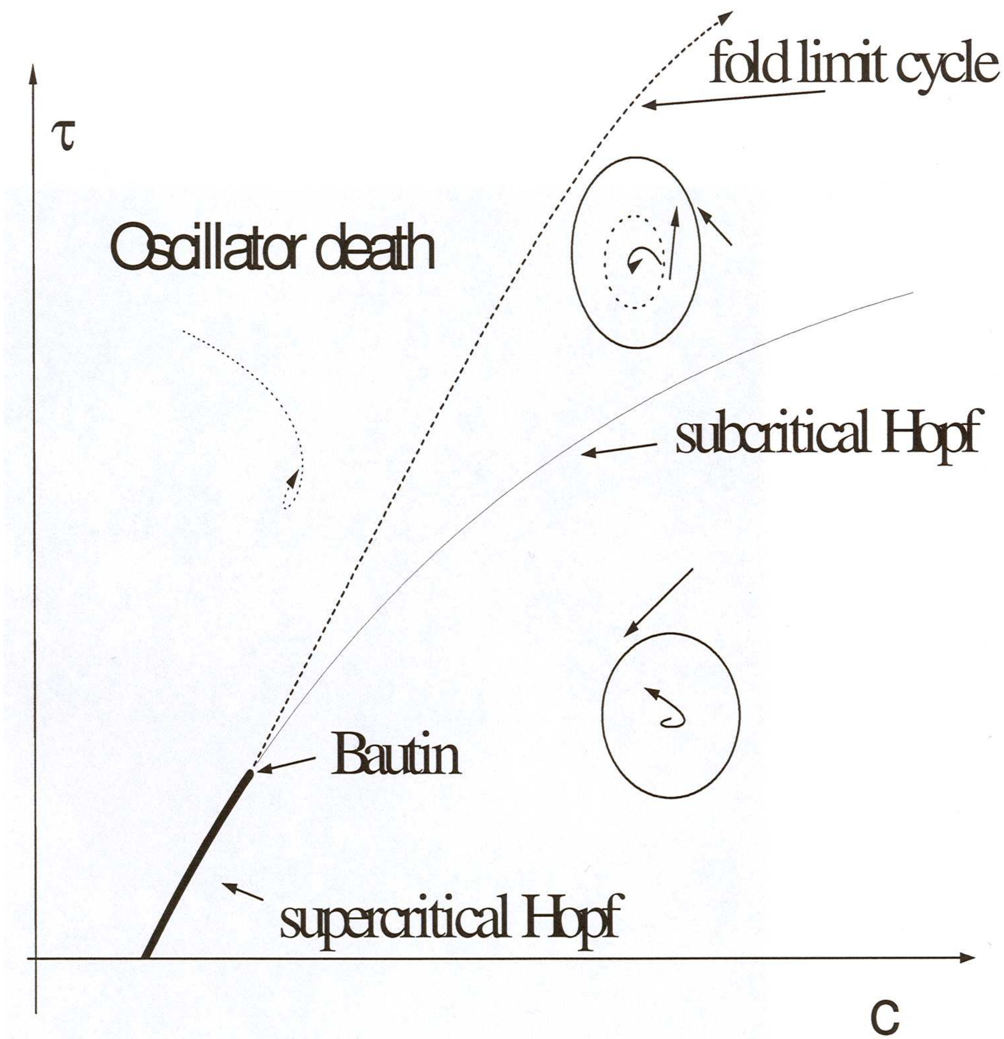
If $l_1 \neq 0$, the system is locally smoothly orbitally equivalent to the system $\dot{\alpha} = iv\alpha + l_1\alpha|\alpha|^2$.
Since everywhere on $\mathcal{B}_{E_0;H}$ we have $d < 0$, then

- $l_1 < 0 \Rightarrow$ supercritical Hopf bifurcation
- $l_1 > 0 \Rightarrow$ subcritical Hopf bifurcation
- $l_1 = 0 \Rightarrow$ generalized Hopf bifurcation

There is at least one value of $c = c_B$ such that bifurcation is of the generalized Hopf type.

$a = 0.25$, $b = \gamma = 0.02$,
$c_B = 0.289024 \in (c_0, c_1) = (0.27, 1.27)$
All three alternatives occur as c is varied.

