Institute of Physics, Belgrade
Scientific Computing Laboratory
Training and Outreach Program for Young Scientists

Scientific Computing Laboratory

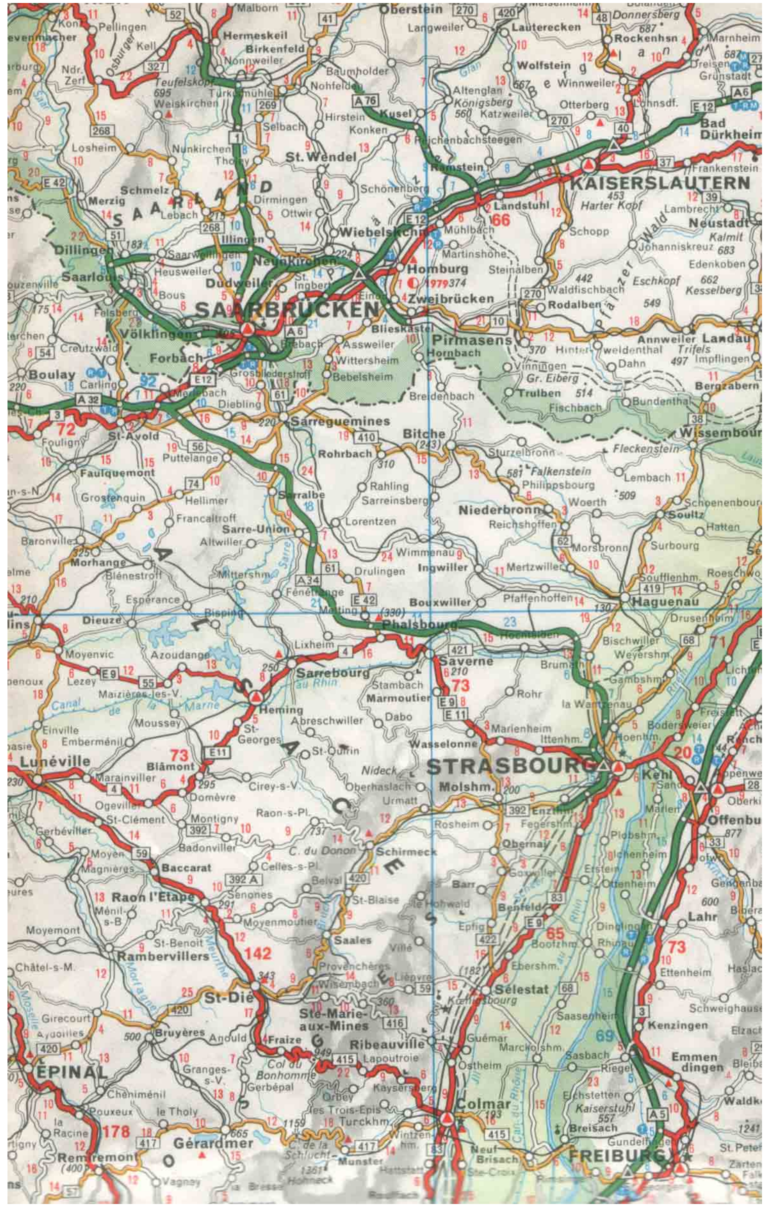
Hilbert's Hotel on the Road to Infinity

Aleksandar Bogojević

Petnica
July 07, 2006

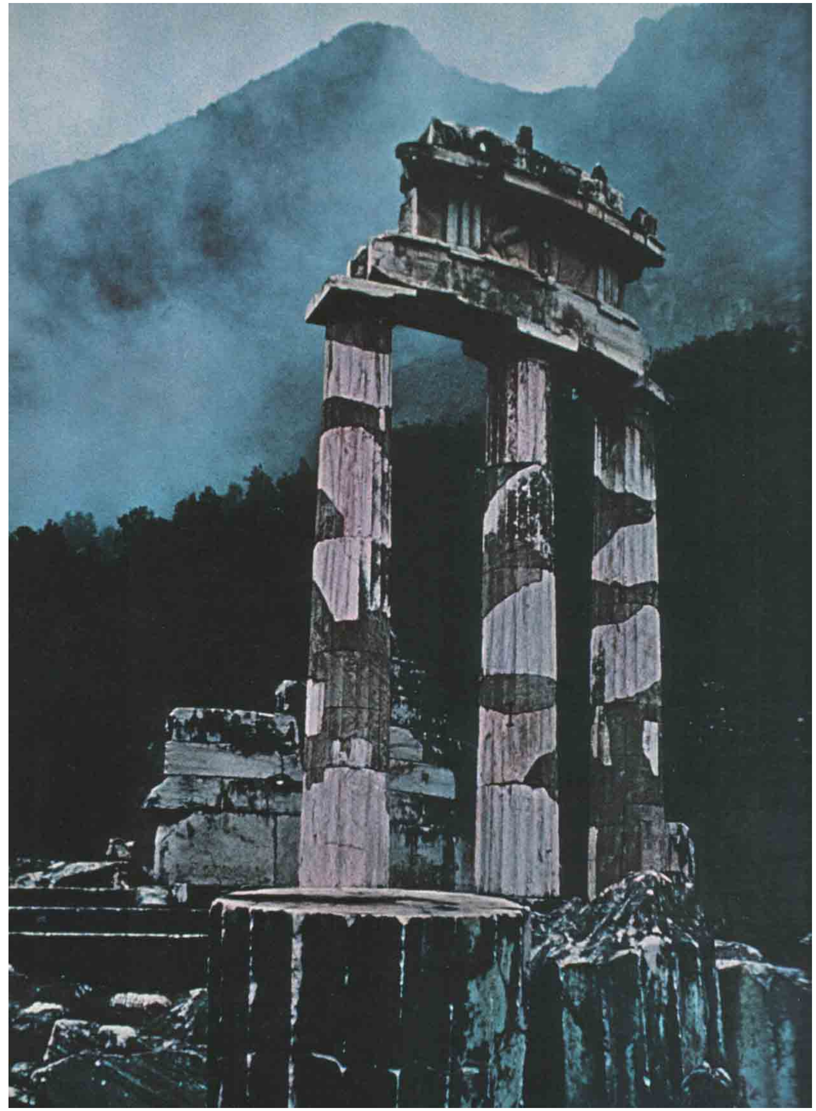
The road to infinity...

- When you were little and on a trip did you ever wonder about all the places you passed through?
- The start and finish of your journey seemed real enough, but what about the places in between?
- Their role on the map was obvious – they made it easier to gauge distances.
- Yet, do these places really exist? Do people really live in them?


SAARBRÜCKEN
KAISERSLAUTERN
STRASBOURG
FREIBURG
Colmar
ÉPINAL
Lunéville
Saverne
