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**BOSE-EINSTEIN CONDENSATES IN
WEAK DISORDER POTENTIALS**

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Abstract

In this thesis we present main features of Bose-Einstein condensates (BEC) in disordered potentials using the mean-field approach. Perturbation theory was applied to a BEC with an arbitrary two-body interaction potential and a disorder with an arbitrary correlation function. In particular, we have considered systems with two different cylindrically-symmetric physical features: anisotropic dipolar interaction and disorder with the anisotropic correlation function, which were afterwards combined.

In the first chapter we give a general introduction to the topics. The second chapter deals with the perturbation calculation of relevant physical quantities, such as condensate density, superfluid density, and sound velocity. The first-order perturbation theory is then applied to the case of a dipolar interaction with anisotropic Lorentz-correlated disorder in the third chapter. The fourth chapter summarizes the results obtained in the thesis and lists research topics for further study. Appendices A and B give technical details on calculation presented throughout the thesis.

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Chapter 1

Introduction

Bose-Einstein Condensate (BEC) is a state of matter of a quantum gas of bosons with a large fraction occupying the lowest energy state (ground state). It is a consequence of quantum statistics, i.e. the fact that bosons are indistinguishable, and permuting them does not produce a different state. This was first noticed by the Indian physicist Satyendra Nath Bose in 1924, whose paper [1] on massless light particles (photons) was generalized by Albert Einstein to the case of massive particles [2]. In that paper, Einstein predicted existence of a critical temperature T_c at which a number of particles in the ground state becomes of the same order as the total number of particles, which is now designated as a critical temperature for a Bose-Einstein condensation phase transition.

In 1937 a superfluid phase of ^4He was found at 2.17K [3, 4]. Its frictionless superfluid motion was connected to BEC [5, 6, 7]. Theoretical study of BECs continued by introducing contact interaction in terms of the s-wave scattering length [8, 9], enabling approximation of all short-range interactions by an effective contact interaction. An ideal BEC in a harmonic trap was also considered [10].

In the 1980s a laser cooling technique was developed, allowing experiments at the temperature of μK range. Finally, in 1995 the first BEC was experimentally observed in dilute vapours of rubidium [11] and sodium [12]. This was culmination of 70 years of the development of various advanced cooling and measurement techniques.

Atoms such as chromium (^{52}Cr) [23] and rubidium (^{87}Rb) [24], which have strong magnetic dipole moment and therefore exhibit long-range interaction, have been Bose-condensed recently. There are also efforts to Bose-condense molecules with a strong electric dipole moment, namely $^{41}\text{K}^{87}\text{Rb}$ [25].

Bose condensates in random disordered potentials were studied in the case of contact interaction [13, 14, 15, 16, 17, 18], as well as for systems with a dominant dipolar interaction [19]. The disorder usually appears as unwanted in wire traps [27, 28], but can be also controlled and tuned using laser speckles [26]. The study of the effect of disorder is now

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a major research topic.

In this thesis we study BEC in a box potential with disorder using the mean-field theory, i.e. Gross-Pitaevskii equation, for dipolar interaction with Lorentz-correlated disorder.

1.1 Dilute Bose gases

Interparticle potential is in general a many-variable function, depending on the position, spin-states, etc. of all particles. However, when considering dilute systems, where the characteristic interaction distance r_0 is much smaller than the average interparticle distance $n^{-1/3}$, interaction potential can be represented as a sum of two-body potentials, neglecting simultaneous interaction of three or more particles.

For isotropic interactions in the low energy regime, a wave function of two scattered particles, for $r \gg r_0$, is characterized by the s-wave scattering length a , a parameter independent of energies. Therefore, long-ranged¹ properties of systems are significantly influenced by short-range isotropic interaction² only through the value of the scattering length a .

The Hamiltonian of the system of particles interacting with a potential $V(\mathbf{r}' - \mathbf{r})$ in an external potential $U(\mathbf{r}, t)$ is given by

$$\begin{aligned}\hat{H}(t) &= \int d^3\mathbf{r} \left(\frac{\hbar^2}{2m} \nabla \hat{\Psi}^\dagger(\mathbf{r}) \nabla \hat{\Psi}(\mathbf{r}) + U(\mathbf{r}, t) \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}(\mathbf{r}) \right) \\ &+ \frac{1}{2} \int d^3\mathbf{r} d^3\mathbf{r}' \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}^\dagger(\mathbf{r}') V(\mathbf{r}' - \mathbf{r}) \hat{\Psi}(\mathbf{r}') \hat{\Psi}(\mathbf{r}).\end{aligned}$$

Here, $\hat{\Psi}^\dagger(\mathbf{r})$ and $\hat{\Psi}(\mathbf{r})$ are creation and annihilation Bose field operators satisfying commutation relations

$$\left[\hat{\Psi}(\mathbf{r}), \hat{\Psi}^\dagger(\mathbf{r}') \right] = \delta(\mathbf{r} - \mathbf{r}'), \quad \left[\hat{\Psi}(\mathbf{r}), \hat{\Psi}(\mathbf{r}') \right] = 0. \quad (1.1)$$

In the Heisenberg picture we have time-dependent field operators and an equation governing their time evolution:

$$\begin{aligned}\hat{\Psi}(\mathbf{r}, t) &= \hat{U}^\dagger(t) \hat{\Psi}(\mathbf{r}) \hat{U}(t), \\ i\hbar \frac{\partial}{\partial t} \hat{\Psi}(\mathbf{r}, t) &= \hat{U}^\dagger(t) \left[\hat{\Psi}(\mathbf{r}), \hat{H}(t) \right] \hat{U}(t) \\ &= \left(-\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{r}, t) + \int d^3\mathbf{r}' \hat{\Psi}^\dagger(\mathbf{r}', t) V(\mathbf{r}' - \mathbf{r}) \hat{\Psi}(\mathbf{r}', t) \right) \hat{\Psi}(\mathbf{r}, t).\end{aligned} \quad (1.2)$$

¹Compared to r_0
²With $a \ll n^{-1/3}$

Let \hat{H}_1 be one-particle Hamiltonian with potential $U(\mathbf{r})$, let $\phi_i(\mathbf{r})$ be its eigenfunctions, and \hat{a}_i corresponding annihilation operators in the Fock space.

At $T = 0$ the system is condensed in the global ground state, which is not known, because, due to interactions, it cannot be constructed in terms of one-particle ground states, unlike for the ideal gas. However, for weak interaction, it is expected that the ground state fraction $\langle \hat{a}_0^\dagger \hat{a}_0 \rangle = n_0$ is still large, making the approximation $\hat{a}_0^\dagger \approx \hat{a}_0 \approx \sqrt{n_0}$ plausible (Bogoliubov approximation). This justifies an expansion $\hat{\Psi} = \Psi + \delta\hat{\Psi}$, where Ψ is a classical field (of the order of $\sqrt{n_0}\phi_0$) and $\delta\hat{\Psi}$ is small quantum correction.

Neglecting completely quantum fluctuations leads to a time-dependent Gross-Pitaevskii equation:

$$i\hbar \frac{\partial \Psi(\mathbf{r}, t)}{\partial t} = \left(-\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{r}, t) + \int d^3\mathbf{r}' V(\mathbf{r}' - \mathbf{r}) \Psi^*(\mathbf{r}', t) \Psi(\mathbf{r}', t) \right) \Psi(\mathbf{r}, t). \quad (1.3)$$

For a time-independent potential $U(\mathbf{r})$ we assume a variable-separated solution

$$\Psi(\mathbf{r}, t) = e^{-i\frac{\mu}{\hbar}t} \psi(\mathbf{r}),$$

where μ is a chemical potential (ground-state energy). This leads to the time-independent Gross-Pitaevskii equation for the ground state $\psi(\mathbf{r})$:

$$\left(-\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{r}) - \mu + \int d^3\mathbf{r}' V(\mathbf{r}' - \mathbf{r}) \psi^*(\mathbf{r}') \psi(\mathbf{r}') \right) \psi(\mathbf{r}) = 0. \quad (1.4)$$

1.2 Superfluid velocity

At zero temperature the system will be in its ground state, and which will be superfluid, according to the Landau criterion of superfluidity. In order to calculate the superfluid velocity, we will first consider a homogenous case and then generalize it.

For the homogeneous case (without the external potential) (1.3) becomes

$$i\hbar \frac{\partial \Psi(\mathbf{r}, t)}{\partial t} = \left(-\frac{\hbar^2}{2m} \nabla^2 + \int d^3\mathbf{r}' V(\mathbf{r}' - \mathbf{r}) \Psi^*(\mathbf{r}', t) \Psi(\mathbf{r}', t) \right) \Psi(\mathbf{r}, t). \quad (1.5)$$

If $\Psi(\mathbf{r}, t)$ is the solution of Eq. (1.5) then so will be the function

$$\Psi'(\mathbf{r}, t) = \Psi(\mathbf{r} - \mathbf{v}_S t, t) e^{\frac{i}{\hbar} \left(m\mathbf{v}_S \mathbf{r} - \frac{m\mathbf{v}_S^2}{2} t \right)}. \quad (1.6)$$

This solution corresponds to a fluid moving uniformly with velocity \mathbf{v}_S . Now one can

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write a total phase factor of a moving Bose condensate as:

$$S(\mathbf{r}, t) = \frac{1}{\hbar} \left[m\mathbf{r}\mathbf{v}_S - \left(\frac{mv_S^2}{2} + \mu \right) t \right]. \quad (1.7)$$

In general, having some potential $U(\mathbf{r}, t)$, one can still calculate the superfluid velocity using the formula

$$\mathbf{v}_S(\mathbf{r}, t) = \frac{\hbar}{m} \nabla S(\mathbf{r}, t). \quad (1.8)$$

On the other hand, multiplying Eq. (1.3) by Ψ^* and subtracting the complex conjugate of the result gives

$$i\hbar \frac{\partial n(\mathbf{r}, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla [\Psi^*(\mathbf{r}, t) \nabla \Psi(\mathbf{r}, t) - \Psi(\mathbf{r}, t) \nabla \Psi^*(\mathbf{r}, t)] \quad (1.9)$$

and using the continuity equation one can identify

$$\mathbf{j}(\mathbf{r}, t) = \frac{\hbar}{2im} [\Psi^*(\mathbf{r}, t) \nabla \Psi(\mathbf{r}, t) - \Psi(\mathbf{r}, t) \nabla \Psi^*(\mathbf{r}, t)] = n \frac{\hbar}{m} \nabla S(\mathbf{r}, t), \quad (1.10)$$

which is consistent with Eq. (1.8).

1.3 Hydrodynamical approach

The continuity equation for a BEC system is given by

$$\frac{\partial n}{\partial t} + \nabla(\mathbf{v}_S n) = 0. \quad (1.11)$$

If we calculate partial derivative with respect to the time of the Eq. (1.7), we obtain

$$\hbar \frac{\partial S}{\partial t} + \frac{mv_S^2}{2} + \mu = 0.$$

Similarly for the superfluid velocity, one can switch to the general case, by taking the chemical potential to be locally dependent on the fluid density, and adding the external potential to the equation. This is valid when the density and the superfluid velocity are slowly varying in space and time. Taking the gradient of the result gives the Euler equation:

$$m \frac{\partial \mathbf{v}_S}{\partial t} + \nabla \left(\frac{mv_S^2}{2} + \mu(n) + U \right) = 0. \quad (1.12)$$

Chapter 2

BEC in disordered potentials

We now consider a system in a random, time-independent, disordered potential $U(\mathbf{r})$. The exact form of the potential is unknown, and it is described only through its statistical properties:

$$\langle U(\mathbf{r}) \rangle = 0, \quad (2.1)$$

$$\langle U(\mathbf{r})U(\mathbf{r}') \rangle = R(\mathbf{r} - \mathbf{r}'), \quad (2.2)$$

Here $\langle \cdot \rangle$ denotes averaging with respect to different disorder realizations i.e. average over an ensemble. However we will assume that spatial averaging of any quantity over distances much larger than the healing length¹ ξ and the characteristic disorder correlation length² σ coincides with the above averaging.

This is a plausible assumption since the disorder correlation is negligible on distances much larger than σ , and the wave-function's response to disorder is negligible over distances much larger than $\sigma + \xi$. Therefore, any quantity that is locally dependent on U and ψ should satisfy the above assumption³. We will use this to redefine the condensate density and the sound velocity in a suitable way.

We will often need to perform Fourier transformation, which we define as:

$$f(\mathbf{k}) = \int d^3\mathbf{r} e^{-i\mathbf{k}\mathbf{r}} f(\mathbf{r}), \quad (2.3)$$

$$f(\mathbf{r}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\mathbf{r}} f(\mathbf{k}). \quad (2.4)$$

¹Healing length is a characteristic distance at which BEC properties are no longer influenced by a fluctuation in the potential. It is given by $\xi^2 = \frac{\hbar^2}{2mng}$.

²At distances much larger than σ the disorder correlation becomes negligible.

³For long-range interactions ψ is not only locally dependent on U because of the interaction term in the GP equation. Then, however, the term can be separated into the near field and the far field, and the latter can be approximated by averaged quantities (in the same way the condensate density is defined below), making the assumption self-consistent.

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Using this definition, we see that

$$\begin{aligned}\langle U(\mathbf{k})U(\mathbf{k}') \rangle &= \int \int d^3\mathbf{r} d^3\mathbf{r}' e^{-i(\mathbf{k}\mathbf{r}+\mathbf{k}'\mathbf{r}')} R(\mathbf{r}-\mathbf{r}') \\ &= (2\pi)^3 \delta(\mathbf{k}+\mathbf{k}') R(\mathbf{k}').\end{aligned}\quad (2.5)$$

2.1 Perturbative solution

Taking the disorder $U(\mathbf{r})$ (hence $R(\mathbf{r})$) small, it is possible to solve the GP equation (1.4) perturbatively. In order to do so, we expand the condensate wave function in a series of terms proportional to powers in U :

$$\psi(\mathbf{r}) = \psi_0 + \psi_1(\mathbf{r}) + \psi_2(\mathbf{r}) + \dots, \quad (2.6)$$

where ψ_0 is a solution of the clean (no disorder) case. The system is homogeneous and the solution can be taken to be real, so, due to homogeneity, ψ_0 is position-independent.