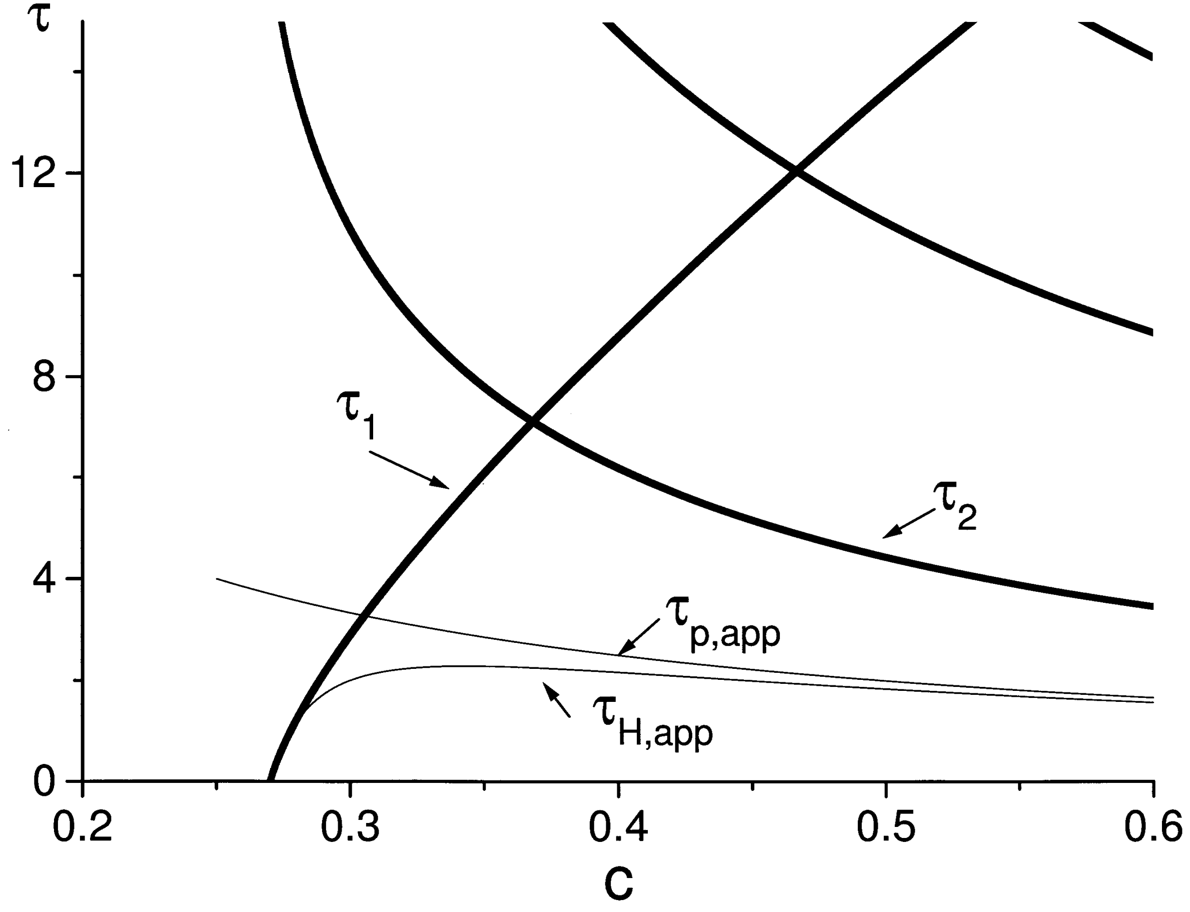
Generalized Hopf bifurcation at (c_B, τ_B)



manifold (plane)

$x_1 = x_2$, $y_1 = y_2$

Bifurcation curves of the exact system and of the approximate system



Conclusion

Two coupled FitzHugh–Nagumo excitable systems — DDE's, and for small time–lag ODE's

The first theorem — boundary beyond which the dynamics is qualitatively different.
The second theorem — three possible types of dynamics for small time–lags.

- stable solution as the only attractor
- oscillatory, when the limit cycle is the only attractor
- bi–stable, stable stationary solution and stable limit cycle