Haguenau
Sarrebourg
St-Dié
Sarreguemines
Pirmasens
Homburg
Zweibrücken
Bitche
Phalsbourg
Sélestat
Offenburg
Kehl
Wissembourg

The road to infinity...

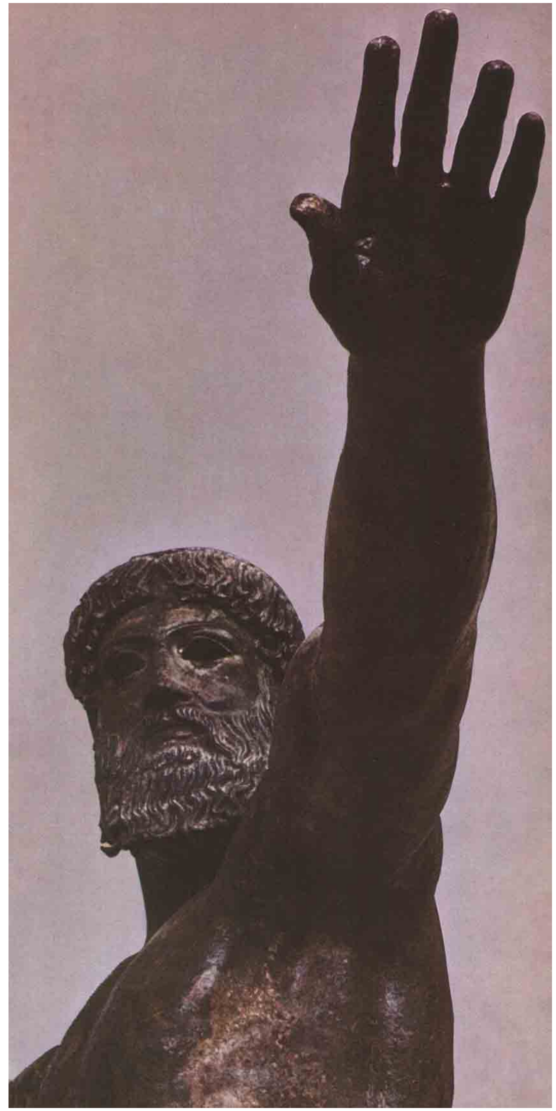
- Don’t worry – even if you do have these feelings they will soon pass. With familiarity, places like Petnica will become real to you, and will no longer seem to be products of some map maker’s fertile imagination.
- When you are young you tend to believe that these kind of strange musings distance you form ‘normal’ people. You don’t share these thoughts with your friends – no point in shouting out your idiosyncrasies.
- Yet, if you have shared these kinds of thoughts you should know that you are not alone – far from it. Some have gone quite far along this road.