Substituting Eq. (2.6) into Eq. (1.4), we obtain:

- 0^{th} order equation:

$$\begin{aligned}-\mu\psi_0 + \psi_0^3 \int d^3\mathbf{r}' V(\mathbf{r}') &= 0, \\ \psi_0^2 &= \frac{\mu}{V(\mathbf{k}=0)}.\end{aligned}\quad (2.7)$$

- 1^{st} order equation:

$$\left(-\frac{\hbar^2}{2m}\nabla^2 - \mu\right)\psi_1(\mathbf{r}) + U(\mathbf{r})\psi_0 + \psi_0^2 \int d^3\mathbf{r}' V(\mathbf{r}'-\mathbf{r})(2\psi_1(\mathbf{r}') + \psi_1(\mathbf{r})) = 0.$$

Its Fourier transform has a form

$$\left(\frac{\hbar^2 k^2}{2m} - \mu\right)\psi_1(\mathbf{k}) + U(\mathbf{k})\psi_0 + \psi_0^2 (2V(\mathbf{k})\psi_1(\mathbf{k}) + V(\mathbf{k}=0)\psi_1(\mathbf{k})) = 0,$$

yielding the solution

$$\psi_1(\mathbf{k}) = -\frac{\psi_0 U(\mathbf{k})}{\frac{\hbar^2 k^2}{2m} + 2\psi_0^2 V(\mathbf{k})}. \quad (2.8)$$

- 2^{nd} order equation:

$$\begin{aligned}\left(-\frac{\hbar^2}{2m}\nabla^2 - \mu\right)\psi_2(\mathbf{r}) + U(\mathbf{r})\psi_1(\mathbf{r}) \\ + \psi_0 \int d^3\mathbf{r}' V(\mathbf{r}'-\mathbf{r}) (2\psi_0\psi_2(\mathbf{r}') + \psi_0\psi_2(\mathbf{r}) + 2\psi_1(\mathbf{r})\psi_1(\mathbf{r}') + \psi_1^2(\mathbf{r}')) = 0,\end{aligned}$$

its Fourier transform (simplified using Eq. (2.7))

$$\frac{\hbar^2 k^2}{2m} \psi_2(\mathbf{k}) + 2\psi_0^2 V(\mathbf{k}) \psi_2(\mathbf{k}) + \int \frac{d^3 \mathbf{k}'}{(2\pi)^3} (U(\mathbf{k} - \mathbf{k}') \psi_1(\mathbf{k}') + \psi_0(2V(\mathbf{k}') + V(\mathbf{k})) \psi_1(\mathbf{k}') \psi_1(\mathbf{k} - \mathbf{k}')) = 0,$$

and the solution

$$\psi_2(\mathbf{k}) = - \int \frac{d^3 \mathbf{k}'}{(2\pi)^3} \frac{U(\mathbf{k} - \mathbf{k}') \psi_1(\mathbf{k}') + \psi_0(2V(\mathbf{k}') + V(\mathbf{k})) \psi_1(\mathbf{k}') \psi_1(\mathbf{k} - \mathbf{k}')}{\frac{\hbar^2 k^2}{2m} + 2\psi_0^2 V(\mathbf{k})}. \quad (2.9)$$

The perturbative procedure can be continued to higher orders straightforwardly.

2.2 Order parameter and condensate depletion

The order parameter is usually defined as an off-diagonal long-range (ODLR) element of the one-particle density matrix. In this case ψ is a real, so

$$\rho(\mathbf{r}, \mathbf{r}') = \langle \psi(\mathbf{r}) \psi(\mathbf{r}') \rangle. \quad (2.10)$$

Choosing appropriately the volume V_0 that satisfies the conditions for switching from ensemble to spatial averaging, and letting $\mathbf{r}' - \mathbf{r}$ be much larger than the volume's linear dimensions yields:

$$\langle \psi(\mathbf{r}) \psi(\mathbf{r}') \rangle \approx \frac{1}{V_0^2} \int_{V_0} d^3 \mathbf{r}_1 d^3 \mathbf{r}_2 \langle \psi(\mathbf{r} + \mathbf{r}_1) \psi(\mathbf{r}' + \mathbf{r}_2) \rangle \quad (2.11)$$

$$= \langle \langle \psi(\mathbf{r}) \rangle \langle \psi(\mathbf{r}') \rangle \rangle \quad (2.12)$$

$$= \langle \psi(\mathbf{r}) \rangle^2. \quad (2.13)$$

In the first line, Eq. (2.11), we have a spatial average of $\langle \psi(\mathbf{r}) \psi(\mathbf{r}') \rangle$ over a small (compared to $\mathbf{r} - \mathbf{r}'$) volume V_0 , which is approximately the same as $\langle \psi(\mathbf{r}) \psi(\mathbf{r}') \rangle$. In the second line, Eq. (2.12), we use the assumption that spatial and disorder averages coincide.

We define the condensate density as

$$n_0 = \langle \psi(\mathbf{r}) \rangle^2 = \psi_0^2 + 2\psi_0 \langle \psi_2 \rangle + \dots, \quad (2.14)$$

taking into account that the total fluid density is the diagonal part of the density matrix,

$$n = \langle \psi(\mathbf{r})^2 \rangle = \psi_0^2 + \langle \psi_1(\mathbf{r}) \psi_1(\mathbf{r}) \rangle + 2\psi_0 \langle \psi_2(\mathbf{r}) \rangle + \dots, \quad (2.15)$$

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the condensate depletion can be expressed as

$$n - n_0 = \langle \psi(\mathbf{r})^2 \rangle - \langle \psi(\mathbf{r}) \rangle^2 = \langle \psi_1(\mathbf{r})\psi_1(\mathbf{r}) \rangle + \dots$$

After expressing $\psi_1(\mathbf{r})$ using its Fourier transform, i.e. Eq. (2.8), the previous expression goes to

$$\begin{aligned} n - n_0 &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\mathbf{r}} \int \frac{d^3\mathbf{k}'}{(2\pi)^3} \langle \psi_1(\mathbf{k}')\psi_1(\mathbf{k} - \mathbf{k}') \rangle \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\mathbf{r}} \int \frac{d^3\mathbf{k}'}{(2\pi)^3} \frac{\psi_0^2 \langle U(\mathbf{k}')U(\mathbf{k} - \mathbf{k}') \rangle}{\left[\frac{\hbar^2\mathbf{k}'^2}{2m} + 2\psi_0^2 V(\mathbf{k}') \right] \left[\frac{\hbar^2(\mathbf{k} - \mathbf{k}')^2}{2m} + 2\psi_0^2 V(\mathbf{k} - \mathbf{k}') \right]} \\ &= n \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{R(\mathbf{k})}{\left[\frac{\hbar^2\mathbf{k}^2}{2m} + 2nV(\mathbf{k}) \right]^2} + \dots, \end{aligned} \quad (2.16)$$

where in the last line we have used Eq. (2.5).

2.3 Equation of state

Substituting Eqs. (2.7)-(2.9) into Eq. (2.15) and solving it for μ by substituting the zeroth order solution into the first order one (proportional to R), gives:

$$\mu_b = nV(\mathbf{k} = 0) - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\frac{\hbar^2\mathbf{k}^2}{2m} R(\mathbf{k})}{\left[\frac{\hbar^2\mathbf{k}^2}{2m} + 2nV(\mathbf{k}) \right]^2}. \quad (2.17)$$

If the density of the system is equal to 0, i.e. there are no particles in the system, the energy needed for a particle to be added is also 0, so $\mu(n = 0) = 0$. In the zeroth order this is automatically true, but for higher orders we have to renormalize the chemical potential using this condition,

$$\mu(n) = \mu_b(n) - \mu_b(0) = \mu_b + \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{R(\mathbf{k})}{\frac{\hbar^2\mathbf{k}^2}{2m}}.$$

This gives the renormalized expression for the equation of state

$$\mu = nV(\mathbf{k} = 0) + 4n \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{V(\mathbf{k})R(\mathbf{k}) \left(\frac{\hbar^2\mathbf{k}^2}{2m} + nV(\mathbf{k}) \right)}{\frac{\hbar^2\mathbf{k}^2}{2m} \left[\frac{\hbar^2\mathbf{k}^2}{2m} + 2nV(\mathbf{k}) \right]^2} + \dots \quad (2.18)$$

For calculating the speed of sound we will need a derivative of this expression with respect

to n , which we give here:

$$\frac{\partial \mu}{\partial n} = V(\mathbf{k} = 0) + 4 \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{\frac{\hbar^2 \mathbf{k}^2}{2m} R(\mathbf{k}) V(\mathbf{k})}{\left[\frac{\hbar^2 \mathbf{k}^2}{2m} + 2nV(\mathbf{k}) \right]^3} + \dots \quad (2.19)$$

2.4 Superfluidity

Without disorder, at $T = 0$, the whole system is superfluid. By introducing weak disorder and moving it with the velocity \mathbf{v} , some part of the fluid will be moving together with it. The normal (non-superfluid) component of the fluid n_N is defined as a part that moves with disorder, while the superfluid component n_S is defined as the fraction of the system that moves with a superfluid velocity \mathbf{k}'_S . Using the total fluid velocity \mathbf{k}_{tot} and the expressions for kinetic energy and momentum of the fluid we will obtain both densities and the change in the superfluid velocity due to disorder.

In the coordinate system in which disorder is moving with velocity \mathbf{v} , the Gross-Pitaevskii equation takes the following time-dependent form:

$$\left[-\frac{\hbar^2}{2m} \nabla_{\mathbf{r}}^2 + U(\mathbf{r} - \mathbf{v}t) + \int d^3 \mathbf{r}' V(\mathbf{r} - \mathbf{r}') \Psi_r^*(\mathbf{r}', t) \Psi_r(\mathbf{r}', t) \right] \Psi_r(\mathbf{r}, t) = i\hbar \frac{\partial \Psi_r(\mathbf{r}, t)}{\partial t}. \quad (2.20)$$

We search for its solution by perturbing the boosted solution without disorder, Eq. (1.6):

$$\Psi_r(\mathbf{r}, t) = \underbrace{e^{i\mathbf{k}_S \mathbf{r}} e^{-\frac{i}{\hbar} \left(\mu + \frac{\hbar^2 k_S^2}{2m} \right) t}}_{\psi_e} \underbrace{(\psi_0 + \psi_{r1}(\mathbf{r}, t) + \dots)}_{\psi}. \quad (2.21)$$

Without disorder ($U = 0$) this reduces to the ordinary real solution (in the rest frame), $\psi_0 = \sqrt{n}$, boosted with the wavevector \mathbf{k}_S , and all corrections are equal to zero. Here μ denotes the chemical potential in the rest frame of the superfluid.

By performing a change of variables:

$$\mathbf{r} = \mathbf{x} + \mathbf{v}t, \quad \Psi(\mathbf{x}, t) = \Psi_r(\mathbf{r}(\mathbf{x}, t), t), \quad \nabla_r \Psi_r = \nabla \Psi,$$

$$\frac{\partial \Psi_r(\mathbf{r}, t)}{\partial t} = \frac{\partial \Psi(\mathbf{x}, t)}{\partial t} - \nabla_r \Psi_r \frac{\partial \mathbf{r}}{\partial t} = \frac{\partial \Psi(\mathbf{x}, t)}{\partial t} - \mathbf{v} \nabla \Psi,$$

the GP equation (2.20) is transformed to

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + i \frac{\hbar^2}{m} \mathbf{k}_v \nabla + U(\mathbf{x}) + \int d^3 \mathbf{x}' V(\mathbf{x} - \mathbf{x}') \Psi^*(\mathbf{x}', t) \Psi(\mathbf{x}', t) \right] \Psi(\mathbf{x}, t) = i\hbar \frac{\partial \Psi(\mathbf{x}, t)}{\partial t}, \quad (2.22)$$

where $\mathbf{k}_v = \frac{m}{\hbar} \mathbf{v}$ is a wavevector corresponding to the velocity of disorder. The ansatz for

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a solution from Eq. (2.21) becomes

$$\Psi(\mathbf{x}, t) = e^{i\mathbf{k}_S \mathbf{x}} e^{-\frac{i}{\hbar} \left(\mu + \frac{\hbar^2 k_S^2}{2m} - \frac{\hbar^2 \mathbf{k}_S \mathbf{k}_v}{m} \right) t} \psi(\mathbf{x}, t), \quad (2.23)$$

$$\psi(\mathbf{x}, t) = \psi_0 + \psi_1(\mathbf{x}, t) + \dots$$

Substituting Eq. (2.23) into Eq. (2.22) and labeling

$$\mathbf{K} = \mathbf{k}_S - \mathbf{k}_v, \quad (2.24)$$

we obtain

$$\begin{aligned} \left[-\frac{\hbar^2}{2m} \nabla^2 - i \frac{\hbar^2}{m} \mathbf{K} \nabla + U(\mathbf{x}) - \mu + \int d^3 \mathbf{x}' V(\mathbf{x} - \mathbf{x}') \psi^*(\mathbf{x}', t) \psi(\mathbf{x}', t) \right] \psi(\mathbf{x}, t) \\ = i \hbar \frac{\partial \psi(\mathbf{x}, t)}{\partial t}. \end{aligned} \quad (2.25)$$

Zeroth-order solution is space-time independent, and is again equal to

$$\psi_0 = \frac{\mu}{V(\mathbf{k} = 0)}. \quad (2.26)$$

Assuming that solutions of all orders up to n -th are time-independent leads to a time-independent term $U(\mathbf{x})\psi_n(\mathbf{x})$ in the linear equation for the solution ψ_{n+1} of the order $n+1$. After Fourier-transforming the equation, $((\mathbf{r}, t) \rightarrow (\mathbf{k}, \omega))$, a term $\delta(\omega)$ appears, making also the solution ψ_{n+1} time-independent. Therefore, using the mathematical induction, all corrections will be time-independent. We conclude that there is a time-independent solution to the Eq. (2.25) satisfying at the same time the following time-independent equation:

$$\left[-\frac{\hbar^2}{2m} \nabla^2 - i \frac{\hbar^2}{m} \mathbf{K} \nabla + U(\mathbf{x}) - \mu + \int d^3 \mathbf{x}' V(\mathbf{x} - \mathbf{x}') \psi^*(\mathbf{x}') \psi(\mathbf{x}') \right] \psi(\mathbf{x}) = 0. \quad (2.27)$$

2.4.1 Local fluid velocity

Superfluid velocity defined by Eq. (1.8) is superfluid velocity in a usual sense. In our present case, with disorder, it encodes the total velocity of both, the superfluid and the normal fluid component.

The fluid wavevector is given by $\mathbf{k}_{tot} = \frac{m}{\hbar} \mathbf{v}_S$, which together with Eq. (1.8) yields:

$$\begin{aligned} \mathbf{k}_{tot} &= \frac{1}{i} \nabla \ln \frac{\Psi}{|\Psi|} = \frac{1}{i} \nabla \ln \sqrt{\frac{\Psi}{\Psi^*}} = \frac{1}{2i} \frac{\Psi^* \nabla \Psi - \Psi \nabla \Psi^*}{|\Psi|^2} \\ &= \frac{1}{2in} (\psi_e^* \psi^* i \mathbf{k}_S \psi_e \psi + \psi_e^* \psi^* \psi_e \nabla \psi + \psi_e \psi i \mathbf{k}_S \psi_e^* \psi^* - \psi_e \psi \psi_e^* \nabla \psi^*) \\ &= \mathbf{k}_S + \frac{1}{2in} (\psi^* \nabla \psi - \psi \nabla \psi^*) \end{aligned} \quad (2.28)$$

$$= \mathbf{k}_S - \Delta \mathbf{k}. \quad (2.29)$$

Here, $\Delta \mathbf{k}$ denotes a change of the fluid wavevector due to disorder. For \mathbf{K} equal to zero ψ is real, and therefore $\Delta \mathbf{k}(\mathbf{x}) = 0$. Expanding $\Delta \mathbf{k}$ using the expression from Eq. (2.28) to linear terms in \mathbf{K} we obtain

$$\Delta \mathbf{k}(\mathbf{x}) = \hat{D}(\mathbf{x}) \mathbf{K} + O(\mathbf{K}^2). \quad (2.30)$$

2.4.2 Two-fluid model

As we said in the beginning of Sec. 2.4 the BEC fluid in the presence of disorder is divided into a macroscopic normal component and a macroscopic superfluid component.

Apart from the notion of the clean-case superfluid wavevector \mathbf{k}_S and the local fluid wavevector $\mathbf{k}_{tot}(\mathbf{x})$ we also define the wavevector \mathbf{k}'_S of the macroscopic superfluid component. For $\mathbf{K} = 0$ the superfluid velocity remains \mathbf{k}_S and its expansion around $\mathbf{K} = 0$ gives:

$$\mathbf{k}'_S = \mathbf{k}_S - \hat{D}_S \mathbf{K} + O(\mathbf{K}^2), \quad (2.31)$$

where \hat{D}_S is derivative of \mathbf{k}'_S with respect to \mathbf{K} .