Delphi

The road to infinity…

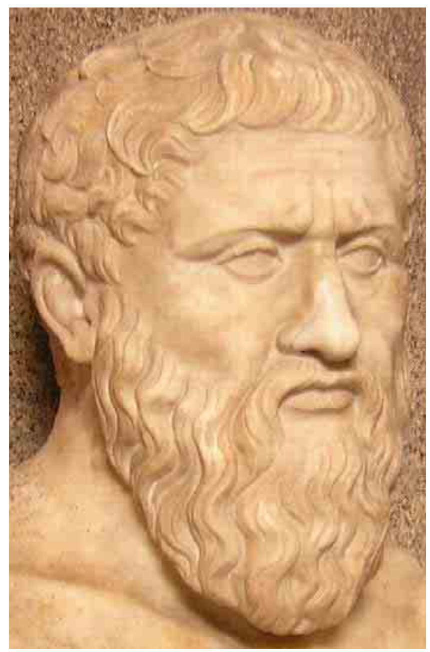
- You shouldn’t be surprised that the road to infinity takes us to ancient Greece that multitude of wondrous island worlds.
- If you’ve never been to Greece you must go there immediately.
- If you have, if you stood near their monuments and saw only ancient ruins then you are beyond help.
- One goes to Greece on a pilgrimage.
- One goes there to get a heat stroke on the Acropolis, to bathe in the same seas as Thales, Pythagoras, Socrates, the seas still controlled by Poseidon.
- One goes there to be invigorated by the same wines that Theseus, Odysseus, and Pericles drank, to eat the Calamata olives as did Aristotle, Demosthenes, Homer…



Poseidon

Zeno

- It is hard to find an important question that wasn't asked 2500 years ago in one of the cities of the Hellenic world, and asking the right question is often the most important step in finding the right answer.
- When counting numbers one starts with zero, when talking about infinity the best place to begin is with Zeno.



Zeno

- Zeno was obsessed with understanding motion, honing his thoughts about motion into a series of wonderful paradoxes. The most famous of these is the one in which he proves that a tortoise will outrun Achilles.
- As years passed Zeno distilled his paradoxes to their essence. His ultimate paradox proves that motion is impossible:

  To go from A to B you need to first pass through the half-way point C_1. To continue on you then have to traverse the distance between C_1 and B, but to do that you must pass through that half-way point C_2. In fact, to get to B you first have to go through an infinity of half-way points C_1, C_2, C_3, C_4, ... You can't traverse infinity in a finite amount of time. Hence, you can never reach B. Motion does not exist – it is just a figment of our imagination.

To get to Petnica from Belgrade you had to pass through a strange place called Lazarevac, then through an even stranger place called Lajkovac – in fact you had to go through an infinity of strange places…

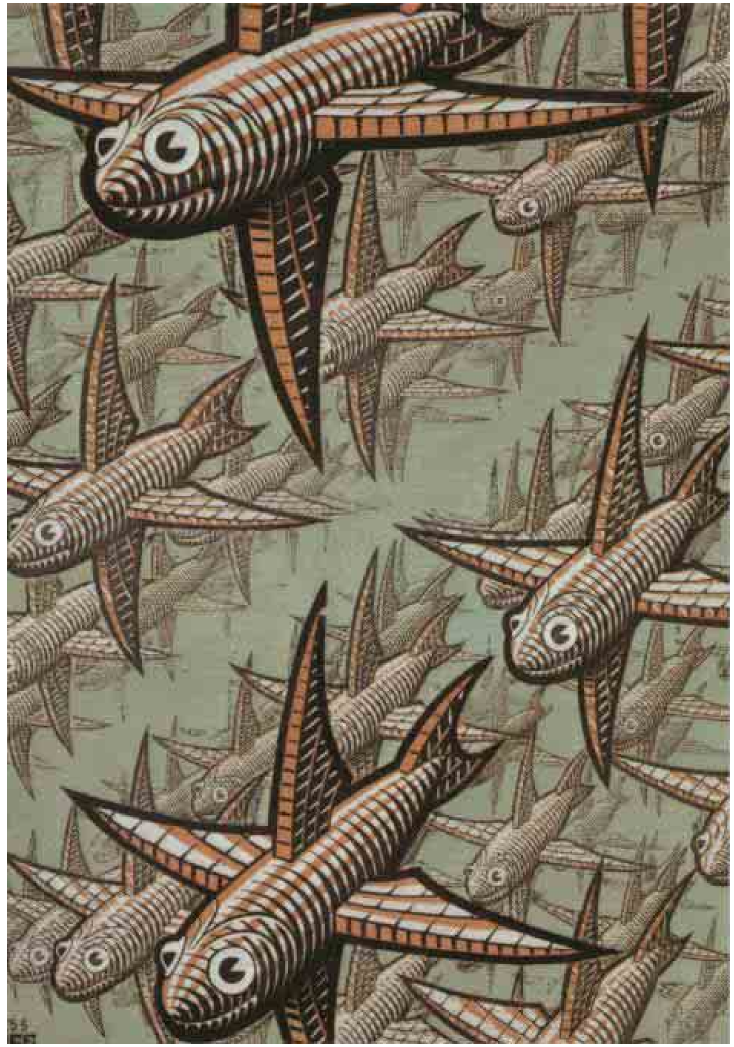
Zeno

- Let us try to understand this paradox. Suppose that it takes us half an hour to go from A to C_1. The next leg of the journey (from C1 to C2) is half the distance, so it should take us half the time, i.e. a quarter of an hour. Continuing in this manner we find that the time needed to go from A to B is

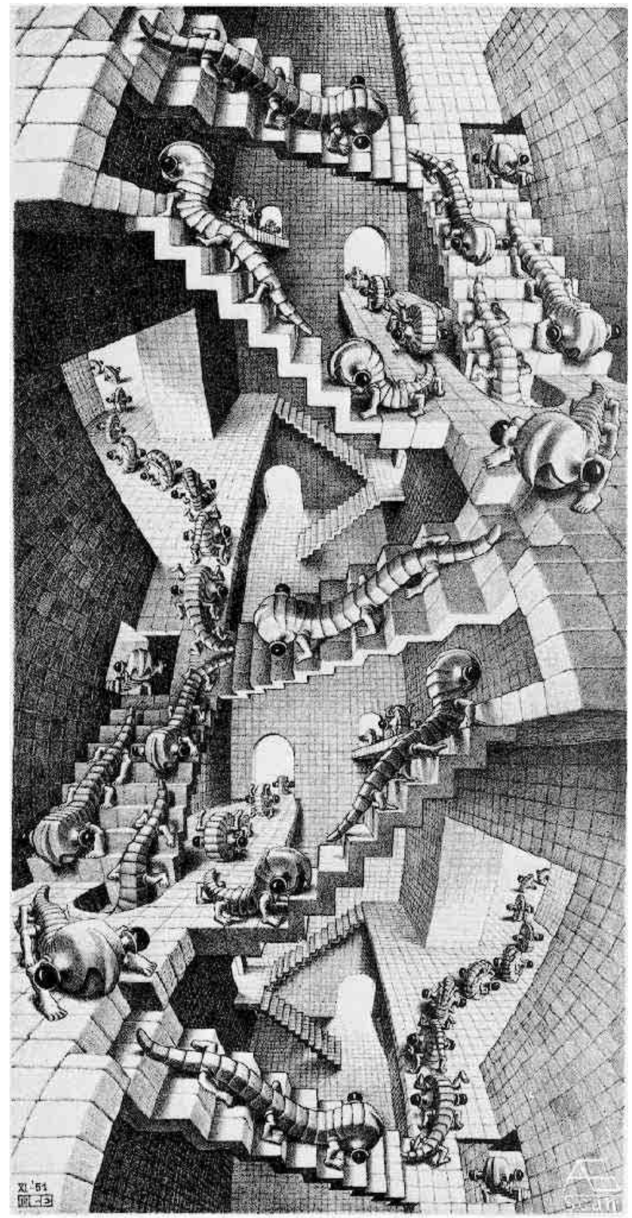
  $T = \frac{1}{2} + \frac{1}{4} + 1/8 + 1/16 + \ldots$