The average momentum density and kinetic energy density can be expanded, using Eqs. (2.24) and (2.29)-(2.31), with respect to \mathbf{k}'_S and \mathbf{k}_v as

$$\langle n(\mathbf{x}) \mathbf{k}_{tot}(\mathbf{x}) \rangle = \left[(\langle n \rangle - \langle n \hat{D} \rangle) (\hat{I} - \hat{D}_S)^{-1} \right] \mathbf{k}'_S + \left[(\langle n \hat{D} \rangle - \langle n \hat{D}_S \rangle) (\hat{I} - \hat{D}_S)^{-1} \right] \mathbf{k}_v, \quad (2.32)$$

and

$$\begin{aligned} \langle n(\mathbf{x}) \mathbf{k}_{tot}^2(\mathbf{x}) \rangle &= \mathbf{k}'_S (\hat{I} - \hat{D}_S^T)^{-1} \left[\langle n \rangle \hat{I} + \langle n \hat{D}^T \hat{D} \rangle - \langle n \hat{D} \rangle - \langle n \hat{D}^T \rangle \right] (\hat{I} - \hat{D}_S)^{-1} \mathbf{k}'_S \\ &+ \mathbf{k}_v (\hat{I} - \hat{D}_S^T)^{-1} \left[\langle n \rangle \hat{D}_S^T \hat{D}_S + \langle n \hat{D}^T \hat{D} \rangle \right. \\ &\quad \left. - \hat{D}_S^T \langle n \hat{D} \rangle - \langle n \hat{D}^T \rangle \hat{D}_S \right] (\hat{I} - \hat{D}_S)^{-1} \mathbf{k}_v \\ &+ 2 \mathbf{k}'_S (\hat{I} - \hat{D}_S^T)^{-1} \left[\langle n \hat{D} \rangle + \langle n \hat{D}^T \rangle \hat{D}_S \right. \\ &\quad \left. - \langle n \rangle \hat{D}_S - \langle n \hat{D}^T \hat{D} \rangle \right] (\hat{I} - \hat{D}_S)^{-1} \mathbf{k}_v. \end{aligned} \quad (2.33)$$

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In the two fluid model the total kinetic energy is the sum of kinetic energies of individual fluids. Therefore, the last (mixed) term in Eq. (2.33) is equal to zero, fixing the macroscopic superfluid velocity change to:

$$\hat{D}_S = (\langle n \rangle - \langle n \hat{D}^T \rangle)^{-1} (\langle n \hat{D} \rangle - \langle n \hat{D}^T \hat{D} \rangle). \quad (2.34)$$

With Eq. (2.32) this leads to a definition of the densities of superfluid and normal components as:

$$\hat{n}_S = (\langle n \rangle - \langle n \hat{D} \rangle) \left[\langle n \rangle - \langle n \hat{D} \rangle - \langle n \hat{D}^T \rangle + \langle n \hat{D}^T \hat{D} \rangle \right]^{-1} (\langle n \rangle - \langle n \hat{D}^T \rangle), \quad (2.35)$$

$$\hat{n}_N = \langle n \rangle \hat{I} - \hat{n}_S. \quad (2.36)$$

The above definitions are such that

$$\langle n(\mathbf{x}) \mathbf{k}_{tot}(\mathbf{x}) \rangle = \hat{n}_S \mathbf{k}'_S + \hat{n}_N \mathbf{k}_v,$$

and it is also easy to check that they satisfy

$$\langle n(\mathbf{x}) \mathbf{k}_{tot}^2(\mathbf{x}) \rangle = \mathbf{k}'_S \hat{n}_S \mathbf{k}'_S + \mathbf{k}_v \hat{n}_N \mathbf{k}_v,$$

making the two-fluid picture self-consistent. Now calculation of the superfluid density reduces to calculating $\hat{D}(\mathbf{x})$.

2.4.3 Calculating \hat{D}

From the definition of $\Delta \mathbf{k}(\mathbf{x})$, Eq. (2.30), we have

$$\Delta k_i = D_{ij}(\mathbf{x}) K_j = (\partial_{K_j} \Delta k_i(\mathbf{x}))_{\mathbf{K}=0} K_j.$$

If we insert Eq. (2.28) and define

$$\mathbf{p}(\mathbf{x}) = (\nabla_{\mathbf{K}} \psi(\mathbf{x}))_{\mathbf{K}=0}, \quad (2.37)$$

the expression for D_{ij} reduces to

$$\begin{aligned} D_{ij}(\mathbf{x}) &= -\frac{1}{2i} \left(\partial_{K_j} \nabla_i \ln \frac{\psi}{\psi^*} \right)_{\mathbf{K}=0} = -\frac{1}{2i} \nabla_i \left(\frac{\partial_{K_j} \psi}{\psi} - \frac{\partial_{K_j} \psi^*}{\psi^*} \right)_{\mathbf{K}=0} \\ &= -\frac{1}{2i} \nabla_i \frac{p_j(\mathbf{x}) - p_j^*(\mathbf{x})}{\psi^0(\mathbf{x})}, \end{aligned}$$

or

$$\hat{D}(\mathbf{x}) = -\nabla \otimes \frac{\text{Im } \mathbf{p}(\mathbf{x})}{\psi^0(\mathbf{x})}. \quad (2.38)$$

Calculating imaginary part of \mathbf{p} can be performed by differentiating GP equation (2.27) with respect to \mathbf{K} and then setting $\mathbf{K} = 0$,

$$-\frac{\hbar^2}{2m} \nabla^2 \mathbf{p}(\mathbf{x}) - \frac{i\hbar^2}{m} \nabla \psi^0(\mathbf{x}) + (U(\mathbf{x}) - \mu) \mathbf{p}(\mathbf{x}) + \int d^3 \mathbf{x}' V(\mathbf{x} - \mathbf{x}') [(\mathbf{p}^*(\mathbf{x}') + \mathbf{p}(\mathbf{x}')) \psi^0(\mathbf{x}') \psi^0(\mathbf{x}) + \psi^0(\mathbf{x}')^2 \mathbf{p}(\mathbf{x})] = 0.$$

Conjugating and subtracting the result from the original expression gives

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{x}) - \mu + \int d^3 \mathbf{x}' V(\mathbf{x} - \mathbf{x}') n^0(\mathbf{x}') \right] \text{Im } \mathbf{p}(\mathbf{x}) = \frac{\hbar^2}{m} \nabla \psi^0(\mathbf{x}). \quad (2.39)$$

To summarize, after finding a solution ψ^0 of GP equation, we calculate $\text{Im } \mathbf{p}(\mathbf{x})$ by solving the linear inhomogeneous equation (2.39). Then it is straightforward to calculate \hat{D} using Eq. (2.38) as well as superfluid and normal component densities using Eqs. (2.35) and (2.36).

2.4.4 Perturbative expansion

In this section we perturbatively solve Eq. (2.39), and calculate Eqs. (2.38) and (2.36).

From now on, when referring to ψ we actually assume ψ^0 , i.e. the solution of time-independent GP equation (1.4), and $\mathbf{p}(\mathbf{k})$ will correspond to the Fourier transform of $\text{Im } \mathbf{p}(\mathbf{x})$. We already know the perturbative expansion for ψ , which is given by Eqs. (2.7)-(2.9).

The Fourier transform of Eq. (2.39) is

$$\left[\frac{\hbar^2 \mathbf{k}^2}{2m} - \mu \right] \mathbf{p}(\mathbf{k}) + \int \frac{d^3 \mathbf{k}'}{(2\pi)^3} [V(\mathbf{k} - \mathbf{k}') n(\mathbf{k} - \mathbf{k}') + U(\mathbf{k} - \mathbf{k}')] \mathbf{p}(\mathbf{k}') = i \frac{\hbar^2 \mathbf{k}}{m} \psi(\mathbf{k}). \quad (2.40)$$

Because ψ_0 is position-independent its Fourier transform will be $\psi_0(\mathbf{k}) = (2\pi)^3 \delta(\mathbf{k}) \psi_0$. Similarly, $n_0(\mathbf{k}) = (2\pi)^3 \delta(\mathbf{k}) \psi_0^2$, since $n_0 = \psi_0^2$. Using this, with $\mu = \psi_0^2 V(\mathbf{k} = 0)$, gives the n-th order equation in the form

$$\frac{\hbar^2 \mathbf{k}^2}{2m} \mathbf{p}_n(\mathbf{k}) + \int \frac{d^3 \mathbf{k}'}{(2\pi)^3} \left[\sum_{i=1}^{n-1} V(\mathbf{k} - \mathbf{k}') n_{n-i}(\mathbf{k} - \mathbf{k}') \mathbf{p}_i(\mathbf{k}') + U(\mathbf{k} - \mathbf{k}') \mathbf{p}_{n-1}(\mathbf{k}') \right] = i \frac{\hbar^2 \mathbf{k}}{m} \psi_n(\mathbf{k}). \quad (2.41)$$

From Eqs. (2.26) and (2.37) it follows that $\mathbf{p}_0 = 0$, and therefore

$$\mathbf{p}_0(\mathbf{k}) = 0. \quad (2.42)$$

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Taking $n = 0$ (the zeroth order) in Eq. (2.41) yields to

$$k^2 \mathbf{p}_0(\mathbf{k}) = 2i(2\pi)^3 \mathbf{k} \delta(\mathbf{k}) \psi_0, \quad (2.43)$$

which has a solution $\mathbf{p}_0(\mathbf{k} \neq 0) = 0$, but is divergent for $\mathbf{k} = 0$. In order to exclude the strange behavior of the distribution given by Eq. (2.43), we take that the solution of the equation is given by Eq. (2.42).

In the first order we have

$$\mathbf{p}_1(\mathbf{k}) = 2i \frac{\mathbf{k}}{k^2} \psi_1(\mathbf{k}), \quad (2.44)$$

and in the second order

$$\mathbf{p}_2(\mathbf{k}) = 2i \left[\frac{\mathbf{k}}{k^2} \psi_2(\mathbf{k}) - \int \frac{d^3 \mathbf{k}'}{(2\pi)^3} \frac{\mathbf{k}'}{k'^2} \frac{[U(\mathbf{k} - \mathbf{k}') + 2\psi_0 V(\mathbf{k} - \mathbf{k}') \psi_1(\mathbf{k} - \mathbf{k}')] \psi_1(\mathbf{k}')}{\frac{\hbar^2 \mathbf{k}^2}{2m}} \right]. \quad (2.45)$$

Expanding Eq. (2.38) in terms of different powers of U leads to

$$\begin{aligned} \hat{D}(\mathbf{x}) &= -\frac{1}{\psi_0} \nabla \otimes \left[\left(1 - \frac{\psi_1(\mathbf{x})}{\psi_0} \right) (\text{Imp} \mathbf{p}_1(\mathbf{x}) + \text{Imp} \mathbf{p}_2(\mathbf{x})) \right] + \dots \\ &= \frac{2}{\psi_0} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\mathbf{x}} \mathbf{k} \otimes \left[\frac{\mathbf{p}_1(\mathbf{k})}{2i} + \frac{\mathbf{p}_2(\mathbf{k})}{2i} - \frac{1}{\psi_0} \int \frac{d^3 \mathbf{k}'}{(2\pi)^3} \frac{\mathbf{p}_1(\mathbf{k}')}{2i} \psi_1(\mathbf{k} - \mathbf{k}') \right] + \dots, \end{aligned}$$

and one immediately gets zeroth, first and second order solutions:

$$\hat{D}_0 = 0, \quad (2.46)$$

$$\hat{D}_1(\mathbf{k}) = \frac{2}{\psi_0} \frac{\mathbf{k} \otimes \mathbf{k}}{k^2} \psi_1(\mathbf{k}), \quad (2.47)$$

and

$$\hat{D}_2(\mathbf{k}) = \frac{2}{\psi_0} \left[\frac{\mathbf{k} \otimes \mathbf{k}}{k^2} \psi_2(\mathbf{k}) - \int \frac{d^3 \mathbf{k}'}{(2\pi)^3} \frac{\mathbf{k} \otimes \mathbf{k}'}{k'^2} \left(\frac{1}{\psi_0} \psi_1(\mathbf{k}') \psi_1(\mathbf{k} - \mathbf{k}') + \frac{U(\mathbf{k} - \mathbf{k}') \psi_1(\mathbf{k}')}{\frac{\hbar^2 \mathbf{k}^2}{2m}} \right) \right]. \quad (2.48)$$

Averages of Eqs. (2.45) and (2.48) lead to expressions of the type

$$k^2 \langle x(\mathbf{k}) \rangle = \delta(\mathbf{k}) \cdot \dots,$$

which is of the same form as Eq. (2.43), and we take the same solution, namely $\langle x(\mathbf{k}) \rangle = 0$. The remaining term in Eq. (2.48), after averaging, has an odd integrand with respect to the integration variable \mathbf{k}' and it is also vanishing. Therefore we conclude that

$$\langle \mathbf{p}_2 \rangle = 0,$$

and

$$\langle \hat{D}_2 \rangle = 0. \quad (2.49)$$

Using the same procedure of ensemble average calculation as in Eq. (2.16), we give below some useful averaged expressions:

$$\begin{aligned} \langle n_1 \hat{D}_1 \rangle &= 2\psi_0 \int \frac{d^3\mathbf{k}d^3\mathbf{k}'}{(2\pi)^6} \frac{2}{\psi_0} \frac{\mathbf{k}' \otimes \mathbf{k}'}{k'^2} \langle \psi_1(\mathbf{k}')\psi_1(\mathbf{k}-\mathbf{k}') \rangle \\ &= 4\psi_0^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\mathbf{k} \otimes \mathbf{k}}{k^2} \frac{R(\mathbf{k})}{\left[\frac{\hbar^2 k^2}{2m} + 2nV(\mathbf{k}) \right]^2}, \end{aligned} \quad (2.50)$$

and

$$\begin{aligned} \langle \hat{D}_1^T \hat{D}_1 \rangle &= 4 \int \frac{d^3\mathbf{k}}{(2\pi)^3} (\mathbf{e}_\mathbf{k} \otimes \mathbf{e}_\mathbf{k})^2 \frac{R(\mathbf{k})}{\left[\frac{\hbar^2 k^2}{2m} + 2nV(\mathbf{k}) \right]^2} \\ &= 4 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \mathbf{e}_\mathbf{k} \otimes \mathbf{e}_\mathbf{k} \frac{R(\mathbf{k})}{\left[\frac{\hbar^2 k^2}{2m} + 2nV(\mathbf{k}) \right]^2} \\ &= \frac{1}{\psi_0^2} \langle n_1 \hat{D}_1 \rangle. \end{aligned} \quad (2.51)$$

Finally, by expanding Eqs. (2.34) and (2.36) and taking into account Eqs. (2.49)-(2.51) we obtain

$$\langle \hat{D} \rangle = 0, \quad (2.52)$$

$$\hat{D}_S = \langle \hat{D}_2 \rangle + \frac{1}{n_0} \langle n_1 \hat{D}_1 \rangle - \langle \hat{D}_1^T \hat{D}_1 \rangle + \dots = o(R), \quad (2.53)$$

$$\begin{aligned} \hat{n}_N &= n_0 \langle \hat{D}_1^T \hat{D}_1 \rangle + \dots \\ &= 4n \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\mathbf{k} \otimes \mathbf{k}}{k^2} \frac{R(\mathbf{k})}{\left[\frac{\hbar^2 k^2}{2m} + 2nV(\mathbf{k}) \right]^2} + \dots \end{aligned} \quad (2.54)$$

These results are valid for arbitrary disorder correlation function $R(\mathbf{k})$, and effective two-particle interaction $V(\mathbf{k})$. Eqs. (2.52) and (2.53) imply that weak disorder strength does not alter neither the average fluid velocity nor the superfluid velocity, as long as the velocities are small and irrespective of the precise nature of disorder and the interparticle interaction.