  The dots indicate that there is no end to this sum.

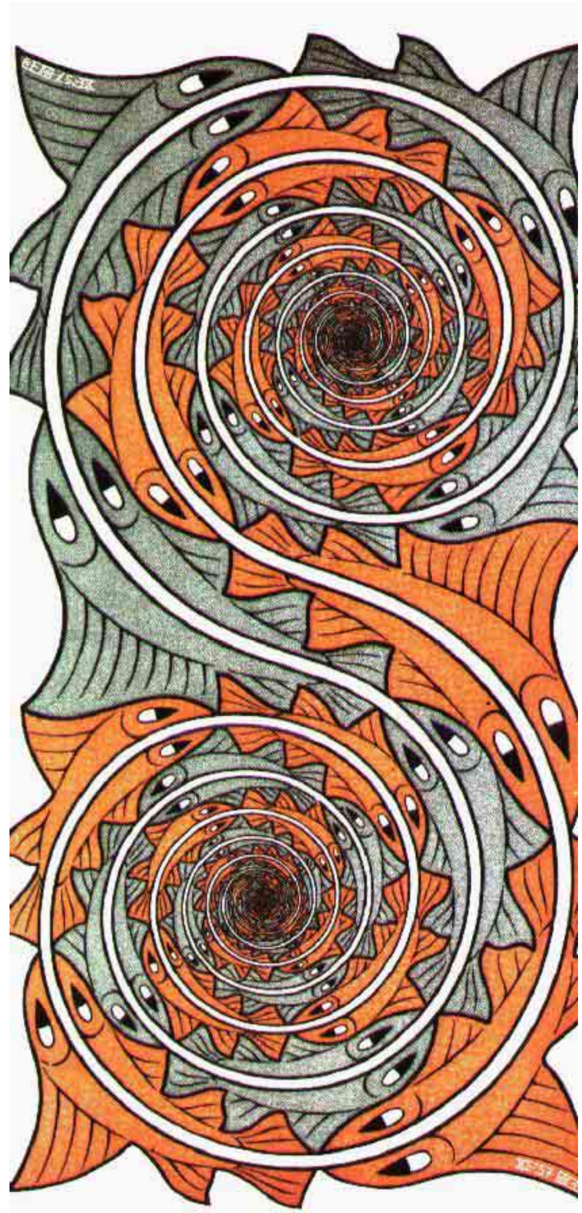
- How can an infinite sum give a finite answer? This is the crux of Zeno's paradox.
- In Zeno's mind (and here he speaks for all of ancient Greece) only finite things make sense. If you share Zeno's belief then you have to admit that he has forced you into a logical impossibility.
- It took us 2000 years to get out of this quandary…



Infinity is not only interesting to philosophers, mathematicians and physicists but also to artists. Few have done a better job in visualizing infinity and symmetry than Escher.