2.4.5 Cylindrical symmetry

In the case of a cylindrically symmetric system, choosing symmetry axis as z axis, denoting by θ and φ the polar and the azimuth angle, and integrating Eq. (2.54) in spherical

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coordinates with respect to φ gives

$$\begin{aligned} \sin \theta \int_0^{2\pi} d\varphi \mathbf{e}_{\mathbf{k}} \mathbf{e}_{\mathbf{k}}^T &= \sin \theta \int_0^{2\pi} d\varphi \begin{pmatrix} \sin^2 \theta \cos^2 \varphi & \sin^2 \theta \sin \varphi \cos \varphi & \sin \theta \cos \theta \cos \varphi \\ \sin^2 \theta \sin \varphi \cos \varphi & \sin^2 \theta \sin^2 \varphi & \sin \theta \cos \theta \sin \varphi \\ \sin \theta \cos \theta \cos \varphi & \sin \theta \cos \theta \sin \varphi & \cos^2 \theta \end{pmatrix} \\ &= \sin \theta \begin{pmatrix} \pi(1 - \cos^2 \theta) & 0 & 0 \\ 0 & \pi(1 - \cos^2 \theta) & 0 \\ 0 & 0 & 2\pi \cos^2 \theta \end{pmatrix}. \end{aligned} \quad (2.55)$$

2.4.6 Spherical symmetry

If both $V(\mathbf{k})$ and $R(\mathbf{k})$ are θ -independent, i.e. we have spherical symmetry, integrating Eq. (2.55) with respect to θ leads to a solution in the second order of disorder strength (first order in R):

$$\hat{n}_N = \frac{4}{3}(n - n')\hat{I}. \quad (2.56)$$

This result shows that the superfluid depletion will be by a factor of 4/3 higher than the condensate depletion.

2.5 Sound velocity

In the mean-field approach, we can try to define sound velocity by perturbing the time-independent solution by a small time-dependent contribution. It is expected that sound waves with wavelengths close the correlation length would scatter and interfere due to disorder hills and valleys, making global sound velocity impossible to define precisely. Locally, the sound waves would have the same speed as in the clean case.

For long-wavelength sound waves (compared to the disorder correlation length), sound velocity can be calculated using the hydrodynamical approach, Eqs. (1.11) and (1.12). Those are macroscopic equations and can be used only for slowly varying quantities that do not depend on the specific microscopic realization. Spatial averaging over distances much larger than the correlation length and much smaller than wavelengths would solve the problem. Assuming that it gives the same result as the disorder average over the ensemble¹, we obtain the hydrodynamic equations for the macroscopic (or averaged) quantities:

$$\frac{\partial n_{\text{mac}}(\mathbf{x}, t)}{\partial t} + \nabla \cdot (\hat{n}_S(\mathbf{x}, t) \mathbf{v}'_S(\mathbf{x}, t)) = 0, \quad (2.57)$$

$$m \frac{\partial \mathbf{v}'_S(\mathbf{x}, t)}{\partial t} + \nabla \cdot \left(\frac{m \mathbf{v}'_S(\mathbf{x}, t)^2}{2} + \mu(n_{\text{mac}}(\mathbf{x}, t)) \right) = 0, \quad (2.58)$$

¹See the beginning of the chapter.

where n_{mac} denotes macroscopic density and the disorder velocity \mathbf{k}_v is taken to be zero.

If we write densities and the superfluid velocity as the sums of averages and their variations,

$$n_{mac}(\mathbf{x}, t) = n + \delta n(\mathbf{x}, t), \quad (2.59)$$

$$\hat{n}_S(\mathbf{x}, t) = \hat{n}_S + \delta \hat{n}_S(\mathbf{x}, t), \quad (2.60)$$

$$\mathbf{v}'_S(\mathbf{x}, t) = \delta \mathbf{v}'_S(\mathbf{x}, t), \quad (2.61)$$

and neglect second-order terms in variations, we get the following system of equations:

$$\frac{\partial \delta n(\mathbf{x}, t)}{\partial t} + \nabla(\hat{n}_S \delta \mathbf{v}'_S(\mathbf{x}, t)) = 0, \quad (2.62)$$

$$\frac{\partial \delta \mathbf{v}'_S(\mathbf{x}, t)}{\partial t} = -\frac{1}{m} \nabla \mu (n + \delta n(\mathbf{x}, t)) = -\frac{1}{m} \frac{\partial \mu}{\partial n} \nabla \delta n(\mathbf{x}, t). \quad (2.63)$$

Taking the time derivative of the first equation and substituting the superfluid velocity variation, expressed from the second equation, gives

$$\frac{\partial^2 \delta n(\mathbf{x}, t)}{\partial t^2} - \frac{1}{m} \frac{\partial \mu}{\partial n} \nabla(\hat{n}_S \nabla \delta n(\mathbf{x}, t)) = 0. \quad (2.64)$$

From here it can be easily seen that the sound velocity in the \mathbf{q} direction is given by

$$c_{\mathbf{q}}^2 = \frac{1}{m} \frac{\partial \mu}{\partial n} \mathbf{q}^T \hat{n}_S \mathbf{q}. \quad (2.65)$$

2.6 Comparison to the result of Huang and Meng

In Ref. [13] Huang and Meng used Bogoliubov transformation to calculate condensate depletion for the case of contact interaction and δ -correlated disorder, obtaining

$$n_{\text{HM}} = \frac{m^{\frac{3}{2}} R \sqrt{\bar{n}}}{4\pi \hbar^3 \sqrt{g}}. \quad (2.66)$$

This result is linear in R , as all our corrections will be, and it is useful to compare relative change of a quantity due to disorder to the relative change of the condensate density. To that end we define a correction of the quantity A as

$$\Delta_A = \lim_{R \rightarrow 0} \frac{\frac{A}{A_0} - A_d}{\frac{n_{\text{HM}}}{n}}. \quad (2.67)$$

Here $A_0 A_d$ is a value of the quantity A in the clean ($R = 0$) system, while A_d denotes possible dimensionless anisotropy factor due to dipolar interaction.

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We will now consider special case when

$$R(\mathbf{k}) = RR_0(\mathbf{k}^T \hat{\sigma} \mathbf{k}), \quad V(\mathbf{k}) = gV_0(\mathbf{e}_\mathbf{k}), \quad (2.68)$$

where R_0 and V_0 are dimensionless functions, R and g are disorder and interaction strengths, and $\hat{\sigma}$ is a tensor of characteristic correlation lengths. For both contact and dipolar interaction the function V_0 does not depend on the module of \mathbf{k} , just on its direction in the case of dipolar interaction.

Performing change of variable $\mathbf{k} \rightarrow \frac{2\sqrt{mng}}{\hbar} \mathbf{k} = \frac{\sqrt{2}}{\xi} \mathbf{k}$, and denoting $\hat{z} = \frac{2\hat{\sigma}}{\xi^2}$, expressions (2.16), (2.18), (2.19) and (2.54) are transformed into

$$n - n_0 = 8\pi n_{\text{HM}} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{R_0(\mathbf{k}^T \hat{z} \mathbf{k})}{[\mathbf{k}^2 + V_0(\mathbf{e}_\mathbf{k})]^2} + \dots, \quad (2.69)$$

$$\mu = ng \left[V_0(\mathbf{e}_\mathbf{k}) + 32\pi \frac{n_{\text{HM}}}{n} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{V_0(\mathbf{e}_\mathbf{k}) R_0(\mathbf{k}^T \hat{z} \mathbf{k}) (\mathbf{k}^2 + \frac{1}{2} V_0(\mathbf{e}_\mathbf{k}))}{\mathbf{k}^2 [\mathbf{k}^2 + V_0(\mathbf{e}_\mathbf{k})]^2} + \dots \right], \quad (2.70)$$

$$\frac{\partial \mu}{\partial n} = g \left[V_0(\mathbf{e}_\mathbf{k}) + 32\pi \frac{n_{\text{HM}}}{n} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{\mathbf{k}^2 R_0(\mathbf{k}^T \hat{z} \mathbf{k}) V_0(\mathbf{e}_\mathbf{k})}{[\mathbf{k}^2 + V_0(\mathbf{e}_\mathbf{k})]^3} + \dots \right], \quad (2.71)$$

$$(n_N)_{ij} = 32\pi n_{\text{HM}} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{R_0(\mathbf{k}^T \hat{z} \mathbf{k}) k_i k_j}{\mathbf{k}^2 [\mathbf{k}^2 + V_0(\mathbf{e}_\mathbf{k})]^2} + \dots \quad (2.72)$$

Taking Eq. (2.67) into account, the previous expressions can be written as

$$- \Delta_{n_0} = 8\pi \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{R_0(\mathbf{k}^T \hat{z} \mathbf{k})}{[\mathbf{k}^2 + V_0(\mathbf{e}_\mathbf{k})]^2}, \quad (2.73)$$

$$\Delta_\mu = 32\pi \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{V_0(\mathbf{e}_\mathbf{k}) R_0(\mathbf{k}^T \hat{z} \mathbf{k}) (\mathbf{k}^2 + \frac{1}{2} V_0(\mathbf{e}_\mathbf{k}))}{\mathbf{k}^2 [\mathbf{k}^2 + V_0(\mathbf{e}_\mathbf{k})]^2}, \quad (2.74)$$

$$\Delta_{\frac{\partial \mu}{\partial n}} = 32\pi \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{\mathbf{k}^2 R_0(\mathbf{k}^T \hat{z} \mathbf{k}) V_0(\mathbf{e}_\mathbf{k})}{[\mathbf{k}^2 + V_0(\mathbf{e}_\mathbf{k})]^3}, \quad (2.75)$$

$$- \Delta_{(n_S)_{ij}} = \Delta_{(n_N)_{ij}} = 32\pi \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{R_0(\mathbf{k}^T \hat{z} \mathbf{k}) k_i k_j}{\mathbf{k}^2 [\mathbf{k}^2 + V_0(\mathbf{e}_\mathbf{k})]^2}. \quad (2.76)$$

Substituting Eqs. (2.75) and (2.76), with Eq. (2.67), in (2.65) we get for the sound velocity

$$\begin{aligned} c_{\mathbf{q}}^2 &= \frac{1}{m} g \left(V_0(\mathbf{q}) + \frac{n_{\text{HM}}}{n} \Delta_{\frac{\partial \mu}{\partial n}} + \dots \right) \mathbf{q}^T n \left(\hat{I} - \frac{n_{\text{HM}}}{n} \hat{\Delta}_{n_N} + \dots \right) \mathbf{q} \\ &= \frac{gn}{m} \left[V_0(\mathbf{q}) + \frac{n_{\text{HM}}}{n} \left(-V_0(\mathbf{q}) \mathbf{q}^T \hat{\Delta}_{n_N} \mathbf{q} + \Delta_{\frac{\partial \mu}{\partial n}} \right) + \dots \right], \end{aligned}$$

and its second-order correction can be written as

$$\Delta_{e_{\mathbf{q}}}^2 = -V_0(\mathbf{q})\mathbf{q}^T \hat{\Delta}_{n_N} \mathbf{q} + \Delta_{\frac{\partial \mu}{\partial n}}. \quad (2.77)$$

2. BEC IN DISORDERED POTENTIALS

Chapter 3

Dipolar interaction and Lorentz-correlated disorder

In this chapter we will consider BEC systems in the presence of two different anisotropies: anisotropic dipolar interaction between the particles and anisotropic disorder potential. The latter is motivated by anisotropy of the laser-speckle potential.

In order to obtain analytical results, we model the disorder correlation function by a cylindrically-symmetric Lorentzian in a Fourier space,

$$R(\mathbf{k}) = \frac{R}{1 + \sigma_\rho^2 k_\rho^2 + \sigma_z^2 k_z^2}. \quad (3.1)$$

This function is not realistic, but the results qualitatively coincide with the case of a Gaussian-correlated disorder, numerically calculated in Ref. [19], and we expect that all phenomena that appear here would also appear for the true laser-speckle correlation function.

Assuming that the van der Waals forces between the atoms can be approximated at low energies by an effective contact interaction, the interaction potential in the presence of an external field that aligns the dipoles in a direction \mathbf{m} takes the form [22]:

$$V_{\text{eff}}(\mathbf{r}) = g\delta(\mathbf{r}) + \frac{C_{dd}}{4\pi r^3} (1 - \cos \phi(\mathbf{m}, \mathbf{r})), \quad (3.2)$$

where $\phi(\mathbf{m}, \mathbf{r})$ represents the angle between vectors \mathbf{m} and \mathbf{r} .

Defining a ratio of a dipole-dipole and contact interaction by

$$\epsilon = \frac{C_{dd}}{3g}, \quad (3.3)$$

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and taking the Fourier transform of Eq. (3.2), we obtain

$$V(\mathbf{k}) = g [1 + \epsilon (3 \cos^2 \phi(\mathbf{m}, \mathbf{k}) - 1)] . \quad (3.4)$$

We now use Eqs. (3.1) and (3.4) to calculate the corrections of physical observables defined in Eqs. (2.73)-(2.76).

Denoting symmetry direction of the disorder by \mathbf{d} , expressions for functions R_0 and V_0 from Eq. (2.68) become

$$R_0(\mathbf{k}^T \hat{z} \mathbf{k}) = \frac{1}{1 + \mathbf{k}^2 (z_\rho^2 \sin^2 \phi(\mathbf{d}, \mathbf{k}) + z_z^2 \cos^2 \phi(\mathbf{d}, \mathbf{k}))} = \frac{1}{1 + \mathbf{k}^2 r^2(\mathbf{d}, \mathbf{k})} , \quad (3.5)$$

$$V_0(e_{\mathbf{k}}) = 1 + \epsilon (3 \cos^2 \phi(\mathbf{m}, \mathbf{k}) - 1) = c^2(\mathbf{m}, \mathbf{k}) , \quad (3.6)$$

with

$$r^2(\mathbf{d}, \mathbf{k}) = z_\rho^2 \sin^2 \phi(\mathbf{d}, \mathbf{k}) + z_z^2 \cos^2 \phi(\mathbf{d}, \mathbf{k}) , \quad (3.7)$$

$$c^2(\mathbf{m}, \mathbf{k}) = 1 + \epsilon (3 \cos^2 \phi(\mathbf{m}, \mathbf{k}) - 1) . \quad (3.8)$$

Inserting Eqs. (3.5) and (3.6) into Eqs. (2.73)-(2.76), writing all integrals in spherical coordinates, and integrating with respect to k leads to

$$-\Delta_{n_0} = \frac{1}{4\pi} \int d\Omega \frac{1}{c(1+cr)^2} , \quad (3.9)$$

$$\Delta_\mu = \frac{1}{2\pi} \int d\Omega \frac{c(3+2cr)}{(1+cr)^2} , \quad (3.10)$$

$$\Delta_{\frac{\partial \mu}{\partial n}} = \frac{1}{4\pi} \int d\Omega \frac{c(3+cr)}{(1+cr)^3} , \quad (3.11)$$

$$\hat{\Delta}_{n_N} = \frac{1}{\pi} \int d\Omega \frac{\mathbf{e}_{\mathbf{k}} \mathbf{e}_{\mathbf{k}}^T}{c(1+cr)^2} , \quad (3.12)$$

where $c = \sqrt{c^2(\mathbf{m}, \mathbf{k})}$ and $r = \sqrt{r^2(\mathbf{d}, \mathbf{k})}$.