XI-'51

When you express infinity in a finite space you often come up with a paradox…



Zeno

- Let us bravely step out into the light. We may not be smarter than Zeno, but we do have that slight 2500 year advantage over him… Let's not be afraid of infinite sums. Note the following

  $T = \frac{1}{2} + \frac{1}{4} + 1/8 + 1/16 + \ldots = \frac{1}{2} + \frac{1}{2} * (\frac{1}{2} + \frac{1}{4} + 1/8 + \ldots) = \frac{1}{2} + \frac{1}{2} * T$

- It follows that $T = 1$, i.e. it will take us a finite amount of time to go from A to B (exactly one hour). This is quite reasonable since it took us half an hour to traverse half that distance.
- Note that we haven't lost our minds playing around with things that are not finite. There is in fact no paradox, however, we can't banish infinities out of our lives. As a consequence, not only A and B exist but also the points in between (such as Petnica).
- Zeno's paradox marks an ultimate intellectual milestone. If the ancient Greeks could have done the above sum the Renaissance would have come 2000 years earlier than it actually did.

David Hilbert

Hilbert's Hotel

- So much about Zeno. We turn now to the paradoxical exploits of a German hotel owner named David Hilbert. Hilbert's hotel is certainly one of the most famous stops on the road to infinity. His hotel is much like any other – only it has an infinite number of rooms.
- Only a week since his hotel has been completed and it is already full. Hilbert's happiness is short lived, however. Some Balkan tourist agency has double booked his rooms. "This may be OK in Montenegro, but this is Germany! The authorities will shut the hotel down unless I solve this problem."
- Hilbert finds a solution: Move the guest in room 1 into room N+1, the one in 2 to N+2, etc. This leaves the first N rooms free for the double booked guests. In honor of this brainwave Hilbert hangs a sign over the reception stating:

  Infinity + N = Infinity

Hilbert's Hotel

- Two days later an angry mob is in front of the hotel. The tourist agency went bankrupt but not before promising that each guest of the hotel can, free of charge, get a room for a relative of their choice. Had Hilbert been Japanese he would have contemplated ritual suicide. Being of a more practical slant of mind Hilbert put before himself the task of finding an infinity of free rooms in his fully booked hotel.
- I wouldn't be telling you this story if Hilbert hadn't come upon a solution:

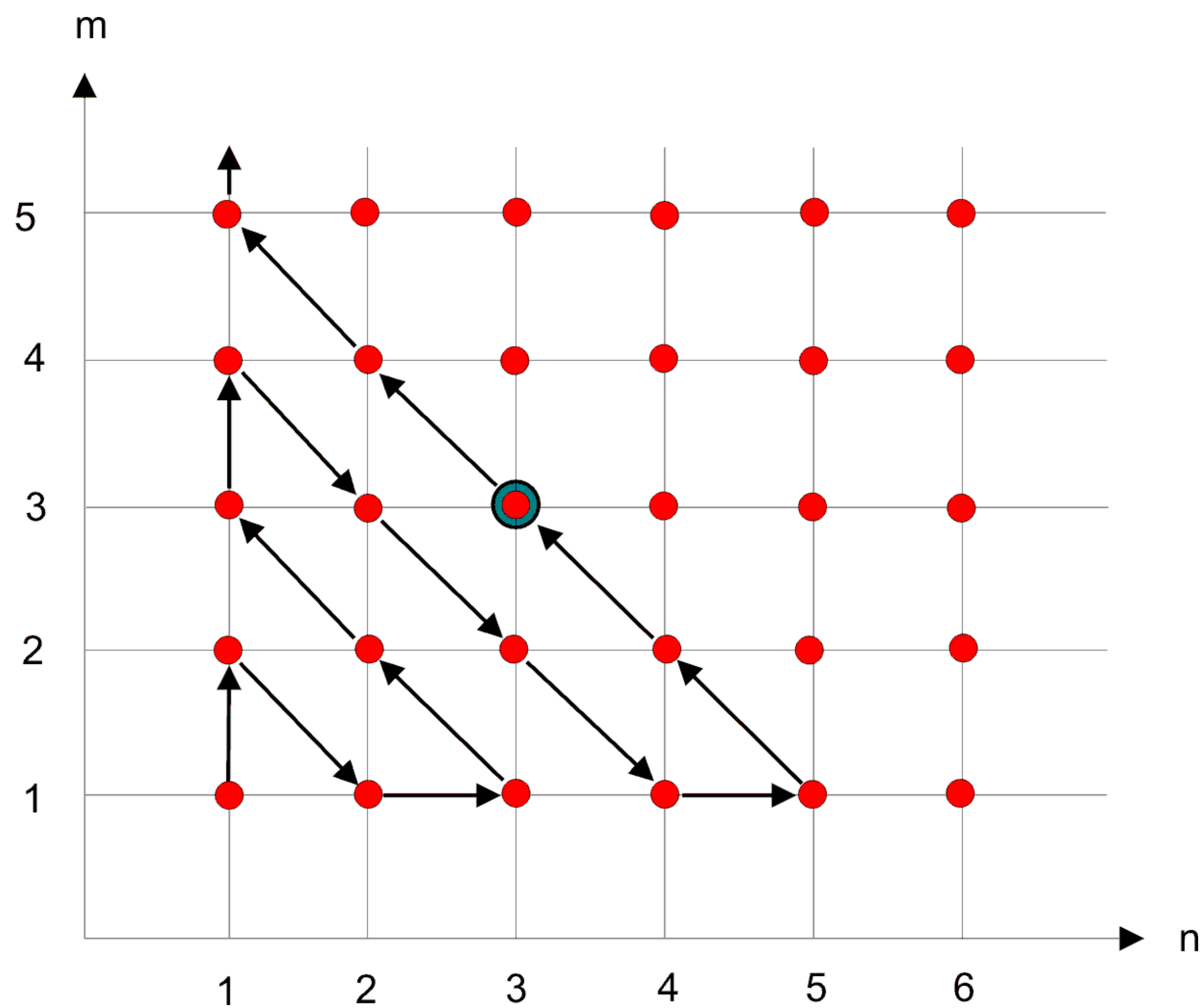
  Move the guest in room K into room 2K. Now that all the odd numbered rooms have been freed move the relative of the guest in 2K into room 2K-1. Having restored peace and quiet Hilbert hangs another sign stating:

  2* Infinity = Infinity

- After such a success he got a bit drunk and added to the sign the following cryptic line: "There are as many even numbers as there are even and odd numbers combined!"

Hilbert's Hotel

- And then the peak tourist season started…
- Soon Hilbert learned the hard way that each true Montenegrin has precisely infinitely many relatives. They all wanted a room in Hilbert's hotel and they wanted it now! They demanded the rooms as their inalienable human right citing, among other documents, the famous UNESCO declaration "Quality tourism for all".
- Hilbert was a law abiding man with great respect for the global community and its declarations. Which hotelier does not respect bureaucracy? However, that same bureaucracy now seemed to spell his doom.
- He didn't give up. With (n,m) he denoted the m-th relative of his current n-th guest. In this notation his current guests were denoted (n,1), i.e. they spanned a horizontal line in the (n,m) plane. If he was to succeed he had to find a way to thread a single line through all the points in that plane. This is precisely what he did!



Hilbert’s solution: Move the third guest’s second relative to room 13 (the thirteenth point along the zigzag line)

Hilbert's Hotel

- Hilbert celebrated this success with the help of some grateful female guests (finitely many of them). Later, when he came to, he added a third sign over his reception:

  Infinity * Infinity = Infinity

  "The number of fractions equals the number of whole numbers."

- Tomorrow morning someone had vandalized his sign adding to it the obscene text:

  "There is but one infinity."

- Hilbert was shocked. "I'm selling this hotel," he cried. "The money was good, but I can't endure having such mathematically illiterate guests!"

Hilbert's Hotel

- OK, so I stretched the truth a bit. Hilbert was not a hotel owner but a great German mathematician. He thought up the idea of a hotel with infinitely many rooms as an illustration to his students.
- I do not wish to even contemplate what he would have done if one of those students had voiced the belief that all infinities are the same.
- Up to now we have indeed been talking about one type of infinity (countable infinity). We denote this infinity (the number of whole numbers) with the symbol $\aleph_0$ (aleph null). It goes without saying that there are other types of infinities. In fact, $\aleph_0$ is the first and smallest infinity there is.