If both c and r contain square roots (the case when both anisotropies are present), we need to rationalize the integrands in Eqs. (3.9)-(3.12) in order to proceed with their integration, obtaining:

$$-\Delta_{n_0} = \int \frac{d\Omega}{4\pi} \left(\frac{1 + c^2 r^2}{c^2 (1 - c^2 r^2)^2} c - \frac{2}{(1 - c^2 r^2)^2} r \right) , \quad (3.13)$$

$$\Delta_\mu = \int \frac{d\Omega}{4\pi} \left(2 \frac{3 - c^2 r^2}{(1 - c^2 r^2)^2} c - 4c^2 \frac{2 - c^2 r^2}{(1 - c^2 r^2)^2} r \right) , \quad (3.14)$$

$$\Delta_{\frac{\partial\mu}{\partial n}} = \int \frac{d\Omega}{4\pi} \left(\frac{3 + c^2 r^2 (6 - c^2 r^2)}{(1 - c^2 r^2)^3} c - \frac{8c^2}{(1 - c^2 r^2)^3} r \right), \quad (3.15)$$

$$\hat{\Delta}_{n_N} = \int \frac{d\Omega}{4\pi} \left(\frac{1 + c^2 r^2}{c^2 (1 - c^2 r^2)^2} c - \frac{2}{(1 - c^2 r^2)^2} r \right) 4\mathbf{e}_k \mathbf{e}_k^T. \quad (3.16)$$

3.1 Cylindrical symmetry

In the case when the direction of dipoles is parallel to the disorder symmetry, i.e. $\mathbf{m} \parallel \mathbf{d}$, the whole system is cylindrically symmetric. Integrating with respect to φ and taking into account Eq. (2.55), expressions (3.9)-(3.16) can be further simplified. Namely, the density correction of normal fluid is separated into two integrals,

$$\hat{\Delta}_{n_N} = -2\Delta_{n_0} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 2I_{sd} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \quad (3.17)$$

The normal component of the fluid can be separated into the parallel and the perpendicular part by

$$\begin{aligned} \Delta_{n_{N\rho}} &= -2\Delta_{n_0} - 2I_{sd}, \\ \Delta_{n_{Nz}} &= 4I_{sd}. \end{aligned}$$

Introducing $\cos\theta = t$ and using the parity of the integrand with respect to t , we get non-rationalized integrals,

$$-\Delta_{n_0} = \int_0^1 dt \frac{1}{c(1+cr)^2}, \quad (3.18)$$

$$\Delta_{\mu} = 2 \int_0^1 dt \frac{c(3+2cr)}{(1+cr)^2}, \quad (3.19)$$

$$\Delta_{\frac{\partial\mu}{\partial n}} = \int_0^1 dt \frac{c(3+cr)}{(1+cr)^3}, \quad (3.20)$$

$$I_{sd} = \int_0^1 dt \frac{t^2}{c(1+cr)^2}, \quad (3.21)$$

and after their rationalization we obtain:

$$-\Delta_{n_0} = \int_0^1 dt \left(\frac{1 + c^2 r^2}{c^2 (1 - c^2 r^2)^2} c - \frac{2}{(1 - c^2 r^2)^2} r \right), \quad (3.22)$$

$$\Delta_{\mu} = \int_0^1 dt \left(2 \frac{3 - c^2 r^2}{(1 - c^2 r^2)^2} c - 4c^2 \frac{2 - c^2 r^2}{(1 - c^2 r^2)^2} r \right), \quad (3.23)$$

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$$\Delta_{\frac{\partial \mu}{\partial n}} = \int_0^1 dt \left(\frac{3 + c^2 r^2 (6 - c^2 r^2)}{(1 - c^2 r^2)^3} c - \frac{8c^2}{(1 - c^2 r^2)^3} r \right), \quad (3.24)$$

$$I_{sd} = \int_0^1 dt t^2 \left(\frac{1 + c^2 r^2}{c^2 (1 - c^2 r^2)^2} c - \frac{2}{(1 - c^2 r^2)^2} r \right). \quad (3.25)$$

Expressions for functions r and c also change to

$$c^2 = 1 - \epsilon + 3\epsilon t^2, \quad (3.26)$$

$$r^2 = z_\rho^2 + (z_z^2 - z_\rho^2) t^2. \quad (3.27)$$

3.2 Contact interaction and cylindrically symmetric Lorentzian disorder

In this section we study special case of contact interaction. In the absence of dipolar interaction we have $\epsilon = 0$, and therefore $c = 1$. Now integrals (3.18)-(3.21) reduce to

$$- \Delta_{n_0} = \int_0^1 dt \frac{1}{(1+r)^2}, \quad (3.28)$$

$$\Delta_\mu = 2 \int_0^1 dt \frac{3+2r}{(1+r)^2}, \quad (3.29)$$

$$\Delta_{\frac{\partial \mu}{\partial n}} = \int_0^1 dt \frac{3+r}{(1+r)^3}, \quad (3.30)$$

$$I_{sd} = \int_0^1 dt \frac{t^2}{(1+r)^2}. \quad (3.31)$$

Making a substitution (Euler substitution, 2.251 from [21])

$$r = xt + z_\rho, \quad (3.32)$$

with $d = z_z^2 - z_\rho^2$, leads to the following integrals of rational functions:

$$- \Delta_{n_0} = \int_0^{z_z - z_\rho} dx \frac{2(d+x^2)z_\rho}{(x^2(-1+z_\rho) + d(1+z_\rho))^2}, \quad (3.33)$$

$$\Delta_\mu = 2 \int_0^{z_z - z_\rho} dx \frac{4z_\rho(3d^2 - 3x^4 + 2(d+x^2)^2 z_\rho)}{(d-x^2)(x^2(-1+z_\rho) + d(1+z_\rho))^2}, \quad (3.34)$$

$$\Delta_{\frac{\partial \mu}{\partial n}} = \int_0^{z_z - z_\rho} dx \frac{2(d+x^2)z_\rho(x^2(-3+z_\rho) + d(3+z_\rho))}{(x^2(-1+z_\rho) + d(1+z_\rho))^3}, \quad (3.35)$$

$$I_{sd} = \int_0^{z_z - z_\rho} dx \frac{32x^2 (d + x^2) z_\rho^3}{(d - x^2)^2 (x^2 (-1 + z_\rho) + d (1 + z_\rho))^2}. \quad (3.36)$$

Their solutions can be expressed in terms of a new function (see Appendix A for details and properties)

$$T(x) = \frac{\arctan \sqrt{x}}{\sqrt{x}}. \quad (3.37)$$

$$-\Delta_{n_0} = \frac{1}{(z_\rho - 1)(z_\rho + 1)} \left[\frac{2z_\rho^2}{(z_\rho + 1)(z_\rho + z_z)} T\left(\frac{z_\rho - 1}{z_\rho + 1} \frac{z_z - z_\rho}{z_z + z_\rho}\right) - \frac{1}{z_z + 1} \right], \quad (3.38)$$

$$\Delta_\mu = -\frac{4(z_\rho^2 - 2)}{(z_\rho^2 - 1)(z_\rho + 1)(z_\rho + z_z)} T\left(\frac{z_\rho - 1}{z_\rho + 1} \frac{z_z - z_\rho}{z_z + z_\rho}\right) + \frac{8}{z_\rho + z_z} T\left(\frac{z_\rho - z_z}{z_\rho + z_z}\right), \quad (3.39)$$

$$\Delta_{\frac{\partial \mu}{\partial n}} = \frac{2}{(-1 + z_\rho^2)^2} \left[\frac{3 + z_z (2 + z_\rho^2)}{2(1 + z_z)^2} + \frac{z_\rho^2 (-4 + z_\rho^2)}{(1 + z_\rho)(z_z + z_\rho)} T\left(\frac{z_\rho - 1}{z_\rho + 1} \frac{z_z - z_\rho}{z_z + z_\rho}\right) \right], \quad (3.40)$$

$$I_{sd} = \frac{1}{(-z_z + z_\rho)(z_z + z_\rho)} \left[-\frac{2 + z_z}{1 + z_z} + \frac{2(-2 + z_\rho^2)}{(1 + z_\rho)(z_z + z_\rho)} T\left(\frac{z_\rho - 1}{z_\rho + 1} \frac{z_z - z_\rho}{z_z + z_\rho}\right) + \frac{4}{z_z + z_\rho} T\left(\frac{z_\rho - z_z}{z_\rho + z_z}\right) \right]. \quad (3.41)$$

From Eq. (2.77) and the fact that in this case $V_0 = 1$, we conclude that the speed of sound can be also separated into the parallel and the perpendicular component:

$$\Delta_{c_\rho^2} = \Delta_{\frac{\partial \mu}{\partial n}} + 2\Delta_{n_0} + 2I_{sd}, \quad (3.42)$$

$$\Delta_{c_z^2} = \Delta_{\frac{\partial \mu}{\partial n}} - 4I_{sd}. \quad (3.43)$$

The calculated corrections are shown in Fig. 3.1. The solid red line in the plot of the superfluid depletion shows critical values of correlation lengths for which the superfluid depletion becomes smaller than the condensate depletion, suggesting that some atoms that are out of the condensate still contribute to the superfluidity.

From Eqs. (3.42) and (3.43) we can see that the superfluid depletion reduces the sound velocity, but the decrease of compressibility increases it. The sound velocity correction becomes negative for some values of correlation lengths. Note that, in both cases, the

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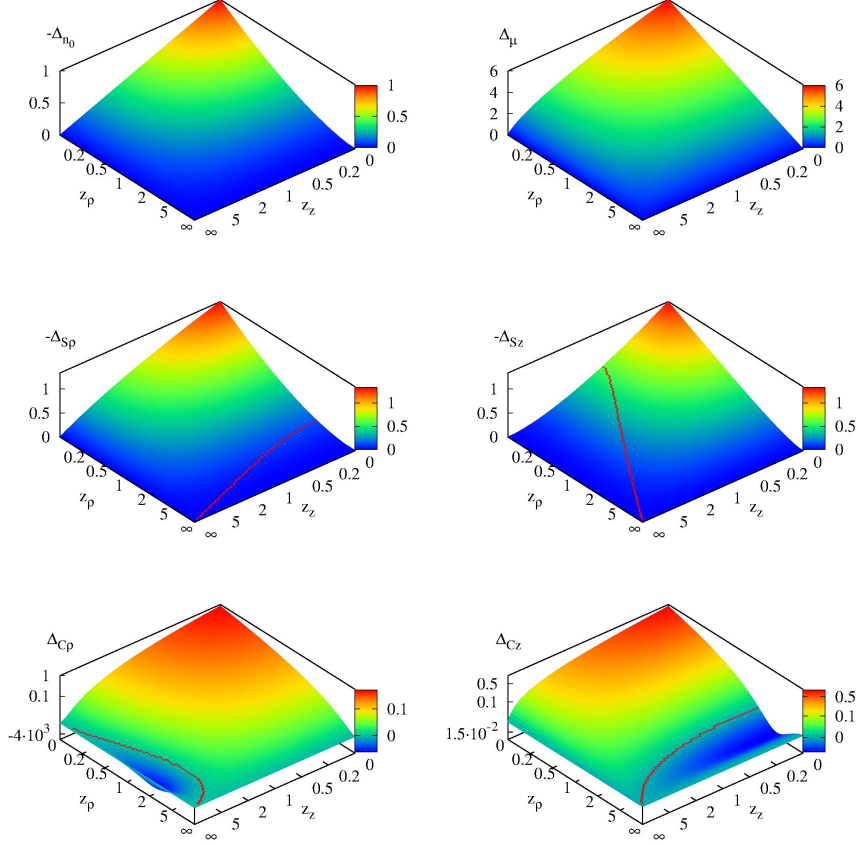


Figure 3.1: Corrections in the case of a pure contact interaction and Lorentz-correlated disorder.

negative correction is two orders of magnitude smaller than the δ -correlation correction. The solid red line denotes vanishing sound velocity correction.

We have also considered a number of different limits, which are discussed in detail in Appendix A.1. Summary of the results is presented in Tables 3.1 - 3.6.

Table 3.1: Condensate depletion for different limiting cases for $\epsilon = 0$ (pure contact interaction).

Limiting case	$-\Delta_{n_0}$
1: $z_z \approx z_\rho$	$\frac{1}{(1+z_\rho)^2} \left(1 + \frac{2}{3(1+z_\rho)}(z_z - z_\rho) + \dots \right)$
2: $z_\rho \ll 1, z_\rho \ll z_z$	$\frac{1}{1+z_z} + \left[\frac{1}{1+z_z} + \frac{1}{z_z} \left(\ln z_\rho + \ln \left(1 + \frac{1}{z_z} \right) - \ln 2 \right) \right] z_\rho^2 + \dots$
3: $1 \gg z_\rho \gg z_z$	$1 - \frac{\pi}{2} z_\rho + 2z_\rho^2 - \frac{\pi}{4} z_\rho \left(\frac{z_z}{z_\rho} \right)^2 - \frac{3\pi}{4} z_\rho^3 + \dots$
4: $z_\rho \gg 1, z_\rho \gg z_z$	$\left[-\frac{1}{1+z_z} + \ln \left(\frac{2z_\rho}{1+z_z} \right) \right] \left(\frac{1}{z_\rho} \right)^2 + \dots$
5: $1 \ll z_\rho \ll z_z$	$\frac{\pi}{2} \left(\frac{1}{z_\rho} \right)^2 \frac{z_\rho}{z_z} + \dots$

Table 3.2: Correction of the chemical potential for different limiting cases for $\epsilon = 0$ (pure contact interaction).

Limiting case	Δ_μ
1: $z_z \approx z_\rho$	$\frac{6+4z_\rho}{(1+z_\rho)^2} - \frac{4(2+z_\rho)}{3(1+z_\rho)^3}(z_z - z_\rho) + \dots$
2: $z_\rho \ll 1, z_\rho \ll z_z$	$\frac{2}{1+z_z} + \frac{4 \ln(1+z_z)}{z_z} + \dots$
3: $1 \gg z_\rho \gg z_z$	$6 - 2\pi z_\rho + \frac{20}{3} z_\rho^2 - \pi z_\rho \left(\frac{z_z}{z_\rho} \right)^2 - \frac{9\pi}{4} z_\rho^3 + \dots$
4: $z_\rho \gg 1, z_\rho \gg z_z$	$\frac{2\pi}{z_\rho} + \dots$
5: $1 \ll z_\rho \ll z_z$	$4 \ln \left(\frac{2z_z}{z_\rho} \right) \frac{1}{z_\rho} \frac{z_\rho}{z_z} + \dots$

Table 3.3: Perpendicular superfluid depletion for different limiting cases for $\epsilon = 0$ (pure contact interaction).