Georg Cantor

An infinity of infinities

- Georg Cantor was the first person to have the courage to compare infinities. His crucial insight was to note that one doesn't need to know how to count in order to, for example, compare the number of apples in one basket and pears in another basket.
- Take one apple and put it next to one pear and continue this process as long as you can. If no fruit is left unpaired then the number of apples is the same as the number of pears. If there remain unpaired apples then their number was greater than the number of pears and vice-versa.
- Even small children can do this, long before they learn how to count. Cantor's inspiration was to see that we are all small kids when it comes to infinity.
- Hilbert's hotel has already given us examples of comparing infinities.

An infinity of infinities

- Now we are ready to compare the number of whole numbers to the number of real numbers in the interval from 0 to 1. Let us assume that it is possible to pair each real number in the unit interval with one and only one whole number. This pairing would look something like:

  $1 \leftrightarrow 0.13249869987...$
  $2 \leftrightarrow 0.78002315221...$
  $3 \leftrightarrow 0.39910010015...$
  $4 \leftrightarrow 0.30166655555...$
  etc..

- Cantor concluded that we necessarily have some real numbers left over. These unpaired numbers can easily be constructed:

  For the first decimal take any digit not equal to the first digit of the first number. For the second take any digit not equal to the second digit of the second number, etc. By continuing this procedure we are guaranteed to get a number that is not in our list – contrary to our original assumption.

- Therefore, the number of real numbers in the unit interval (we denote this by c) is larger than $\aleph_0$. There truly is more than one infinity.

An infinity of infinities

- It is fairly easy to show that the whole real line has the same number of points as the unit interval. Similarly, a surface has the same number of points as a line. Try to prove this – the reasoning is no more complex than was the case with Hilbert's hotel.
- Now you may wonder if there exist other infinities except $\aleph_0$ and c. Yes, and (as you might expect) there are infinitely many of them.
- Cantor proved that there exists not only $\aleph_0$ but also $\aleph_1$ and $\aleph_2$ and $\aleph_3$ and so forth – each greater than the previous. He did this by constructing an explicit procedure of how to generate $\aleph_{n+1}$ from $\aleph_n$.

An infinity of infinities

- The aim of our short drive along the road to infinity is to catch a brief tantalizing view of things infinite. Not long ago people used to fear infinity, but then we learned how to play around with infinities.
- But are infinites really all that safe? Their dark side can be seen the moment you pose the following simple question. Where in the series $\aleph_0$, $\aleph_1$, $\aleph_2$, $\aleph_3$, ... Should we put c?
- We've shown that c is greater than $\aleph_0$ so one naturally assumes that c is in fact $\aleph_1$, this is what Canto thought.
- He might have been correct – who knows? To learn the answer we need to meet the dark prince of mathematics Kurt Goedel.

Kurt Goedel

Goedel's legacy

- In 1930 Kurt Goedel proved that Cantor's conjecture can't be disproved.
- Goedel's life work was to trace the very boundaries of mathematics as a formal system, i.e. a system based on axioms.
- Three decades after Goedel Paul Cohen proved the reverse statement: Cantor's conjecture also can't be proved.
- Cantor may be right or wrong, we don't know which. What we do know is that we'll never know.
- Several centuries ago people had a fear of infinity, believing that it is substantially at odds with our finite minds. They believed that to contemplate infinity is to travel the road to insanity.
- Later we found that infinity can even be useful. Now we've discovered the true face of infinity – it is a milestone delineating the limits of what is knowable.

Goedel's legacy

- People like to play games, but they certainly do not like to lose, but nothing irritates us so much as to be told that there is something we will never be able to do.
- When faced with such an untenable situation Man sooner or later chooses to cheat, to change the rules. If formal systems such as mathematics constrain us then at some point we are bound to create some other type of system.
- At this moment we have no idea how to do this or what that system would be like. We are in the same situation that Zeno faced a couple of thousand years ago. It might even take us a similar length of time, but one thing I feel sure of is that we will find a way out.
- Life in the Universe we inhabit is an intellectual game, one in which it is not OK just to participate but in which we must win. Infinity is our birthright – human minds are finite, but the Humanity's potential is infinite. This is the religious belief upon which science lies.