Limiting case	$-\Delta_{S_\rho}$
1: $z_z \approx z_\rho$	$\frac{4}{3(1+z_\rho)^2} - \frac{8}{15(1+z_\rho)^3}(z_z - z_\rho) + \dots$
2: $z_\rho \ll 1, z_\rho \ll z_z$	$\frac{2(z_z-2)}{z_z^2} + \frac{4 \ln(1+z_z)}{z_z^2} + \dots$
3: $1 \gg z_\rho \gg z_z$	$\frac{4}{3} - \frac{3\pi}{4} z_\rho + \frac{16}{5} z_\rho^2 - \frac{\pi}{8} z_\rho \left(\frac{z_z}{z_\rho} \right)^2 - \frac{5\pi}{4} z_\rho^3 + \dots$
4: $z_\rho \gg 1, z_\rho \gg z_z$	$\frac{2}{z_\rho^2} + \dots$
5: $1 \ll z_\rho \ll z_z$	$\pi \frac{1}{z_\rho^2} \frac{z_\rho}{z_z} + \dots$

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Table 3.4: Parallel superfluid depletion for different limiting cases for $\epsilon = 0$ (pure contact interaction).

Limiting case	$-\Delta_{S_z}$
1: $z_z \approx z_\rho$	$\frac{4}{3(1+z_\rho)^2} - \frac{8}{5(1+z_\rho)^3}(z_z - z_\rho) + \dots$
2: $z_\rho \ll 1, z_\rho \ll z_z$	$\frac{4(2+z_z)}{z_z^2(1+z_z)} - \frac{8 \ln(1+z_z)}{z_z^3} + \dots$
3: $1 \gg z_\rho \gg z_z$	$\frac{4}{3} - \frac{\pi}{2}z_\rho + \frac{8}{5}z_\rho^2 - \frac{3\pi}{4}z_\rho \left(\frac{z_z}{z_\rho}\right)^2 - \frac{\pi}{2}z_\rho^3 + \dots$
4: $z_\rho \gg 1, z_\rho \gg z_z$	$4 \left(-\frac{(2+z_z)}{1+z_z} + \ln \frac{8z_\rho}{1+z_z} \right) \frac{1}{z_\rho^2} + \dots$
5: $1 \ll z_\rho \ll z_z$	$4 \frac{1}{z_\rho^2} \left(\frac{z_\rho}{z_z}\right)^2 + \dots$

Table 3.5: Perpendicular component of the sound velocity for different limiting cases for $\epsilon = 0$ (pure contact interaction).

Limiting case	Δ_{c_ρ}
1: $z_z \approx z_\rho$	$\frac{5-z_\rho}{6(1+z_\rho)^3} - \frac{16+z_\rho}{15(1+z_\rho)^4}(z_z - z_\rho) + \dots$
2: $z_\rho \ll 1, z_\rho \ll z_z$	$\frac{4+3z_z(2+z_z)}{2z_z^2(1+z_z)^2} - \frac{2 \ln(1+z_z)}{z_z^3} + \dots$
3: $1 \gg z_\rho \gg z_z$	$\frac{5}{6} - \frac{5\pi}{8}z_\rho + \frac{17}{5}z_\rho^2 - \frac{7\pi}{16}z_\rho \left(\frac{z_z}{z_\rho}\right)^2 - \frac{13\pi}{8}z_\rho^3 + \dots$
4: $z_\rho \gg 1, z_\rho \gg z_z$	$\frac{1}{2} \left(-2 + \frac{z_z}{(1+z_z)^2} + \ln \frac{2z_\rho}{1+z_z} \right) \frac{1}{z_\rho^2} + \dots$
5: $1 \ll z_\rho \ll z_z$	$-\frac{\pi}{4} \frac{1}{z_\rho^2} \frac{z_\rho}{z_z} + \dots$

Table 3.6: Parallel component of the sound velocity for different limiting cases for $\epsilon = 0$ (pure contact interaction).

Limiting case	Δ_{c_z}
1: $z_z \approx z_\rho$	$\frac{5-z_\rho}{6(1+z_\rho)^3} - \frac{8-7z_\rho}{15(1+z_\rho)^4}(z_z - z_\rho) + \dots$
2: $z_\rho \ll 1, z_\rho \ll z_z$	$\frac{-8+z_z(-12+z_z(-1+2z_z))}{2z_z^2(1+z_z)^2} + \frac{4 \ln(1+z_z)}{z_z^3} + \dots$
3: $1 \gg z_\rho \gg z_z$	$\frac{5}{6} - \frac{3\pi}{4}z_\rho + \frac{21}{5}z_\rho^2 - \frac{\pi}{8}z_\rho \left(\frac{z_z}{z_\rho}\right)^2 - 2\pi z_\rho^3 + \dots$
4: $z_\rho \gg 1, z_\rho \gg z_z$	$\frac{1}{2} \left(\frac{8+z_z(13+4z_z)}{(1+z_z)^2} + 3 \ln \frac{1+z_z}{2z_\rho} \right) \frac{1}{z_\rho^2} + \dots$
5: $1 \ll z_\rho \ll z_z$	$\frac{\pi}{4} \frac{1}{z_\rho^2} \frac{z_\rho}{z_z} + \dots$

3.3 Dipolar interacton and isotropic Lorentzian disorder

In this section we study special case of dipolar interaction with isotropic disorder, which amounts to $r = z = \text{const.}$ We again have cylindrical symmetry and integrals (3.18)-(3.21) reduce to

$$-\Delta_{n_0} = \int_0^1 dt \frac{1}{c(1+zc)^2}, \quad (3.44)$$

$$\Delta_\mu = 2 \int_0^1 dt \frac{c(3+2zc)}{(1+zc)^2}, \quad (3.45)$$

$$\Delta_{\frac{\partial\mu}{\partial n}} = \int_0^1 dt \frac{c(3+zc)}{(1+zc)^3}, \quad (3.46)$$

$$I_{sd} = \int_0^1 dt \frac{t^2}{c(1+zc)^2}. \quad (3.47)$$

Making a substitution similar to the one in Eq. (3.32)

$$c = xt + \delta, \quad (3.48)$$

with $\delta = \sqrt{1-\epsilon}$, leads to the following integrals of rational functions:

$$-\Delta_{n_0} = \int_0^{\sqrt{3-2\delta^2}-\delta} dx \frac{-2x^2 + 6\epsilon}{(x^2(-1+z\delta) + 3(\epsilon + z\delta\epsilon))^2}, \quad (3.49)$$

$$\Delta_\mu = 2 \int_0^{\sqrt{3-2\delta^2}-\delta} dx \frac{4\delta^2(x^2+3\epsilon)^2(x^2(-3+2z\delta) + 9\epsilon + 6z\delta\epsilon)}{(-x^2+3\epsilon)^2(x^2(-1+z\delta) + 3(\epsilon + z\delta\epsilon))^2}, \quad (3.50)$$

$$\Delta_{\frac{\partial\mu}{\partial n}} = \int_0^{\sqrt{3-2\delta^2}-\delta} dx \frac{2\delta^2(x^2+3\epsilon)^2(x^2(-3+z\delta) + 3(3+z\delta)\epsilon)}{(-x^2+3\epsilon)(x^2(-1+z\delta) + 3(\epsilon + z\delta\epsilon))^3}, \quad (3.51)$$

$$I_{sd} = \int_0^{\sqrt{3-2\delta^2}-\delta} dx \frac{32x^2\delta^2}{(-x^2+3\epsilon)(x^2(-1+z\delta) + 3(\epsilon + z\delta\epsilon))^2}. \quad (3.52)$$

We present the results using the T -function, with $c = \frac{z\delta-1}{z\delta+1}$ and $\epsilon = \frac{4\lambda}{1-2\lambda+3\lambda^2}$:

$$-\Delta_{n_0} = \frac{z(1-\lambda)}{(-1+z^2\delta^2)(1-\lambda+z\delta(1+\lambda))} + \frac{(-1+\lambda)}{\delta(-1+z\delta)(1+z\delta)^2} T(c\lambda), \quad (3.53)$$

$$\Delta_\mu = \frac{2(-1+\lambda+2z\delta(-1-\lambda+z\delta(1-\lambda+z\delta(1+\lambda))))}{z(-1+z^2\delta^2)(1-\lambda+z\delta(1+\lambda))} \quad (3.54)$$

$$+ \frac{2(-1+\lambda)}{z^2\delta} T(-\lambda) + \frac{2(-1+\lambda)}{z^2\delta(-1+z\delta)(1+z\delta)^2} T(c\lambda). \quad (3.55)$$

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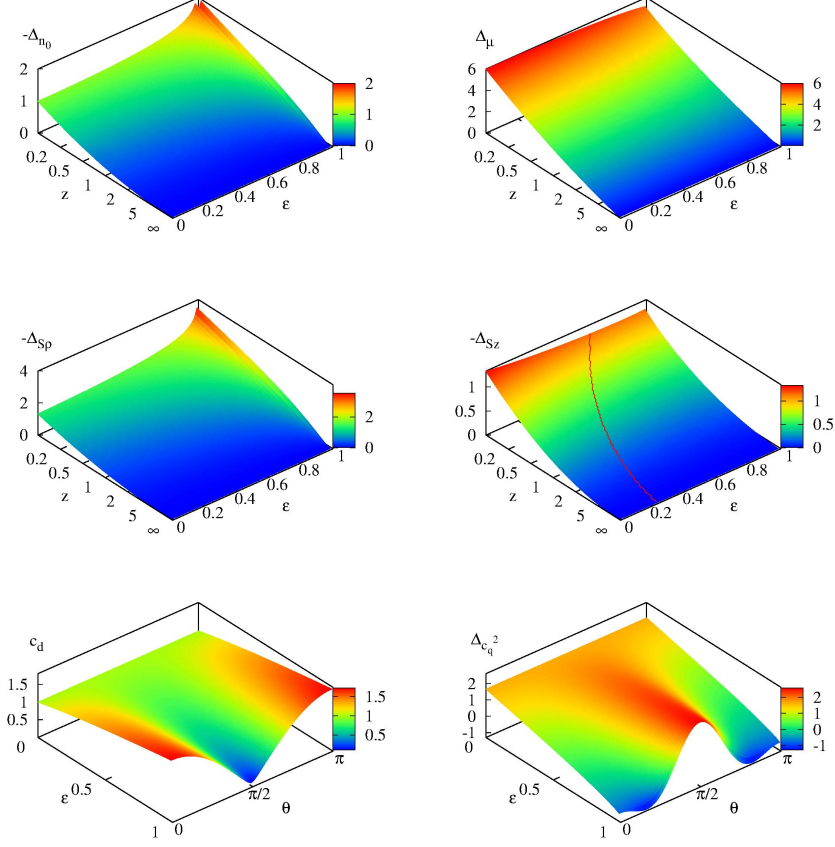


Figure 3.2: Corrections for the case of dipolar interaction and isotropic Lorentz-correlated disorder.

$$\Delta_{\frac{\partial \mu}{\partial n}} = \frac{(-1 + \lambda) (1 - \lambda + z^2 \delta^2 (2 - 2\lambda + 3z\delta(1 + \lambda)))}{z (-1 + z^2 \delta^2)^2 (1 - \lambda + z\delta(1 + \lambda))^2} \quad (3.56)$$

$$+ \frac{(1 - \lambda)}{z^2 \delta} T(-\lambda) - \frac{(-1 + 4z^2 \delta^2) (-1 + \lambda)}{z^2 \delta (-1 + z\delta)^2 (1 + z\delta)^3} T(c\lambda), \quad (3.57)$$

$$I_{sd} = \frac{(-1 + \lambda)^3}{4z\delta^2 \lambda (1 - \lambda + z\delta(1 + \lambda))} - \frac{(-1 + \lambda)^3}{4z^2 \delta^3 \lambda} T(-\lambda) + \frac{(-1 + \lambda)^3}{4z^2 \delta^3 (\lambda + z\delta \lambda)} T(c\lambda) \quad (3.58)$$

The above results are shown in Fig. 3.2. We stress that condensate depletion and depletion of the perpendicular component of superfluid increase as ϵ is increased and diverge for $\epsilon \rightarrow 1$. However, parallel component of the superfluid depletion decreases with increasing ϵ . The solid red line in the plot shows where the condensate depletion and the depletion of perpendicular superfluid component are equal.

3.4 Dipoles parallel to disorder symmetry

If all dipoles are parallel to the disorder symmetry, the whole system is cylindrically symmetric, as discussed in Sec. 3.1. This enables us to use the Euler substitutions given by Eqs. (3.32) and (3.48) in each term of the integrals (3.22)-(3.25). In this way, they are reduced to two integrals of rational function. However, for $c^2 r^2 = 1$ those diverge, but these divergences cancel out because original integrals (3.18)-(3.21) are well defined. The analytical results are presented detail in Appendix B.

3.4.1 Critical values of ϵ

The ratio of the parallel superfluid depletion and the condensate depletion increases with increasing z_ρ , but decreases with increasing z_z and ϵ . The ratio of the perpendicular superfluid depletion and the condensate depletion obeys complementary dependence laws. In Fig. 3.3 we plot the values of ϵ for which condensate depletion becomes larger then the parallel superfluid depletion or smaller then the perpendicular superfluid depletion.

Note that for certain values of correlation lengths the parallel superfluid density is always larger then the condensate density, regardless of the dipolar interaction strength.

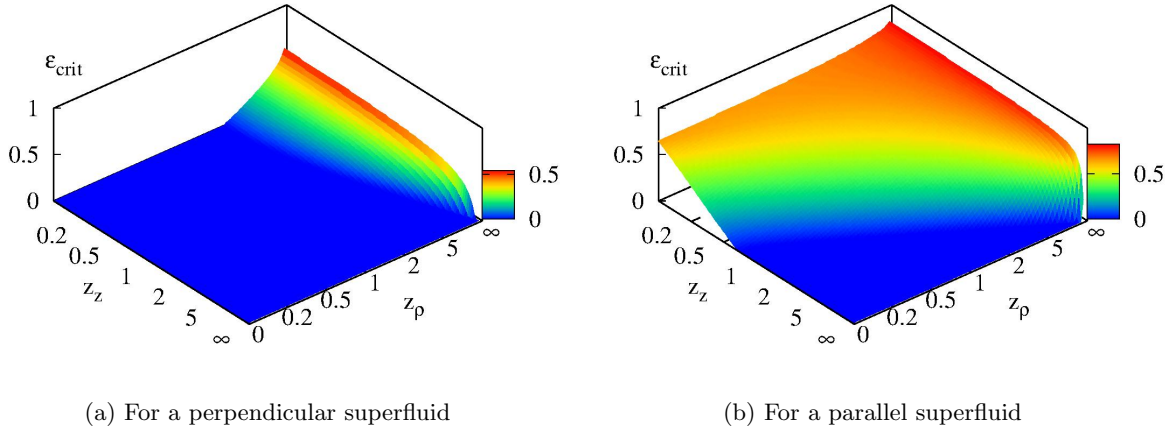


Figure 3.3: Critical values of ϵ for which superfluid and condensate depletion are equal.

3. DIPOLAR INTERACTION AND LORENTZ-CORRELATED DISORDER

Chapter 4

Conclusions

In this thesis we have studied Bose-Einstein condensates in disordered potentials, without the overall trap, using the Gross-Pitaevskii equation. To this end we have given the precise definition of the condensate density and the superfluid density and developed perturbation theory for their calculation for the case of arbitrary correlation function and interaction.

We have then applied this perturbation theory to the case of anisotropic disorder and dipolar interaction, obtaining qualitative, experimentally relevant results. In particular, we have calculated corrections due to disorder of condensate density, compressibility, superfluid density and the sound velocity, and studied their dependence on the strength of the dipolar interaction and the disorder correlation length. We stress that the anisotropy of the sound velocity should be measurable in Bragg spectroscopy experiments, while the anisotropic superfluid density should affect the collective excitations of a harmonically trapped condensate.

In future research we plan several extensions. To check the existing non-perturbative calculations [20], we will automatize calculation of higher perturbative corrections in the simplest case of contact interaction and δ -correlated disorder. We will also work on obtaining quantitative results for the laser-speckle potential using numerical simulations. Using finite-temperature Green's function we hope to extend the definition of the superfluid density, expecting that the thermally depleted component would move with the disorder, making the above definition general. We plan to investigate influence of the disorder on the critical temperature, and check the existence of a new phase of matter where the superfluidity is present without (global) condensation, and the opposite, where the condensation is present, but without superfluidity.

4. CONCLUSIONS

Appendix A

The T-function

In Sec. 3.2 We have defined

$$T(x) = \frac{\arctan \sqrt{x}}{\sqrt{x}},$$

where $|x| < 1$. Here we present some of the relevant properties of this function.

This is a well defined real-valued function for $x > 0$, but for $x < 0$ it has two branches and we have to choose one branch of the square root. We take the one for which $\sqrt{-1} = i$, and then for negative arguments we have

$$T(x) = \frac{\arctan i\sqrt{-x}}{i\sqrt{-x}} = \frac{\operatorname{arctanh}\sqrt{-x}}{\sqrt{-x}} = \frac{1}{2\sqrt{-x}} \ln \frac{1 + \sqrt{-x}}{1 - \sqrt{-x}}.$$

Taylor expansion of T around zero and 1 are given by

$$T(x) = 1 - \frac{x}{3} + \frac{x^2}{5} - \frac{x^3}{7} + \dots, \quad (\text{A.1})$$

$$T(1-x) = \frac{\pi}{4} + \frac{\pi-2}{8}x + \frac{3\pi-8}{32}x^2 + \dots. \quad (\text{A.2})$$

We also need behaviour of the function when its argument tends to -1 :

$$T(-1+x) = \frac{1}{2\sqrt{1-x}} \ln \frac{1 + \sqrt{1-x}}{1 - \sqrt{1-x}} = \ln 2 - \frac{\ln x}{2} + \left(\frac{\ln 2}{2} - \frac{1}{4} - \frac{\ln x}{4} \right) x + \dots. \quad (\text{A.3})$$

First and second derivatives of the function are equal to

$$T'(x) = \frac{1}{2x} \left(\frac{1}{1+x} - T(x) \right),$$

$$T''(x) = \frac{3}{4x^2} \left(-\frac{3+5x}{3(1+x)^2} + T(x) \right).$$

And, finally, we give a plot of the function in Fig. A.1.

A. THE T-FUNCTION

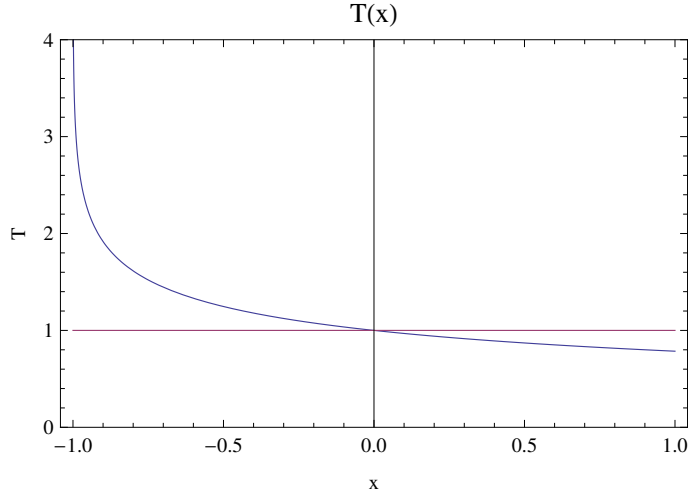


Figure A.1: Function $T(x)$.

A.1 Limiting cases of the T-function

Since we are interested in the different limits of physical quantities with respect to z_ρ and z_z , we have to distinguish areas of interest in the $z_\rho - z_z$ plane where arguments of the T -function in Sec. 3.2, $\frac{z_\rho-1}{z_\rho+1} \frac{z_z-z_\rho}{z_z+z_\rho}$ and $\frac{z_\rho-z_z}{z_\rho+z_z}$, approach values 0, -1 and 1 . In Fig. A.2 we show the mentioned areas for the argument $\frac{z_\rho-1}{z_\rho+1} \frac{z_z-z_\rho}{z_z+z_\rho}$ of the T -function.

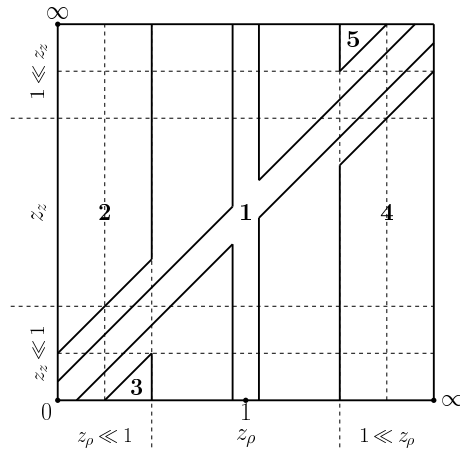


Figure A.2: Different limit areas for the argument $\frac{z_\rho-1}{z_\rho+1} \frac{z_z-z_\rho}{z_z+z_\rho}$ of the T -function.

In the diagonal part of the area 1 we have $z_z \approx z_\rho$, and functions are expanded to Taylor series given by Eq. (A.1).

In the area 2 we have $z_z \gg z_\rho$ and $1 \gg z_\rho$, and expand T -functions around -1 , Eq. (A.3).

In the area 3 we have $z_\rho \gg z_z$ and $1 \gg z_\rho$, and expand both T -functions in Taylor series around 1, Eq. (A.2).

In the area 4 we have $z_\rho \gg z_z$ and $z_\rho \gg 1$, and expand $T\left(\frac{z_\rho-1}{z_\rho+1} \frac{z_z-z_\rho}{z_z+z_\rho}\right)$ around -1 and $T\left(\frac{z_\rho-z_z}{z_\rho+z_z}\right)$ around 1.

In the area 5 we have $z_z \gg z_\rho$ and $z_\rho \gg 1$, and expand $T\left(\frac{z_\rho-z_z}{z_\rho+z_z}\right)$ around -1 and $T\left(\frac{z_\rho-1}{z_\rho+1} \frac{z_z-z_\rho}{z_z+z_\rho}\right)$ around 1.

For small (or large) values of z_z (compared to z_ρ) we can freely expand $T\left(\frac{z_\rho-z_z}{z_\rho+z_z}\right)$ around 1 or -1 , regardless of the value of the z_ρ . However, for $T\left(\frac{z_\rho-1}{z_\rho+1} \frac{z_z-z_\rho}{z_z+z_\rho}\right)$ we have to take care that we do not make Taylor expansion around a value that is close to -1 . If we denote $s = \text{sign}(z_z - z_\rho)$, for a correlation length much larger than the other we get

$$T\left(\frac{z_\rho-1}{z_\rho+1} \cdot \frac{z_z-z_\rho}{z_z+z_\rho}\right) = T\left(s \frac{z_\rho-1}{z_\rho+1}\right) - 2sT'\left(s \frac{z_\rho-1}{z_\rho+1}\right) \frac{z_\rho-1}{z_\rho+1} \left(\frac{z_\rho}{z_z}\right)^s + \dots \quad (\text{A.4})$$

This expression cannot be used in areas 2 and 4.

A. THE T-FUNCTION

Appendix B

Results for dipolar interaction with anisotropic disorder

General results for integrals (3.22)-(3.25) can be obtained by making Euler substitutions (3.48) in the first and (3.32) in the second summand. To do so, we first make a change of variable $t \rightarrow t/\sqrt{3\epsilon}$.

All the results can be expressed in terms of quantities ϵ , z_z and z_ρ , but use the following combinations of these quantities in order to simplify the calculation:

$$\delta = \sqrt{1 - \epsilon}, \quad (\text{B.1})$$

$$d = \frac{z_z^2 - z_\rho^2}{3\epsilon}, \quad (\text{B.2})$$

$$a = \frac{1}{2} (z_\rho^2 + d\delta^2), \quad (\text{B.3})$$

$$b = d + \frac{1}{4} (z_\rho^2 - d\delta^2)^2, \quad (\text{B.4})$$

$$l_c = \frac{-\delta + \sqrt{1 + 2\epsilon}}{\sqrt{3\epsilon}}, \quad (\text{B.5})$$

$$l_r = \frac{z_z - z_\rho}{\sqrt{3\epsilon}}. \quad (\text{B.6})$$

Here, $-a \pm \sqrt{b}$ are solutions of the equation $1 - c^2 r^2 = 0$ with respect to dt^2 , while l_c and l_r are upper limits of integration in Eqs. (3.22)-(3.25) after performing Euler substitutions. We also define new functions,

$$A(x, y) = T(x + \sqrt{y}) + T(x - \sqrt{y}), \quad (\text{B.7})$$

$$B(x, y) = \sqrt{b} (T(x + \sqrt{y}) - T(x - \sqrt{y})), \quad (\text{B.8})$$

B. RESULTS FOR DIPOLAR INTERACTION WITH ANISOTROPIC DISORDER

where T -function is assumed to be complex valued, and slightly redefined:

$$T(x) = \lim_{\epsilon \rightarrow 0} \frac{1}{2} \left(\frac{\arctan \sqrt{x+i\epsilon}}{\sqrt{x+i\epsilon}} + \frac{\arctan \sqrt{x-i\epsilon}}{\sqrt{x-i\epsilon}} \right). \quad (\text{B.9})$$

This enables expressing solutions in terms of real functions of real variables. Newly defined T -function is the same as the old one everywhere in the complex plane, except for $x < -1$, where both have discontinuities, but the new one is real-valued.

Several expressions will appear repeatedly in all four integrals, so we define

$$C_r = \frac{dl_r (d + l_r^2) z_\rho^2}{3(a^2 - b) b \sqrt{\epsilon} ((a^2 - b)(d - l_r^2)^4 + 8adl_r^2 (d - l_r^2)^2 z_\rho^2 + 16d^2 l_r^4 z_\rho^4)}, \quad (\text{B.10})$$

$$C_c = \frac{d\delta^2 l_c (1 + l_c^2)}{3(a^2 - b) b \sqrt{\epsilon} (16d^2 \delta^4 l_c^4 + (a^2 - b)(l_c^2 - 1)^4 + 8ad\delta^2 l_c (l_c^2 - 1)^2)}, \quad (\text{B.11})$$

$$x_r = \frac{4dl_r^2 (b + a(-a + z_\rho^2))}{(a^2 - b)(d + l_r^2)^2}, \quad y_r = \frac{16bd^2 l_r^4 z_\rho^4}{(a^2 - b)^2 (d + l_r^2)^4}, \quad (\text{B.12})$$

$$A_r = A(x_r, y_r), \quad B_r = B(x_r, y_r), \quad (\text{B.13})$$

$$x_c = \frac{4(-a^2 + b + ad\delta^2) l_c^2}{(a^2 - b)(1 + l_c^2)^2}, \quad y_c = \frac{16bd^2 \delta^4 l_c^4}{(a^2 - b)^2 (1 + l_c^2)^4}, \quad (\text{B.14})$$

$$A_c = A(x_c, y_c), \quad B_c = B(x_c, y_c). \quad (\text{B.15})$$

With the above definitions, the solutions are:

$$\begin{aligned} -\Delta_{n_0} &= \sqrt{3}d(-a^2 + b)(d - l_r^2)^2 - 4adl_r^2 z_\rho^2 C_r \\ &+ \sqrt{3}d(-4ad^2 \delta^4 l_c^2 + a(a^2 + 3b)(-1 + l_c^2)^2 - (a^2 + b)d\delta^2(1 - 6l_c^2 + l_c^4)) C_c \\ &+ \frac{(a^2 - 3b)d^2 l_r z_\rho^2}{2\sqrt{3}(a^2 - b)^2 b \sqrt{\epsilon} (d + l_r^2)} A_r \\ &+ \frac{d(d + l_r^2)(-(a^2 - b)^2 + a(a^2 - 2b)z_\rho^2)}{4\sqrt{3}(a^2 - b)b^2 \sqrt{\epsilon} l_r z_\rho^2} B_r \\ &+ \frac{dl_c(a^4 - b(b - 3d) - a^2 d - a^3 z_\rho^2 + abz_\rho^2)}{2\sqrt{3}(a^2 - b)^2 b \sqrt{\epsilon} (1 + l_c^2)} A_c \\ &+ \frac{d(1 + l_c^2)(2a(-b(b - 2d) + a^2(b - d)) + b(-a^2 + b)z_\rho^2)}{8\sqrt{3}(a^2 - b)b^2 \sqrt{\epsilon} l_c (2a - z_\rho^2)} B_c, \end{aligned} \quad (\text{B.16})$$

$$\begin{aligned}
\Delta_\mu &= 2\sqrt{3} \left((a^2 - b) (a (d - l_r^2)^2 + 4dl_r^2 z_\rho^2) - d\delta^2 \left((a^2 + b) (d - l_r^2)^2 + 4adl_r^2 z_\rho^2 \right) \right) C_r \\
&+ 2\sqrt{3}d \left(b (-1 + l_c^2)^2 + a^2 (1 + 6l_c^2 + l_c^4) - 4al_c^2 z_\rho^2 \right) C_c \\
&+ \frac{dl_r (a^4 + a^2d - b(b + 3d) - a^3 z_\rho^2 + abz_\rho^2)}{\sqrt{3} (a^2 - b)^2 b\sqrt{\epsilon} (d + l_r^2)} A_r \\
&+ \frac{(d + l_r^2) (2a (a^2(b + d) - b(b + 2d)) + b (-a^2 + b) z_\rho^2)}{4\sqrt{3} (a^2 - b) b^2 \sqrt{\epsilon} l_r z_\rho^2} B_r \\
&+ \frac{d (-a^2(2b + d) + b(2b + 3d)) \delta^2 l_c}{\sqrt{3} (a^2 - b)^2 b\sqrt{\epsilon} (1 + l_c^2)} A_c \\
&+ \frac{\left((a^2 - b)^2 (b + d) + ad (-a^2(b + d) + b(b + 2d)) \delta^2 \right) (1 + l_c^2)}{2\sqrt{3} (a^2 - b) b^2 d \delta^2 \sqrt{\epsilon} l_c} B_c \\
&+ \frac{8l_r}{\sqrt{3}d\sqrt{\epsilon}} T \left(-\frac{l_r^2}{d} \right), \tag{B.17}
\end{aligned}$$

$$\begin{aligned}
I_{sd} &= \frac{(a^2 - b) (a (d - l_r^2)^2 + 4dl_r^2 z_\rho^2)}{\sqrt{3}\epsilon} C_r \\
&+ \frac{(a^2 - b) (-4dl_c^2 + a^2 (1 + l_c^2)^2 - b (1 - 6l_c^2 + l_c^4) - a (1 + l_c^2)^2 z_\rho^2)}{\sqrt{3}\epsilon} C_c \\
&- \frac{adl_r z_\rho^2}{6\sqrt{3} (a^2 - b) b\epsilon^{3/2} (d + l_r^2)} A_r \\
&+ \frac{(d + l_r^2) (2a^3 - 2ab - 2a^2 z_\rho^2 + bz_\rho^2)}{24\sqrt{3} b^2 \epsilon^{3/2} l_r z_\rho^2} B_r \\
&+ \frac{d\delta^2 l_c (b + a (-a + z_\rho^2))}{6\sqrt{3} (a^2 - b) b\epsilon^{3/2} (1 + l_c^2)} A_c \\
&+ \frac{(-a^2(b - 2d) + b(b - d)) (1 + l_c^2)}{24\sqrt{3} b^2 \epsilon^{3/2} l_c (2a - z_\rho^2)} B_c, \tag{B.18}
\end{aligned}$$

The solution for $\Delta_{\frac{\partial\mu}{\partial(n)}}$ is more complex. We can write it in the form:

$$\begin{aligned}
\Delta_{\frac{\partial\langle\mu\rangle}{\partial(n)}} &= \frac{3\sqrt{3}\epsilon p_r}{2l_r dz_\rho^2 (d + l_r^2)} C_r^2 + \frac{3\sqrt{3}\epsilon p_c}{2\delta^2 l_c (1 + l_c^2)} C_c^2 \\
&+ \frac{dl_r p_{Ar}}{4\sqrt{3} (a^2 - b)^3 b^2 \sqrt{\epsilon} (d + l_r^2)} A_r + \frac{(d + l_r^2) p_{Br}}{16\sqrt{3} (a^2 - b)^2 b^3 \sqrt{\epsilon} l_r z_\rho^2} B_r \\
&+ \frac{p_{Ac}}{4\sqrt{3} (a^2 - b)^3 b^2 \sqrt{\epsilon} (1 + l_c^2)} A_c + \frac{(1 + l_c^2) p_{Bc}}{16\sqrt{3} (a^2 - b)^2 b^3 \sqrt{\epsilon} l_c (2a - z_\rho^2)} B_c,
\end{aligned}$$

B. RESULTS FOR DIPOLAR INTERACTION WITH ANISOTROPIC DISORDER

where p_r -s are polynomials given by:

$$\begin{aligned}
p_r = & 2a^7bd^6 + 2a^5b^2d^6 - 10a^3b^3d^6 + 6ab^4d^6 - 6a^7d^7 + 20a^5bd^7 + 18a^3b^2d^7 \\
& + 12a^7bd^5l_r^2 - 52a^5b^2d^5l_r^2 + 68a^3b^3d^5l_r^2 - 28ab^4d^5l_r^2 - 36a^7d^6l_r^2 + 152a^5bd^6l_r^2 \\
& + 192ab^3d^6l_r^2 - 72a^5d^7l_r^2 + 176a^3bd^7l_r^2 - 8ab^2d^7l_r^2 + 30a^7bd^4l_r^4 - 98a^5b^2d^4l_r^4 \\
& - 38ab^4d^4l_r^4 - 90a^7d^5l_r^4 + 172a^5bd^5l_r^4 + 14a^3b^2d^5l_r^4 - 96ab^3d^5l_r^4 - 128a^3bd^6l_r^4 \\
& + 40a^7bd^3l_r^6 - 88a^5b^2d^3l_r^6 + 56a^3b^3d^3l_r^6 - 8ab^4d^3l_r^6 - 120a^7d^4l_r^6 + 80a^5bd^4l_r^6 \\
& - 128ab^3d^4l_r^6 + 144a^5d^5l_r^6 - 992a^3bd^5l_r^6 + 1424ab^2d^5l_r^6 + 384a^3d^6l_r^6 - 768abd^6l_r^6 \\
& - 98a^5b^2d^2l_r^8 + 106a^3b^3d^2l_r^8 - 38ab^4d^2l_r^8 - 90a^7d^3l_r^8 + 172a^5bd^3l_r^8 + 14a^3b^2d^3l_r^8 \\
& - 128a^3bd^4l_r^8 - 256ab^2d^4l_r^8 + 12a^7bdl_r^{10} - 52a^5b^2dl_r^{10} + 68a^3b^3dl_r^{10} - 28ab^4dl_r^{10} \\
& + 152a^5bd^2l_r^{10} - 308a^3b^2d^2l_r^{10} + 192ab^3d^2l_r^{10} - 72a^5d^3l_r^{10} + 176a^3bd^3l_r^{10} \\
& + 2a^5b^2l_r^{12} - 10a^3b^3l_r^{12} + 6ab^4l_r^{12} - 6a^7dl_r^{12} + 20a^5bdl_r^{12} + 18a^3b^2dl_r^{12} \\
& - 2a^6bd^6z_\rho^2 + 2a^4b^2d^6z_\rho^2 + 2a^2b^3d^6z_\rho^2 - 2b^4d^6z_\rho^2 + 6a^6d^7z_\rho^2 - 21a^4bd^7z_\rho^2 \\
& + 4a^2b^2d^7z_\rho^2 + 11b^3d^7z_\rho^2 - 12a^6bd^5l_r^2z_\rho^2 + 44a^4b^2d^5l_r^2z_\rho^2 - 52a^2b^3d^5l_r^2z_\rho^2 \\
& + 20b^4d^5l_r^2z_\rho^2 + 36a^6d^6l_r^2z_\rho^2 - 58a^4bd^6l_r^2z_\rho^2 + 128a^2b^2d^6l_r^2z_\rho^2 - 106b^3d^6l_r^2z_\rho^2 \\
& - 30a^6bd^4l_r^4z_\rho^2 + 30a^4b^2d^4l_r^4z_\rho^2 + 30a^2b^3d^4l_r^4z_\rho^2 - 30b^4d^4l_r^4z_\rho^2 + 90a^6d^5l_r^4z_\rho^2 \\
& + 213a^4bd^5l_r^4z_\rho^2 - 484a^2b^2d^5l_r^4z_\rho^2 + 181b^3d^5l_r^4z_\rho^2 - 288a^4d^6l_r^4z_\rho^2 + 560a^2bd^6l_r^4z_\rho^2 \\
& + 112b^2d^6l_r^4z_\rho^2 - 40a^6bd^3l_r^6z_\rho^2 - 24a^4b^2d^3l_r^6z_\rho^2 + 168a^2b^3d^3l_r^6z_\rho^2 - 104b^4d^3l_r^6z_\rho^2 \\
& + 120a^6d^4l_r^6z_\rho^2 + 500a^4bd^4l_r^6z_\rho^2 - 960a^2b^2d^4l_r^6z_\rho^2 + 340b^3d^4l_r^6z_\rho^2 - 576a^4d^5l_r^6z_\rho^2 \\
& + 1184a^2bd^5l_r^6z_\rho^2 - 608b^2d^5l_r^6z_\rho^2 - 30a^6bd^2l_r^8z_\rho^2 + 30a^4b^2d^2l_r^8z_\rho^2 + 30a^2b^3d^2l_r^8z_\rho^2 \\
& - 30b^4d^2l_r^8z_\rho^2 + 90a^6d^3l_r^8z_\rho^2 + 213a^4bd^3l_r^8z_\rho^2 - 484a^2b^2d^3l_r^8z_\rho^2 + 181b^3d^3l_r^8z_\rho^2 \\
& - 288a^4d^4l_r^8z_\rho^2 + 560a^2bd^4l_r^8z_\rho^2 + 112b^2d^4l_r^8z_\rho^2 - 12a^6bdl_r^{10}z_\rho^2 + 44a^4b^2dl_r^{10}z_\rho^2 \\
& - 52a^2b^3dl_r^{10}z_\rho^2 + 20b^4dl_r^{10}z_\rho^2 + 36a^6d^2l_r^{10}z_\rho^2 - 58a^4bd^2l_r^{10}z_\rho^2 + 128a^2b^2d^2l_r^{10}z_\rho^2 \\
& - 106b^3d^2l_r^{10}z_\rho^2 - 2a^6bl_r^{12}z_\rho^2 + 2a^4b^2l_r^{12}z_\rho^2 + 2a^2b^3l_r^{12}z_\rho^2 - 2b^4l_r^{12}z_\rho^2 - 308a^3b^2d^6l_r^2 \\
& + 168a^3b^2d^4l_r^6 + 30a^7bd^2l_r^8 - 96ab^3d^3l_r^8 - 36a^7d^2l_r^{10} + 2a^7bl_r^{12} - 256ab^2d^6l_r^4 \\
& - 8ab^2d^3l_r^{10} - 32ab^3dl_r^{12} - 32ab^3d^7 + 11b^3dl_r^{12}z_\rho^2 + 106a^3b^3d^4l_r^4 + 6a^6dl_r^{12}z_\rho^2 \\
& - 21a^4bdl_r^{12}z_\rho^2 + 4a^2b^2dl_r^{12}z_\rho^2,
\end{aligned}$$

$$\begin{aligned}
p_c = & -3a^6b + a^4b^2 + 7a^2b^3 - 5b^4 + 6a^6d - 21a^4bd + 4a^2b^2d + 11b^3d - 82a^6bl_c^2 \\
& -22a^2b^3l_c^2 + 34b^4l_c^2 + 108a^6dl_c^2 - 246a^4bdl_c^2 + 16a^2b^2dl_c^2 - 70b^3dl_c^2 - 301a^6bl_c^4 \\
& +233a^2b^3l_c^4 - 11b^4l_c^4 + 378a^6dl_c^4 - 171a^4bdl_c^4 - 964a^2b^2dl_c^4 - 11b^3dl_c^4 - 288a^4d^2l_c^4 \\
& +112b^2d^2l_c^4 - 444a^6bl_c^6 + 20a^4b^2l_c^6 + 524a^2b^3l_c^6 - 100b^4l_c^6 + 552a^6dl_c^6 + 236a^4bdl_c^6 \\
& +268b^3dl_c^6 - 960a^4d^2l_c^6 + 2016a^2bd^2l_c^6 - 288b^2d^2l_c^6 - 301a^6bl_c^8 + 79a^4b^2l_c^8 \\
& +378a^6dl_c^8 - 171a^4bdl_c^8 - 964a^2b^2dl_c^8 - 11b^3dl_c^8 - 288a^4d^2l_c^8 + 560a^2bd^2l_c^8 \\
& +70a^4b^2l_c^{10} - 22a^2b^3l_c^{10} + 34b^4l_c^{10} + 108a^6dl_c^{10} - 246a^4bdl_c^{10} \\
& +a^4b^2l_c^{12} + 7a^2b^3l_c^{12} - 5b^4l_c^{12} + 6a^6dl_c^{12} - 21a^4bdl_c^{12} + 4a^2b^2dl_c^{12} + 11b^3dl_c^{12} \\
& +4a^3b^2z_\rho^2 - 2ab^3z_\rho^2 + 52a^5bl_c^2z_\rho^2 - 40a^3b^2l_c^2z_\rho^2 - 12ab^3l_c^2z_\rho^2 - 72a^5dl_c^2z_\rho^2 \\
& +176a^3bdl_c^2z_\rho^2 - 8ab^2dl_c^2z_\rho^2 + 226a^5bl_c^4z_\rho^2 - 196a^3b^2l_c^4z_\rho^2 - 30ab^3l_c^4z_\rho^2 - 11b^4l_c^8 \\
& -288a^5dl_c^4z_\rho^2 + 480a^3bdl_c^4z_\rho^2 + 192ab^2dl_c^4z_\rho^2 + 344a^5bl_c^6z_\rho^2 + 70a^4b^2l_c^2 - 82a^6bl_c^{10} \\
& -304a^3b^2l_c^6z_\rho^2 - 40ab^3l_c^6z_\rho^2 - 432a^5dl_c^6z_\rho^2 + 480a^3bdl_c^6z_\rho^2 + 528ab^2dl_c^6z_\rho^2 - 3a^6bl_c^{12} \\
& +384a^3d^2l_c^6z_\rho^2 - 768abd^2l_c^6z_\rho^2 + 226a^5bl_c^8z_\rho^2 - 196a^3b^2l_c^8z_\rho^2 - 30ab^3l_c^8z_\rho^2 \\
& -288a^5dl_c^8z_\rho^2 + 480a^3bdl_c^8z_\rho^2 + 192ab^2dl_c^8z_\rho^2 + 52a^5bl_c^{10}z_\rho^2 - 40a^3b^2l_c^{10}z_\rho^2 \\
& -12ab^3l_c^{10}z_\rho^2 - 72a^5dl_c^{10}z_\rho^2 + 176a^3bdl_c^{10}z_\rho^2 - 8ab^2dl_c^{10}z_\rho^2 - 2a^5bl_c^{12}z_\rho^2 \\
& -70b^3dl_c^{10} + 16a^2b^2dl_c^{10} - 2a^5bz_\rho^2 + 560a^2bd^2l_c^4 + 233a^2b^3l_c^8 \\
& +4a^3b^2l_c^{12}z_\rho^2 - 2ab^3l_c^{12}z_\rho^2 - 2208a^2b^2dl_c^6 + 79a^4b^2l_c^4 + 112b^2d^2l_c^8,
\end{aligned}$$

$$\begin{aligned}
p_{Ar} = & -2a^6b + 2a^4b^2 + 2a^2b^3 - 2b^4 + 6a^6d - 23a^4bd + 36a^2b^2d - 19b^3d \\
& +21b^2d^2 + 2a^5bz_\rho^2 - 4a^3b^2z_\rho^2 + 2ab^3z_\rho^2 - 6a^5dz_\rho^2 + 16a^3bdz_\rho^2 \\
& +6a^4d^2 - 15a^2bd^2 - 10ab^2dz_\rho^2,
\end{aligned}$$

$$\begin{aligned}
p_{Br} = & -4a^5b^2 + 8a^3b^3 - 4ab^4 - 9a^5bd + 26a^3b^2d - 17ab^3d + 12a^5d^2 - 33a^3bd^2 \\
& +2a^4b^2z_\rho^2 - 4a^2b^3z_\rho^2 + 2b^4z_\rho^2 - 4a^4bdz_\rho^2 + 12a^2b^2dz_\rho^2 - 8b^3dz_\rho^2 + 33ab^2d^2,
\end{aligned}$$

$$\begin{aligned}
p_{Ac} = & 8a^5b^2l_c - 16a^3b^3l_c + 8ab^4l_c + 8a^5bdl_c - 32a^3b^2dl_c + 24ab^3dl_c - 12a^5d^2l_c \\
& +32a^3bd^2l_c - 44ab^2d^2l_c - 4a^4b^2l_cz_\rho^2 + 8a^2b^3l_cz_\rho^2 - 4b^4l_cz_\rho^2 - 3a^4bdl_cz_\rho^2 \\
& -11b^3dl_cz_\rho^2 + 6a^4d^2l_cz_\rho^2 - 15a^2bd^2l_cz_\rho^2 + 21b^2d^2l_cz_\rho^2 + 14a^2b^2dl_cz_\rho^2,
\end{aligned}$$

B. RESULTS FOR DIPOLAR INTERACTION WITH ANISOTROPIC DISORDER

$$\begin{aligned} p_{Bc} = & 4a^6b^2 - 4a^4b^3 - 4a^2b^4 + 4b^5 + 7a^6bd - 21a^4b^2d + 5a^2b^3d + 9b^4d - 12a^6d^2 \\ & - 30a^2b^2d^2 - 13b^3d^2 - 4a^5b^2z_\rho^2 + 8a^3b^3z_\rho^2 - 4ab^4z_\rho^2 - 7a^5bdz_\rho^2 + 22a^3b^2dz_\rho^2 \\ & + 12a^5d^2z_\rho^2 - 33a^3bd^2z_\rho^2 + 33ab^2d^2z_\rho^2 + 31a^4bd^2 - 15ab^3dz_\rho^2. \end{aligned}$$

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